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Operations on level graphs of bipolar fuzzy graphs

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Abstract. We define the notion of level graphs of bipolar fuzzy graphs and use its to characterizations of various classical and new operations on bipolar fuzzy graphs.

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1 Introduction

The theory of graphs is an extremely useful tool for solving numerous problems in different areas such as geometry, algebra, operations research, optimization, and computer science. In many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it is natural to deal with the uncertainty using the methods of fuzzy sets, and fuzzy logic. But, the using of fuzzy graphs as models of various systems (social, economics systems, communication networks and others) leads to difficulties. In many domains, we deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. The bipolar fuzzy sets as an extension of fuzzy sets were introduced by Zhang [20, 21] in 1994. In a bipolar fuzzy set, the membership degree range is $[-1, 1]$, the member degree 0 of an element shows that the element is irrelevant to the corresponding property. If membership degree of an element is positive, it means that the element somewhat satisfies the property, and a negative membership degree shows that the element somewhat satisfies the implicit counter-property. The bipolar fuzzy graph model is more precise, flexible, and compatible as compared to the classical and fuzzy graph models. This is the motivation to generalize the notion of fuzzy graphs to the notion of bipolar fuzzy graphs. In 1965, Zadeh [19] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Now, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical, life science, management sciences, engineering, statistics, graph theory, signal processing, pattern recognition, computer networks and expert systems. Fuzzy graphs and fuzzy analogs of several graph theoretical notions were discussed by Rosenfeld [13], whose basic idea was introduced by Kauffmann [7] in 1973. Rosenfeld considered the fuzzy relations between fuzzy sets and developed the structure of fuzzy graphs. Some operations on fuzzy graphs were introduced by Mordeson and Peng [11]. Akram and

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Dudek [3] generalized some operations to interval-valued fuzzy graphs. The concept of intuitionistic fuzzy graphs was introduced by Shannon and Atanassov [16], they investigated some of their properties in [17]. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [12]. Akram introduced the concept of bipolar fuzzy graphs in [1], he discussed the concept of isomorphism of these graphs, and investigated some of their important properties, also defined some operations on bipolar fuzzy graphs (see also [2, 4–6]).

In this paper, we define the notion of level graphs of a bipolar fuzzy graph and investigate some of their properties. Next we show that level graphs can be used to the characterization of various products of two bipolar fuzzy graphs.

2 Preliminaries

In this section, we review some definitions that are necessary for this paper.

Let V be a nonempty set. Denote by $\widetilde{V^2}$ the collection of all 2-element subsets of V. A pair (V, E), where $E \subseteq \widetilde{V^2}$, is called a *graph*.

Further, for simplicity, the subsets of the form $\{x, y\}$ will be denoted by xy.

Definition 1. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs and let $V = V_1 \times V_2$.

- The *union of graphs* G_1^* and G_2^* is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.
- The graph $(V_1 \cup V_2, E_1 \cup E_2 \cup E')$, where E' is the set of edges joining vertices of V_1 and V_2, is denoted by $G_1^* + G_2^*$ and is called the *join of graphs* G_1^* and G_2^*.
- The *Cartesian product of graphs* G_1^* and G_2^*, denoted by $G_1^* \times G_2^*$, is the graph (V, E) with
$E = \{(x, x_2)(x, y_2) \mid x \in V_1\,,\ x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \mid z \in V_2\,,\ x_1y_1 \in E_1\}$.
- The *cross product of graphs* G_1^* and G_2^*, denoted by $G_1^* * G_2^*$, is the graph (V, E) such that $E = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1\,,\ x_2y_2 \in E_2\}$.
- The *lexicographic product of graphs* G_1^* and G_2^*, denoted by $G_1^* \bullet G_2^*$, is the graph (V, E) such that
$E = \{(x, x_2)(x, y_2) \mid x \in V_1,\ x_2y_2 \in E_2\} \cup \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1,\ x_2y_2 \in E_2\}$.
- The *strong product of graphs* G_1^* and G_2^*, denoted by $G_1^* \boxtimes G_2^*$, is the graph (V, E) such that
$E = \{(x, x_2)(x, y_2) \mid x \in V_1\,,\ x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \mid z \in V_2\,,\ x_1y_1 \in E_1\} \cup \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1\,,\ x_2y_2 \in E_2\}$.
- The *composition of graphs* G_1^* and G_2^*, denoted by $G_1^*[G_2^*]$, is the graph (V, E) such that
$E = \{(x, x_2)(x, y_2) \mid x \in V_1\,,\ x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \mid z \in V_2\,,\ x_1y_1 \in E_1\} \cup \{(x_1, x_2)(y_1, y_2) \mid x_2, y_2 \in V_2,\ ,\ x_2 \neq y_2\,,\ x_1y_1 \in E_1\}$.

One can find the corresponding examples clarifying the above concepts in [9,10, 14,15,18].

Definition 2. Let X be a set, a mapping $A = (\mu_A^N, \mu_A^P) : X \to [-1,0] \times [0,1]$ is called a *bipolar fuzzy set* on X. For every $x \in X$, the value $A(x)$ is written as $(\mu_A^N(x), \mu_A^P(x))$.

We use the *positive membership degree* $\mu_A^P(x)$ to denote the satisfaction degree of elements x to the property corresponding to a bipolar fuzzy set A, and the *negative membership degree* $\mu_A^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set A.

Definition 3. A *fuzzy graph* of a graph $G^* = (V, E)$ is a pair $G = (\sigma, \mu)$, where σ and μ are fuzzy sets on V and $\widetilde{V^2}$, respectively, such that $\mu(x, y) \le \min(\sigma(x), \sigma(y))$ for all $xy \in E$ and $\mu(xy) = 0$ for $xy \in \widetilde{V^2} \setminus E$.

Let $G^* = (V, E)$ be a crisp graph and let A, B be bipolar fuzzy sets on V and E, respectively. The pair (A, B) is called a *bipolar fuzzy pair* of a graph G^*.

Definition 4. ([1]) A *bipolar fuzzy graph* of a graph $G^* = (V, E)$ is a bipolar fuzzy pair $G = (A, B)$ of G^*, where $A = (\mu_A^N, \mu_A^P)$ and $B = (\mu_B^N, \mu_B^P)$ are such that

$$\mu_B^P(xy) \le \min(\mu_A^P(x), \mu_A^P(y)), \quad \mu_B^N(xy) \ge \max(\mu_A^N(x), \mu_A^N(y)) \quad \text{for all } xy \in E.$$

A fuzzy graph (σ, μ) of a graph G^* can be considered as an bipolar fuzzy graph $G = (A, B)$, where $\mu_A^N(x) = 0$ for all $x \in V$, $\mu_B^N(xy) = 0$ for all $xy \in E$ and $\mu_B^P = \mu$, $\mu_A^P = \sigma$.

Definition 5. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy pair of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Consider two bipolar fuzzy sets $A = (\mu_A^N, \mu_A^P)$ and $B = (\mu_B^N, \mu_B^P)$.

- The *union* $G_1 \cup G_2$ is defined as the pair (A, B) of bipolar fuzzy sets determined on the union of graphs G_1^* and G_2^* such that

$$\text{(i)} \qquad \mu_A^P(x) = \begin{cases} \mu_{A_1}^P(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}^P(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \max(\mu_{A_1}^P(x), \mu_{A_2}^P(x)) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(ii)} \qquad \mu_A^N(x) = \begin{cases} \mu_{A_1}^N(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}^N(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \min(\mu_{A_1}^N(x), \mu_{A_2}^N(x)) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(iii)} \qquad \mu_B^P(xy) = \begin{cases} \mu_{B_1}^P(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \mu_{B_2}^P(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \max(\mu_{B_1}^P(xy), \mu_{B_2}^P(xy)) & \text{if } xy \in E_1 \cap E_2, \end{cases}$$

(iv) $\mu_B^N(xy)=\begin{cases}\mu_{B_1}^N(xy) & \text{if } xy\in E_1 \text{ and } xy\notin E_2\\ \mu_{B_2}^N(xy) & \text{if } xy\in E_2 \text{ and } xy\notin E_1\\ \min(\mu_{B_1}^N(xy),\mu_{B_2}^N(xy)) & \text{if } xy\in E_1\cap E_2.\end{cases}$

- The *join* G_1+G_2 is the pair (A,B) of bipolar fuzzy sets defined on the join $G_1^*+G_2^*$ such that

(i) $\mu_A^P(x)=\begin{cases}\mu_{A_1}^P(x) & \text{if } x\in V_1 \text{ and } x\notin V_2\\ \mu_{A_2}^P(x) & \text{if } x\in V_2 \text{ and } x\notin V_1\\ \max(\mu_{A_1}^P(x),\mu_{A_2}^P(x)) & \text{if } x\in V_1\cap V_2,\end{cases}$

(ii) $\mu_A^N(x)=\begin{cases}\mu_{A_1}^N(x) & \text{if } x\in V_1 \text{ and } x\notin V_2\\ \mu_{A_2}^N(x) & \text{if } x\in V_2 \text{ and } x\notin V_1\\ \min(\mu_{A_1}^N(x),\mu_{A_2}^N(x)) & \text{if } x\in V_1\cap V_2,\end{cases}$

(iii) $\mu_B^P(xy)=\begin{cases}\mu_{B_1}^P(xy) & \text{if } xy\in E_1 \text{ and } xy\notin E_2\\ \mu_{B_2}^P(xy) & \text{if } xy\in E_2 \text{ and } xy\notin E_1\\ \max(\mu_{B_1}^P(xy),\mu_{B_2}^P(xy)) & \text{if } xy\in E_1\cap E_2\\ \min(\mu_{A_1}^P(x),\mu_{A_2}^P(x)) & \text{if } xy\in E',\end{cases}$

(iv) $\mu_B^N(xy)=\begin{cases}\mu_{B_1}^N(xy) & \text{if } xy\in E_1 \text{ and } xy\notin E_2\\ \mu_{B_2}^N(xy) & \text{if } xy\in E_2 \text{ and } xy\notin E_1\\ \min(\mu_{B_1}^N(xy),\mu_{B_2}^N(xy)) & \text{if } xy\in E_1\cap E_2\\ \max(\mu_{A_1}^N(x),\mu_{A_2}^N(y)) & \text{if } xy\in E'.\end{cases}$

- The *Cartesian product* $G_1\times G_2$ is the pair (A,B) of bipolar fuzzy sets defined on the Cartesian product $G_1^*\times G_2^*$ such that

(i) $\mu_A^P(x_1,x_2)=\min(\mu_{A_1}^P(x_1),\mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2)=\max(\mu_{A_1}^N(x_1),\mu_{A_2}^N(x_2))$ for all $(x_1,x_2)\in V_1\times V_2$,

(ii) $\mu_B^P((x,x_2)(x,y_2))=\min(\mu_{A_1}^P(x),\mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x,x_2)(x,y_2))=\max(\mu_{A_1}^N(x),\mu_{B_2}^N(x_2y_2))$ for all $x\in V_1$ and $x_2y_2\in E_2$,

(iii) $\mu_B^P((x_1,z)(y_1,z))=\min(\mu_{B_1}^P(x_1y_1),\mu_{A_2}^P(z))$,
$\mu_B^N((x_1,z)(y_1,z))=\max(\mu_{B_1}^N(x_1y_1),\mu_{A_2}^N(z))$ for all $z\in V_2$ and $x_1y_1\in E_1$.

- The *cross product* G_1*G_2 is the pair (A,B) of bipolar fuzzy sets defined on the cross product $G_1^**G_2^*$ such that

(i) $\mu_A^P(x_1,x_2)=\min(\mu_{A_1}^P(x_1),\mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2)=\max(\mu_{A_1}^N(x_1),\mu_{A_2}^N(x_2))$ for all $(x_1,x_2)\in V_1\times V_2$,

(ii) $\mu_B^P((x_1,x_2)(y_1,y_2))=\min(\mu_{B_1}^P(x_1y_1),\mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x_1,x_2)(y_1,y_2))=\max(\mu_{B_1}^N(x_1y_1),\mu_{B_2}^N(x_2y_2))$ for all $x_1y_1\in E_1$ and for all $x_2y_2\in E_2$.

- The *lexicographic product* $G_1\bullet G_2$ is the pair (A,B) of bipolar fuzzy sets defined on the lexigographic product $G_1^*\bullet G_2^*$ such that

(i) $\mu_A^P(x_1,x_2)=\min(\mu_{A_1}^P(x_1),\mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2)=\max(\mu_{A_1}^N(x_1),\mu_{A_2}^N(x_2))$ for all $(x_1,x_2)\in V_1\times v_2$,

(ii) $\mu_B^P((x,x_2)(x,y_2))=\min(\mu_{A_1}^P(x),\mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x,x_2)(x,y_2))=\max(\mu_{A_1}^N(x),\mu_{B_2}^N(x_2y_2))$ for all $x\in V_1$ and for all $x_2y_2\in E_2$,

(iii) $\mu_B^P((x_1,x_2)(y_1,y_2))=\min(\mu_{B_1}^P(x_1y_1),\mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x_1,x_2)(y_1,y_2))=\max(\mu_{B_1}^N(x_1y_1),\mu_{B_2}^N(x_2y_2))$ for all $x_1y_1\in E_1$ and for all $x_2y_2\in E_2$.

- The *strong product* $G_1\boxtimes G_2$ of G_1 is the pair (A,B) of bipolar fuzzy sets defined on the strong product $G_1^*\boxtimes G_2^*$ such that

(i) $\mu_A^P(x_1,x_2)=\min(\mu_{A_1}^P(x_1),\mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2)=\max(\mu_{A_1}^N(x_1),\mu_{A_2}^N(x_2))$ for all $(x_1,x_2)\in V_1\times V_2$,

(ii) $\mu_B^P((x,x_2)(x,y_2))=\min(\mu_{A_1}^P(x),\mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x,x_2)(x,y_2))=\max(\mu_{A_1}^N(x),\mu_{B_2}^N(x_2y_2))$ for all $x\in V_1$ and for all $x_2y_2\in E_2$,

(iii) $\mu_B^P((x_1,z)(y_1,z))=\min(\mu_{B_1}^P(x_1y_1),\mu_{A_2}^P(z))$,
$\mu_B^N((x_1,z)(y_1,z))=\max(\mu_{B_1}^N(x_1y_1),\mu_{A_2}^N(z))$ for all $z\in V_2$ and for all $x_1y_1\in E_1$,

(iv) $\mu_B^P((x_1,x_2)(y_1,y_2))=\min(\mu_{B_1}^P(x_1y_1),\mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x_1,x_2)(y_1,y_2))=\max(\mu_{B_1}^N(x_1y_1),\mu_{B_2}^N(x_2y_2))$ for all $x_1y_1\in E_1$ and for all $x_2y_2\in E_2$.

- The *composition* $G_1[G_2]$ is the pair (A,B) of bipolar fuzzy sets defined on the composition $G_1^*[G_2^*]$ such that

(i) $\mu_A^P(x_1,x_2)=\min(\mu_{A_1}^P(x_1),\mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2)=\max(\mu_{A_1}^N(x_1),\mu_{A_2}^N(x_2))$ for all $(x_1,x_2)\in V_1\times V_2$,

(ii) $\mu_B^P((x,x_2)(x,y_2))=\min(\mu_{A_1}^P(x),\mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x,x_2)(x,y_2))=\max(\mu_{A_1}^N(x),\mu_{B_2}^N(x_2y_2))$ for all $x\in V_1$ and for all $x_2y_2\in E_2$,

(iii) $\mu_B^P((x_1,z)(y_1,z))=\min(\mu_{B_1}^P(x_1y_1),\mu_{A_2}^P(z))$,
$\mu_B^N((x_1,z)(y_1,z))=\max(\mu_{B_1}^N(x_1y_1),\mu_{A_2}^N(z))$ for all $z\in V_2$ and for all $x_1y_1\in E_1$,

(iv) $\mu_B^P((x_1,x_2)(y_1,y_2))=\min(\mu_{A_2}^P(x_2),\mu_{A_2}^P(y_2),\mu_{B_1}^P(x_1y_1))$,
$\mu_B^N((x_1,x_2)(y_1,y_2))=\max(\mu_{A_2}^N(x_2),\mu_{A_2}^N(y_2),\mu_{B_1}^N(x_1y_1))$ for all $x_2,y_2\in V_2$, where $x_2\neq y_2$ and for all $x_1y_1\in E_1$.

3 Level graphs of bipolar fuzzy graphs

In this section we define the level graph of a bipolar fuzzy graph and discuss some important operations on bipolar fuzzy graphs by characterizing these operations by their level counterparts graphs.

Definition 6. Let $A : X \to [-1,0] \times [0,1]$ be a bipolar fuzzy set on X. The set $A_{(a,b)} = \{x \in X \mid \mu_A^P(x) \geq b\,,\ \mu_A^N(x) \leq a\}$, where $(a,b) \in [-1,0] \times [0,1]$, is called the (a,b)-*level set* of A.

The following theorem is important in this paper. It is substantial modification of the transfer principle for fuzzy sets described in [8].

Theorem 1. *Let V be a set, and $A = (\mu_A^N, \mu_A^P)$ and $B = (\mu_B^N, \mu_B^P)$ be bipolar fuzzy sets on V and $\widetilde{V^2}$, respectively. Then $G = (A,B)$ is a bipolar fuzzy graph if and only if $(A_{(a,b)}, B_{(a,b)})$, called the (a,b)-level graph of G, is a graph for each pair $(a,b) \in [-1,0] \times [0,1]$.*

Proof. Let $G = (A,B)$ be a bipolar fuzzy graph. For every $(a,b) \in [-1,0] \times [0,1]$, if $xy \in B_{(a,b)}$, then $\mu_B^N(xy) \leq a$ and $\mu_B^P(xy) \geq b$. Since G is a bipolar fuzzy graph,

$$a \geq \mu_B^N(xy) \geq \max(\mu_A^N(x), \mu_A^N(y))$$

and

$$b \leq \mu_B^P(xy) \leq \min(\mu_A^P(x), \mu_A^P(y)),$$

and so $a \geq \mu_A^N(x)$, $a \geq \mu_A^N(y)$, $b \leq \mu_A^P(x)$, $b \leq \mu_A^P(y)$, that is, $x, y \in A_{(a,b)}$. Therefore, $(A_{(a,b)}, B_{(a,b)})$ is a graph for each $(a,b) \in [-1,0] \times [0,1]$.

Conversely, let $(A_{(a,b)}, B_{(a,b)})$ be a graph for all $(a,b) \in [-1,0] \times [0,1]$. For every $xy \in \widetilde{V^2}$, let $\mu_B^N(xy) = a$ and $\mu_B^P(xy) = b$. Then $xy \in B_{(a,b)}$. Since $(A_{(a,b)}, B_{(a,b)})$ is a graph, we have $x, y \in A_{(a,b)}$; hence $\mu_A^N(x) \leq a$, $\mu_A^P(x) \geq b$, $\mu_A^N(y) \leq a$ and $\mu_A^P(x) \geq b$. Therefore,

$$\mu_B^N(xy) = a \geq \max(\mu_A^N(x), \mu_A^N(y))$$

and

$$\mu_B^P(xy) = b \leq \min(\mu_A^P(x), \mu_A^P(y)),$$

that is $G = (A,B)$ is a bipolar fuzzy graph. $\square$

Theorem 2. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A,B)$ is the Cartesian product of G_1 and G_2 if and only if for each pair $(a,b) \in [-1,0] \times [0,1]$ the (a,b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the Cartesian product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the Cartesian product of bipolar fuzzy graphs G_1 and G_2. For every $(a, b) \in [-1, 0] \times [0, 1]$, if $(x, y) \in A_{(a,b)}$, then

$$\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) = \mu^P_A(x, y) \geq b$$

and

$$\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) = \mu^N_A(x, y) \leq a,$$

hence $x \in (A_1)_{(a,b)}$ and $y \in (A_2)_{(a,b)}$; that is $(x, y) \in (A_1)_{(a,b)} \times (A_2)_{(a,b)}$. Therefore, $A_{(a,b)} \subseteq (A_1)_{(a,b)} \times (A_2)_{(a,b)}$. Now if $(x, y) \in (A_1)_{(a,b)} \times (A_2)_{(a,b)}$, then $x \in (A_1)_{(a,b)}$ and $y \in (A_2)_{(a,b)}$. It follows that $\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) \geq b$ and $\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) \leq a$. Since (A, B) is the Cartesian product of G_1 and G_2, $\mu^P_A(x, y) \geq b$ and $\mu^N_A(x, y) \leq a$; that is $(x, y) \in A_{(a,b)}$. Therefore, $(A_1)_{(a,b)} \times (A_2)_{(a,b)} \subseteq A_{(a,b)}$ and so $(A_1)_{(a,b)} \times (A_2)_{(a,b)} = A_{(a,b)}$.

We now prove $B_{(a,b)} = E$, where E is the edge set of the Cartesian product $(G_1)_{(a,b)} \times (G_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$. Then, $\mu^P_B((x_1, x_2)(y_1, y_2)) \geq b$ and $\mu^N_B((x_1, x_2)(y_1, y_2)) \leq a$. Since (A, B) is the Cartesian product of G_1 and G_2, one of the following cases holds:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$,

(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$.

For the case (i), we have

$$\mu^P_B((x_1, x_2)(y_1, y_2)) = \min(\mu^P_{A_1}(x_1), \mu^P_{B_2}(x_2y_2)) \geq b,$$

$$\mu^N_B((x_1, x_2)(y_1, y_2)) = \max(\mu^N_{A_1}(x_1), \mu^N_{B_2}(x_2y_2)) \leq a,$$

and so $\mu^P_{A_1}(x_1) \geq b$, $\mu^N_{A_1}(x_1) \leq a$, $\mu^P_{B_2}(x_2y_2) \geq b$ and $\mu^N_{B_2}(x_2y_2) \leq a$. It follows that $x_1 = y_1 \in (A_1)_{(a,b)}$, $x_2y_2 \in (B_2)_{(a,b)}$; that is $(x_1, x_2)(y_1, y_2) \in E$. Similarly, for the case (ii), we conclude that $(x_1, x_2)(y_1, y_2) \in E$. Therefore, $B_{(a,b)} \subseteq E$. For every $(x, x_2)(x, y_2) \in E$, $\mu^P_{A_1}(x) \geq b$, $\mu^N_{A_1}(x) \leq a$, $\mu^P_{B_2}(x_2y_2) \geq b$ and $\mu^N_{B_2}(x_2y_2) \leq a$. Since (A, B) is the Cartesian product of G_1 and G_2, we have

$$\mu^P_B((x, x_2)(x, y_2)) = \min(\mu^P_{A_1}(x), \mu^P_{B_2}(x_2y_2)) \geq b,$$

$$\mu^N_B((x, x_2)(x, y_2)) = \max(\mu^N_{A_1}(x), \mu^N_{B_2}(x_2y_2)) \leq a.$$

Therefore, $(x, x_2)(x, y_2) \in B_{(a,b)}$. Similarly, for every $(x_1, z)(y_1, z) \in E$, we have $(x_1, z)(y_1, z) \in B_{(a,b)}$. Therefore, $E \subseteq B_{(a,b)}$, and so $B_{(a,b)} = E$.

Conversely, suppose that the (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the Cartesian product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$ for all $(a, b) \in [-1, 0] \times [0, 1]$. Let $\min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2)) = b$ and $\max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2)) = a$ for some $(x_1, x_2) \in V_1 \times V_2$. Then $x_1 \in (A_1)_{(a,b)}$ and $x_2 \in (A_2)_{(a,b)}$. By the hypothesis, $(x_1, x_2) \in A_{(a,b)}$, hence

$$\mu^P_A((x_1, x_2)) \geq b = \min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2))$$

and
$$\mu_A^N((x_1,x_2)) \le a = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)).$$
Now let $\mu_A^N(x_1,x_2) = c$ and $\mu_A^P(x_1,x_2) = d$, then we have $(x_1,x_2) \in A_{(c,d)}$. Since $(A_{(c,d)}, B_{(c,d)})$ is the Cartesian product of levels $((A_1)_{(c,d)}, (B_1)_{(c,d)})$ and $((A_2)_{(c,d)}, (B_2)_{(c,d)})$, then $x_1 \in (A_1)_{(c,d)}$ and $x_2 \in (A_2)_{(c,d)}$. Hence,
$$\mu_{A_1}^P(x_1) \ge d = \mu_A^P(x_1,x_2), \qquad \mu_{A_1}^N(x_1) \le c = \mu_A^N(x_1,x_2),$$
$$\mu_{A_2}^P(x_2) \ge d = \mu_A^P(x_1,x_2) \quad \text{and} \quad \mu_{A_2}^N(x_2) \le c = \mu_A^N(x_1,x_2).$$
It follows that
$$\min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2)) \ge \mu_A^P(x_1,x_2)$$
and
$$\max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)) \le \mu_A^N(x_1,x_2).$$
Therefore,
$$\mu_A^P(x_1,x_2) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2))$$
and
$$\mu_A^N(x_1,x_2) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)) \quad \text{for all } (x_1,x_2) \in V_1 \times V_2.$$

Similarly, for every $x \in V_1$ and every $x_2y_2 \in E_2$, let
$$\min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) = b, \qquad \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) = a,$$
$$\mu_B^P((x,x_2)(x,y_2)) = d \quad \text{and} \quad \mu_B^N((x,x_2)(x,y_2)) = c.$$
Then we have $\mu_{A_1}^P(x) \ge b$, $\mu_{B_2}^P(x_2y_2) \ge b$, $\mu_{A_1}^N(x) \le a$, $\mu_{B_2}^N(x_2y_2) \le a$ and $(x,x_2)(x,y_2) \in B_{(c,d)}$, i.e., $x \in (A_1)_{(a,b)}$, $x_2y_2 \in (B_2)_{(a,b)}$ and $(x,x_2)(x,y_2) \in B_{(c,d)}$. Since $(A_{(a,b)}, B_{(a,b)})$ (respectively, $(A_{(c,d)}, B_{(c,d)})$) is the Cartesian product of levels $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$ (respectively, $((A_1)_{(c,d)}, (B_1)_{(c,d)})$ and $((A_2)_{(c,d)}, (B_2)_{(c,d)})$), we have $(x,x_2)(x,y_2) \in B_{(a,b)}$, $x \in (A_1)_{(c,d)}$, and $x_2y_2 \in (B_2)_{(c,d)}$, which implies $(x,x_2)(x,y_2) \in B_{(a,b)}$, $\mu_{A_1}^P(x) \ge d$, $\mu_{A_1}^N(x) \le c$, $\mu_{B_2}^P(x_2y_2) \ge d$ and $\mu_{B_2}^N(x_2y_2) \le c$. It follows that
$$\mu_B^N((x,x_2)(x,y_2)) \le a = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)),$$
$$\mu_B^P((x,x_2)(x,y_2)) \ge b = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)),$$
$$\min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) \ge d = \mu_B^P((x,x_2)(x,y_2)),$$
and
$$\max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) \le c = \mu_B^N((x,x_2)(x,y_2)).$$
Therefore,
$$\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)),$$
$$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2))$$

for all $x \in V_1$ and $x_2y_2 \in E_2$.

As above we can show that

$$\mu_B^P((x_1, z)(y_1, z)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{A_2}^P(z)),$$

$$\mu_B^N((x_1, z)(y_1, z)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{A_2}^N(z))$$

for all $z \in V_2$ and for all $x_1y_1 \in E_1$. This completes the proof. □

Now by Theorem 1 and Theorem 2 we have the following corollary.

Corollary 1. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are bipolar fuzzy graphs, then the Cartesian product $G_1 \times G_2$ is a bipolar fuzzy graph.*

Theorem 3. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the composition of G_1 and G_2 if and only if for each $(a, b) \in [-1, 0] \times [0, 1]$ the (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the composition of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the composition of bipolar fuzzy graphs G_1 and G_2. By the definition of $G_1[G_2]$ and the same argument as in the proof of Theorem 2, we have $A_{(a,b)} = (A_1)_{(a,b)} \times (A_2)_{(a,b)}$. Now we prove $B_{(a,b)} = E$, where E is the edge set of the composition $(G_1)_{(a,b)}[(G_2)_{(a,b)}]$ for all $(a, b) \in [-1, 0] \times [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$. Then $\mu_B^P((x_1, x_2)(y_1, y_2)) \geq b$ and $\mu_B^N((x_1, x_2)(y_1, y_2)) \leq a$. Since $G = (A, B)$ is the composition $G_1[G_2]$, one of the following cases holds:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$,

(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$,

(iii) $x_2 \neq y_2$ and $x_1y_1 \in E_1$.

For the cases (i) and (ii), similarly as in the cases of (i) and (ii) in the proof of Theorem 2, we obtain $(x_1, x_2)(y_1, y_2) \in E$. For the case (iii), we have

$$\mu_B^P((x_1, x_2)(y_1, y_2)) = \min(\mu_{A_2}^P(x_2), \mu_{A_2}^P(y_2), \mu_{B_1}^P(x_1y_1)) \geq b,$$

$$\mu_B^N((x_1, x_2)(y_1, y_2)) = \max(\mu_{A_2}^N(x_2), \mu_{A_2}^N(y_2), \mu_{B_1}^N(x_1y_1)) \leq a.$$

Thus, $\mu_{A_2}^P(x_2) \geq b$, $\mu_{A_2}^P(y_2) \geq b$, $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{A_2}^N(x_2) \leq a$, $\mu_{A_2}^N(y_2) \leq a$ and $\mu_{B_1}^N(x_1y_1) \leq a$. It follows that $x_2, y_2 \in (A_2)_{(a,b)}$ and $x_1y_1 \in (B_1)_{(a,b)}$; that is $(x_1, x_2)(y_1, y_2) \in E$. Therefore, $B_{(a,b)} \subseteq E$.

For every $(x, x_2)(x, y_2) \in E$, $\mu_{A_1}^P(x) \geq b$, $\mu_{A_1}^N(x) \leq a$, $\mu_{B_2}^P(x_2y_2)) \geq b$ and $\mu_{B_2}^N(x_2y_2) \leq a$. Since $G = (A, B)$ is the composition $G_1[G_2]$, we have

$$\mu_B^P((x, x_2)(x, y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) \geq b,$$

$$\mu_B^N((x, x_2)(x, y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) \leq a.$$

Therefore, $(x,x_1)(x,y_2) \in B_{(a,b)}$. Similarly, for every $(x_1,z)(y_1,z) \in E$, we have $(x,x_2)(x,y_2) \in B_{(a,b)}$. For every $(x_1,x_2)(y_1,y_2) \in E$, where $x_2 \neq y_2$, is $x_1 \neq y_1$, $\mu^P_{B_1}(x_1y_1) \geq b$, $\mu^N_{B_1}(x_1y_1) \leq a$, $\mu^P_{A_2}(y_2) \geq b$, $\mu^N_{A_2}(y_2) \leq a$, $\mu^P_{A_2}(x_2) \geq b$ and $\mu^N_{A_2}(x_2) \leq a$. Since $G = (A,B)$ is the composition $G_1[G_2]$, we have

$$\mu^P_B((x_1,x_2)(y_1,y_2)) = \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2), \mu^P_{B_1}(x_1y_1)) \geq b,$$

$$\mu^N_B((x_1,x_2)(y_1,y_2)) = \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2), \mu^N_{B_1}(x_1y_1)) \leq a,$$

hence $(x_1,x_2)(y_1,y_2) \in B_{(a,b)}$. Therefore $E \subseteq B_{(a,b)}$, and so $E = B_{(a,b)}$.

Conversely, suppose that $(A_{(a,b)}, B_{(a,b)})$, where $(a,b) \in [-1,0] \times [0,1]$, is the composition of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. By the definition of the composition and the proof of Theorem 2, we have

(i) $\mu^P_A(x_1,x_2) = \min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2))$,
$\mu^N_A(x_1,x_2) = \max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2))$ for all $(x_1,x_2) \in V_1 \times V_2$,

(ii) $\mu^P_B((x,x_2)(x,y_2)) = \min(\mu^P_{A_1}(x), \mu^P_{B_2}(x_2y_2))$,
$\mu^N_B((x,x_2)(x,y_2)) = \max(\mu^N_{A_1}(x), \mu^N_{B_2}(x_2y_2))$ for all $x \in V_1$ and $x_2y_2 \in E_2$,

(iii) $\mu^P_B((x_1,z)(y_1,z)) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{A_2}(z))$,
$\mu^N_B((x_1,z)(y_1,z)) = \max(\mu^N_{B_1}(x_1y_1), \mu^N_{A_2}(z))$ for all $z \in V_2$ and $x_1y_1 \in E_1$.

Similarly, by the same argumentation as in the proof of Theorem 2, we obtain

$$\mu^P_B((x_1,x_2)(y_1,y_2)) = \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2), \mu^P_{B_1}(x_1y_1)),$$

$$\mu^N_B((x_1,x_2)(y_1,y_2)) = \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2), \mu^N_{B_1}(x_1y_1))$$

for all $x_2, y_2 \in V_2$ $(x_2 \neq y_2)$ and for all $x_1y_1 \in E_1$. This completes the proof. □

Corollary 2. *If $G_1 = (A_1,B_1)$ and $G_2 = (A_2,B_2)$ are bipolar fuzzy graphs, then their composition $G_1[G_2]$ is a bipolar fuzzy graph.*

Theorem 4. *Let $G_1 = (A_1,B_1)$ and $G_2 = (A_2,B_2)$ be bipolar fuzzy graphs of $G^*_1 = (V_1,E_1)$ and $G^*_2 = (V_2,E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A,B)$ is the union of G_1 and G_2 if and only if each (a,b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A,B)$ be the union of bipolar fuzzy graphs G_1 and G_2. We show that $A_{(a,b)} = (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$ for each $(a,b) \in [-1,0] \times [0,1]$. Let $x \in A_{(a,b)}$, then $x \in V_1 \setminus V_2$ or $x \in V_2 \setminus V_1$. If $x \in V_1 \setminus V_2$, then $\mu^P_{A_1}(x) = \mu^P_A(x) \geq b$ and $\mu^N_{A_1}(x) = \mu^N_A(x) \leq a$, which implies $x \in (A_1)_{(a,b)}$. Analogously $x \in V_2 \setminus V_1$ implies $x \in (A_2)_{(a,b)}$. Therefore, $x \in (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$, and so $A_{(a,b)} \subseteq (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$.

Now let $x \in (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$. Then we have $x \in (A_1)_{(a,b)}$ and $x \notin (A_2)_{(a,b)}$ or $x \in (A_2)_{(a,b)}$ and $x \notin (A_1)_{(a,b)}$. For the first case, we have $\mu^P_A(x) = \mu^P_{A_1}(x) \geq b$ and $\mu^N_A(x) = \mu^N_{A_1}(x) \leq a$, which implies $x \in A_{(a,b)}$. For the second case, we have

$\mu_A^P(x) = \mu_{A_2}^P(x) \geq b$ and $\mu_A^N(x) = \mu_{A_2}^N(x) \leq a$. Hence $x \in A_{(a,b)}$. Consequently, $(A_1)_{(a,b)} \cup (A_2)_{(a,b)} \subseteq A_{(a,b)}$. To prove that $B_{(a,b)} = (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$ consider $xy \in B_{(a,b)}$. Then $xy \in E_1 \setminus E_2$ or $xy \in E_2 \setminus E_1$. For $xy \in E_1 \setminus E_2$ we have $\mu_{B_1}^P(xy) = \mu_B^P(xy) \geq b$ and $\mu_{B_1}^N(xy) = \mu_B^N(xy) \leq a$. Thus $xy \in (B_1)_{(a,b)}$. Similarly $xy \in E_2 \setminus E_1$ gives $xy \in (B_2)_{(a,b)}$. Therefore $B_{(a,b)} \subseteq (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$. If $xy \in (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$, then $xy \in (B_1)_{(a,b)} \setminus (B_2)_{(a,b)}$ or $xy \in (B_2)_{(a,b)} \setminus (B_1)_{(a,b)}$. For the first case, $\mu_B^P(xy) = \mu_{B_1}^P(xy) \geq b$ and $\mu_B^N(xy) = \mu_{B_1}^N(xy) \leq a$, hence $xy \in B_{(a,b)}$. In the second case we obtain $xy \in B_{(a,b)}$. Therefore, $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \subseteq B_{(a,b)}$.

Conversely, let for all $(a,b) \in [-1,0] \times [0,1]$ the level graph $(A_{(a,b)}, B_{(a,b)})$ be the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. Let $x \in V_1$, $\mu_{A_1}^P(x) = b$, $\mu_{A_1}^N(x) = a$, $\mu_A^P(x) = d$ and $\mu_A^N(x) = c$. Then $x \in (A_1)_{(a,b)}$ and $x \in A_{(c,d)}$. But by the hypothesis $x \in A_{(a,b)}$ and $x \in (A_1)_{(c,d)}$. Thus, $\mu_A^P(x) \geq b$, $\mu_A^N(x) \leq a$, $\mu_{A_1}^P(x) \geq d$ and $\mu_{A_1}^N(x) \leq c$. Therefore, $\mu_{A_1}^P(x) \leq \mu_A^P(x)$, $\mu_A^N(x) \geq \mu_{A_1}^N(x)$, $\mu_{A_1}^P(x) \geq \mu_A^P(x)$ and $\mu_{A_1}^N(x) \leq \mu_A^N(x)$. Hence $\mu_{A_1}^P(x) = \mu_A^P(x)$ and $\mu_A^N(x) = \mu_{A_1}^N(x)$. Similarly, for every $x \in V_2$, we get $\mu_{A_2}^P(x) = \mu_A^P(x)$ and $\mu_A^N(x) = \mu_{A_2}^N(x)$. Thus, we conclude that

$$\text{(i)} \begin{cases} \mu_A^P(x) = \mu_{A_1}^P(x) & \text{if } x \in V_1 \\ \mu_A^P(x) = \mu_{A_2}^P(x) & \text{if } x \in V_2, \end{cases}$$

$$\text{(ii)} \begin{cases} \mu_A^N(x) = \mu_{A_1}^N(x) & \text{if } x \in V_1 \\ \mu_A^N(x) = \mu_{A_2}^N(x) & \text{if } x \in V_2. \end{cases}$$

By a similar method as above, we obtain

$$\text{(iii)} \begin{cases} \mu_B^P(xy) = \mu_{B_1}^P(xy) & \text{if } xy \in E_1 \\ \mu_B^P(xy) = \mu_{B_2}^P(xy) & \text{if } xy \in E_2, \end{cases}$$

$$\text{(iv)} \begin{cases} \mu_B^N(xy) = \mu_{B_1}^N(xy) & \text{if } xy \in E_1 \\ \mu_B^N(xy) = \mu_{B_2}^N(xy) & \text{if } xy \in E_2. \end{cases}$$

This completes the proof. □

Corollary 3. *If G_1 and G_2 are bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, in which $V_1 \cap V_2 = \varnothing$, then $G_1 \cup G_2$ is a bipolar fuzzy graph.*

Theorem 5. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A, B)$ is the join of G_1 and G_2 if and only if each (a,b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the join of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the join of bipolar fuzzy graphs G_1 and G_2. Then by the definition and the proof of Theorem 4, $A_{(a,b)} = (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$. We show that $B_{(a,b)} = (B_1)_{(a,b)} \cup (B_2)_{(a,b)} \cup E'_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$, where $E'_{(a,b)}$ is the set of all edges joining the vertices $(A_1)_{(a,b)}$ and $(A_2)_{(a,b)}$.

From the proof of Theorem 4 it follows that $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \subseteq B_{(a,b)}$. If $xy \in E'_{(a,b)}$, then $\mu^P_{A_1}(x) \geq b$, $\mu^N_{A_1}(x) \leq a$, $\mu^P_{A_2}(y) \geq b$ and $\mu^N_{A_2}(y) \leq a$. Hence

$$\mu^P_B(xy) = \min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) \geq b$$

and

$$\mu^N_B(xy) = \max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) \leq a.$$

It follows that $xy \in B_{(a,b)}$. Therefore, $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \cup E'_{(a,b)} \subseteq B_{(a,b)}$. For every $xy \in B_{(a,b)}$, if $xy \in E_1 \cup E_2$, then $xy \in (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$, by the proof of Theorem 4. If $x \in V_1$ and $y \in V_2$, then

$$\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) = \mu^P_B(xy) \geq b$$

and

$$\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) = \mu^N_B(xy) \leq a,$$

hence $x \in (A_1)_{(a,b)}$ and $y \in (A_2)_{(a,b)}$. So, $xy \in E'_{(a,b)}$. Therefore, $B_{(a,b)} \subseteq (B_1)_{(a,b)} \cup (B_2)_{(a,b)} \cup E'_{(a,b)}$.

Conversely, let each level graph $(A_{(a,b)}, B_{(a,b)})$ be the join of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. From the proof of Theorem 4, we have

$$\text{(i)} \begin{cases} \mu^P_A(x) = \mu^P_{A_1}(x) & \text{if } x \in V_1 \\ \mu^P_A(x) = \mu^P_{A_2}(x) & \text{if } x \in V_2, \end{cases}$$

$$\text{(ii)} \begin{cases} \mu^N_A(x) = \mu^N_{A_1}(x) & \text{if } x \in V_1 \\ \mu^N_A(x) = \mu^N_{A_2}(x) & \text{if } x \in V_2, \end{cases}$$

$$\text{(iii)} \begin{cases} \mu^P_B(xy) = \mu^P_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu^P_B(xy) = \mu^P_{B_2}(xy) & \text{if } xy \in E_2, \end{cases}$$

$$\text{(iv)} \begin{cases} \mu^N_B(xy) = \mu^N_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu^N_B(xy) = \mu^N_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$$

Let $x \in V_1$, $y \in V_2$, $\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) = b$, $\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) = a$, $\mu^P_B(xy) = d$ and $\mu^N_B(xy) = c$. Then $x \in (A_1)_{(a,b)}$, $y \in (A_2)_{(a,b)}$ and $xy \in B_{(c,d)}$. It follows that $xy \in B_{(a,b)}$, $x \in (A_1)_{(c,d)}$ and $y \in (A_2)_{(c,d)}$. So, $\mu^P_B(xy) \geq b$, $\mu^N_B(xy) \leq a$, $\mu^P_{A_1}(x) \geq d$, $\mu^N_{A_1}(x) \leq c$, $\mu^P_{A_2}(y) \geq d$ and $\mu^N_{A_2}(y) \leq c$. Therefore,

$$\mu^P_B(xy) \geq b = \min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) \geq d = \mu^P_B(xy),$$

$$\mu^N_B(xy) \leq a = \max(\mu^N_{A_1}(x), \mu^P_{A_2}(y)) \leq c = \mu^N_B(xy).$$

Thus,

$$\mu^P_B(xy) = \min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)), \quad \mu^N_B(xy) = \max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)),$$

as desired. $\square$

Theorem 6. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the cross product of G_1 and G_2 if and only if each (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the cross product of $((A_1)_{(a,b)}, (B_1)_{(a,b)}$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the cross product of G_1 and G_2. By the definition of the Cartesian product $G_1 \times G_2$ and the proof of Theorem 2, we have $A_{(a,b)} = (A_1)_{(a,b)} \times (A_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. We show that

$$B_{(a,b)} = \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in (B_1)_{(a,b)}\ ,\ x_2y_2 \in (B_2)_{(a,b)}\}$$

for all $(a, b) \in [-1, 0] \times [0, 1]$. Indeed, if $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$, then

$$\mu_B^P((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) \geq b,$$

$$\mu_B^N((x_1, x_2)(y_1, y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^P(x_2y_2)) \leq a,$$

hence $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{B_2}^P(x_2y_2) \geq b$, $\mu_{B_1}^N(x_1y_1) \leq a$ and $\mu_{B_2}^P(x_2y_2) \leq a$. So, $x_1y_1 \in (B_1)_{(a,b)}$ and $x_2y_2 \in (B_2)_{(a,b)}$. Now if $x_1y_1 \in (B_1)_{(a,b)}$ and $x_2y_2 \in (B_2)_{(a,b)}$, then $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{B_1}^N(x_1y_1) \leq a$, $\mu_{B_2}^P(x_2y_2) \geq b$ and $\mu_{B_2}^N(x_2y_2) \leq a$. It follows that

$$\mu_B^P((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) \geq b,$$

$$\mu_B^N((x_1, x_2)(y_1, y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) \leq a,$$

because $G = (A, B)$ is the cross product $G_1 * G_2$. Therefore, $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$.

Conversely, let each (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ be the cross product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. In view of the fact that the cross product $(A_{(a,b)}, B_{(a,b)})$ has the same vertex set as the cartesian product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$, and by the proof of Theorem 2, we have

$$\mu_A^P((x_1, x_2)) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2)),$$

$$\mu_A^N((x_1, x_2)) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)),$$

for all $(x_1, x_2) \in V_1 \times V_2$.

Let $\min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) = b$, $\max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) = a$, $\mu_B^P((x_1, x_2)(y_1, y_2)) = d$ and $\mu_B^N((x_1, x_2)(y_1, y_2)) = c$ for $x_1y_1 \in E_1$, $x_2y_2 \in E_2$. Then $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{B_2}^P(x_2y_2) \geq b$, $\mu_{B_1}^N(x_1y_1) \leq a$, $\mu_{B_2}^N(x_2y_2) \leq a$ and $(x_1, x_2)(y_1, y_2) \in B_{(c,d)}$, hence $x_1y_1 \in (B_1)_{(a,b)}$, $x_2y_2 \in (B_2)_{(a,b)}$, and consequently, $x_1y_1 \in (B_1)_{(c,d)}$, $x_2y_2 \in (B_2)_{(c,d)}$ since $B_{(c,d)} = \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in (B_1)_{(c,d)}\ ,\ x_2y_2 \in (B_2)_{(c,d)}\}$. It follows that $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$, $\mu_{B_1}^P(x_1y_1) \geq d$, $\mu_{B_1}^N(x_1y_1) \leq c$, $\mu_{B_2}^P(x_2y_2) \geq d$ and $\mu_{B_2}^N(x_2y_2) \leq c$. Therefore,

$$\mu_B^P((x_1, x_2)(y_1, y_2)) = d \leq \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) = b \leq \mu_B^P((x_1, x_2)(y_1, y_2)),$$

$$\mu_B^N((x_1, x_2)(y_1, y_2)) = c \geq \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) = a \geq \mu_B^N((x_1, x_2)(y_1, y_2)).$$

Hence

$$\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)),$$

$$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)),$$

which completes our proof. □

Corollary 4. *The cross product of two bipolar fuzzy graphs is a bipolar fuzzy graph.*

Theorem 7. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the lexicographic product of G_1 and G_2 if and only if $G_{(a,b)} = (G_1)_{(a,b)} \bullet (G_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$.*

Proof. Let $G = (A, B) = G_1 \bullet G_2$. By the definition of the Cartesian product $G_1 \times G_2$ and the proof of Theorem 2, we have $A_{(a,b)} = (A_1)_{(a,b)} \times (A_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. We show that $B_{(a,b)} = E_{(a,b)} \cup E'_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$, where $E_{(a,b)} = \{(x, x_2)(x, y_2) \,|\, x \in V_1\,,\ x_2y_2 \in (B_2)_{(a,b)}\}$ is the subset the edge set of the direct product $(G_1)_{(a,b)} \times (G_2)_{(a,b)}$, and $E'_{(a,b)} = \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in (B_1)_{(a,b)}\,,\ x_2y_2 \in (B_2)_{(a,b)}\}$ is the edge set of the cross product $(G_1)_{(a,b)} * (G_2)_{(a,b)}$. For every $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$, $x_1 = y_1$, $x_2y_2 \in E_2$ or $x_1y_1 \in E_1$, $x_2y_2 \in E_2$. If $x_1 = y_1$, $x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E_{(a,b)}$, by the definition of the Cartesian product and the proof of Theorem 2. If $x_1y_1 \in E_1$, $x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E'_{(a,b)}$, by the definition of the cross product and the proof of Theorem 6. Therefore, $B_{(a,b)} \subseteq E_{(a,b)} \cup E'_{(a,b)}$. From the definition of the Cartesian product and the proof of Theorem 2, we conclude that $E_{(a,b)} \subseteq B_{(a,b)}$, and also from the definition of the cross product and the proof of Theorem 6, we obtain $E'_{(a,b)} \subseteq B_{(a,b)}$. Therefore, $E_{(a,b)} \cup E'_{(a,b)} \subseteq B_{(a,b)}$.

Conversely, let $G_{(a,b)} = (A_{(a,b)}, B_{(a,b)}) = (G_1)_{(a,b)} \bullet (G_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. We know that $(G_1)_{(a,b)} \bullet (G_2)_{(a,b)}$ has the same vertex set as the Cartesian product $(G_1)_{(a,b)} \times (G_2)_{(a,b)}$. Now by the proof of Theorem 2, we have

$$\mu_A^P((x_1, x_2)) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2)),$$

$$\mu_A^N((x_1, x_2)) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2))$$

for all $(x_1, x_2) \in V_1 \times V_2$.

Assume that for some $x \in V_1$ and $x_2y_2 \in E_2$ is $\min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) = b$, $\max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) = a$, $\mu_B^P((x, x_2)(x, y_2)) = d$ and $\mu_B^N((x, x_2)(x, y_2)) = c$. Then, in view of the definitions of the Cartesian and lexicographic products, we have

$$(x, x_2)(x, y_2) \in (B_1)_{(a,b)} \bullet (B_2)_{(a,b)} \Leftrightarrow (x, x_2)(x, y_2) \in (B_1)_{(a,b)} \times (B_2)_{(a,b)},$$

$$(x, x_2)(x, y_2) \in (B_1)_{(c,d)} \bullet (B_2)_{(c,d)} \Leftrightarrow (x, x_2)(x, y_2) \in (B_1)_{(c,d)} \times (B_2)_{(c,d)}.$$

From this, by the same argument as in the proof of Theorem 2, we can conclude

$$\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_A^P(x), \mu_{B_2}^P(x_2y_2)),$$

$$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_A^N(x), \mu_{B_2}^N(x_2y_2)).$$

Suppose now that we have $\mu_B^P((x_1,x_2)(y_1,y_2)) = d$, $\mu_B^N((x_1,x_2)(y_1,y_2)) = c$, $\min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) = b$, $\max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) = a$ for $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. Then, in view of the definitions of the cross product and the lexicographic product, we have

$$(x_1,x_2)(y_1,y_2) \in (B_1)_{(a,b)} \bullet (B_2)_{(a,b)} \Leftrightarrow (x_1,x_2)(y_1,y_2) \in (B_1)_{(a,b)} * (B_2)_{(a,b)},$$

$$(x_1,x_2)(y_1,y_2) \in (B_1)_{(c,d)} \bullet (B_2)_{(c,d)} \Leftrightarrow (x_1,x_2)(y_1,y_2) \in (B_1)_{(c,d)} * (B_2)_{(c,d)}.$$

By the same argument as in the proof of Theorem 6, we can conclude

$$\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)),$$

$$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)),$$

which completes the proof. □

Corollary 5. *The lexicographic product of two bipolar fuzzy graphs is a bipolar fuzzy graph.*

Lemma 1. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, such that $V_1 = V_2$, $A_1 = A_2$ and $E_1 \cap E_2 = \varnothing$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if $(A_{(a,b)}, B_{(a,b)})$ is the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$ for all $(a,b) \in [-1,0] \times [0,1]$.*

Proof. Let $G = (A, B)$ be the union of bipolar fuzzy graphs G_1 and G_2. Then by the definition of the union and the fact that $V_1 = V_2$, $A_1 = A_2$, we have $A = A_1 = A_2$, hence $A_{(a,b)} = (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$. We now show that $B_{(a,b)} = (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$. For every $xy \in (B_1)_{(a,b)}$ we have $\mu_B^P(xy) = \mu_{B_1}^P(xy) \geq b$ and $\mu_B^N(xy) = \mu_{B_1}^N(xy) \leq a$, hence $xy \in B_{(a,b)}$. Therefore, $(B_1)_{(a,b)} \subseteq B_{(a,b)}$. Similarly, we obtain $(B_2)_{(a,b)} \subseteq B_{(a,b)}$. Thus, $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \subseteq B_{(a,b)}$. For every $xy \in B_{(a,b)}$ either $xy \in E_1$ or $xy \in E_2$. If $xy \in E_1$, $\mu_{B_1}^P(xy) = \mu_B^P(xy) \geq b$ and $\mu_{B_1}^N(xy) = \mu_B^N(xy) \leq a$ and hence $xy \in (B_1)_{(a,b)}$. If $xy \in E_2$, we have $xy \in (B_2)_{(a,b)}$. Therefore, $B_{(a,b)} \subseteq (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$.

Conversely, suppose that the (a,b)-level graph $(A_{(a,b)}, B_{(a,b)})$ be the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. Let $\mu_A^P(x) = b$, $\mu_A^N(x) = a$, $\mu_{A_1}^P(x) = d$ and $\mu_{A_1}^N(x) = c$ for some $x \in V_1 = V_2$. Then $x \in A_{(a,b)}$ and $x \in (A_1)_{(c,d)}$, so $x \in (A_1)_{(a,b)}$ and $x \in A_{(c,d)}$, because $A_{(a,b)} = (A_1)_{(a,b)}$ and $A_{(c,d)} = (A_1)_{(c,d)}$. It follows that $\mu_{A_1}^P(x) \geq b$, $\mu_{A_1}^N(x) \leq a$, $\mu_A^P(x) \geq d$ and $\mu_A^N(x) \leq c$. Therefore, $\mu_{A_1}^P(x) \geq \mu_A^P(x)$, $\mu_{A_1}^N(x) \leq \mu_A^N(x)$, $\mu_A^P(x) \geq \mu_{A_1}^P(x)$ and $\mu_A^N(x) \leq \mu_{A_1}^N(x)$. So, $\mu_A^P(x) = \mu_{A_1}^P(x)$ and $\mu_A^N(x) = \mu_{A_1}^N(x)$. Since $A_1 = A_2$, $V_1 = V_2$, then $A = A_1 = A_1 \cup A_2$.

By a similar method, we conclude that

(i) $\begin{cases} \mu_B^P(xy) = \mu_{B_1}^P(xy) & \text{if } xy \in E_1 \\ \mu_B^P(xy) = \mu_{B_2}^P(xy) & \text{if } xy \in E_2, \end{cases}$

(ii) $\begin{cases} \mu_B^N(xy) = \mu_{B_1}^N(xy) & \text{if } xy \in E_1 \\ \mu_B^N(xy) = \mu_{B_2}^N(xy) & \text{if } xy \in E_2. \end{cases}$

This completes the proof. □

Theorem 8. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the strong product of G_1 and G_2 if and only if each $G_{(a,b)}$, where $(a, b) \in [-1, 0] \times [0, 1]$, is the strong product of $(G_1)_{(a,b)}$ and $(G_2)_{(a,b)}$.*

Proof. According to the definitions of the strong product, the cross product and the Cartesian product, we obtain $G_1 \boxtimes G_2 = (G_1 \times G_2) \cup (G_1 * G_2)$ and

$$(G_1)_{(a,b)} \boxtimes (G_2)_{(a,b)} = ((G_1)_{(a,b)} \times (G_2)_{(a,b)}) \cup ((G_1)_{(a,b)} * (G_2)_{(a,b)})$$

for all $(a, b) \in [-1, 0] \times [0, 1]$. Now by Theorem 6, Theorem 2 and Lemma 1, we see that

$$\begin{aligned} G = G_1 \boxtimes G_2 &\Longleftrightarrow G = (G_1 \times G_2) \cup (G_1 * G_2) \\ &\Longleftrightarrow G_{(a,b)} = (G_1 \times G_2)_{(a,b)} \cup (G_1 * G_2)_{(a,b)} \\ &\Longleftrightarrow G_{(a,b)} = ((G_1)_{(a,b)} \times (G_2)_{(a,b)}) \cup ((G_1)_{(a,b)} * (G_2)_{(a,b)}) \\ &\Longleftrightarrow G_{(a,b)} = (G_1)_{(a,b)} \boxtimes (G_2)_{(a,b)} \end{aligned}$$

for all $(a, b) \in [-1, 0] \times [0, 1]$. □

Corollary 6. *The strong product of two bipolar fuzzy graphs is a bipolar fuzzy graph.*

4 Conclusion

Graph theory is one of the branches of modern mathematics applied to many areas of mathematics, science, and technology. In computer science, graphs are used to represent networks of communication, computational devices, image segmentation, clustering and the flow of computation. In many cases, some aspects of a graph theoretic problem may be uncertain, and we deal with bipolar information. Bipolarity is met in many areas such as knowledge representation, reasoning with conditions, inconsistency handling, constraint satisfaction problem, decision, learning, etc. In this paper, we define the notion of level graph of a bipolar fuzzy graph and investigate some of their properties. We define three kinds of new operations of bipolar fuzzy graphs and discuss these operations and some defined important operations on bipolar fuzzy graphs by characterizing these operations by their level counterparts graphs.

References

[1] Akram M. *Bipolar fuzzy graphs* Information Sci., 2011, **181**, 5548–5564.

[2] Akram M. *Bipolar fuzzy graphs with applications.* Knowledge Based Systems, 2013, **39**, 1–8.

[3] Akram M., Dudek W. A. *Interval-valued fuzzy graphs.* Comput. Math. Appl., 2011, **61**, 289–299.

[4] Akram M., Dudek W. A. *Regular bipolar fuzzy graphs.* Neural Computing Appl., 2012, **21**, 197–205.

[5] Akram M., Dudek W. A., Sarwar S. *Properties of bipolar fuzzy hypergraphs.* Italian J. Pure Appl. Math., 2013, **31**, 426–458.

[6] Akram M., Li S., Shum K. P. *Antipodal bipolar fuzzy graphs.* Italian J. Pure Appl. Math., 2013, **31**, 425–438.

[7] Kauffman A. *Introduction a la theorie des sous-emsembles Flous.* Masson et cie, Vol. 1, 1973.

[8] Kondo M., Dudek W. A. *On the transfer principle in fuzzy theory.* Mathware and Soft Computing, 2005, **12**, 41–55.

[9] Lee K. M. *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets.* J. Fuzzy Logic Intell. Systems, 2004, **14**, 125–129.

[10] Lee K. M. *Bipolar-valued fuzzy sets and their basic operations.* Proc. Internat. Confer. on Intelligent Technologies, Bangkok, Thailand, 2000, 307–317.

[11] Mordeson J. N., Peng C. S. *Operations on fuzzy graphs.* Information Sci., 1994, **79**, 159–170.

[12] Parvathi R., Karunambigai N. G., Atanassov K. T. *Operations on intuitionistic fuzzy graphs.* Fuzzy Systems, FUZZ-IEEE 2009, IEEE International Conference, 2009, 1396–1401.

[13] Rosenfeld A. *Fuzzy graphs.* in: L. A. Zadeh, Fu, M. Shimura (Eds.), Fuzzy Sets and their Applications, Academic Press, New York, 1975, 77–95.

[14] Sabiduss G. *Graphs with given group and given graph theoretical properties.* Canad. J. Math., 1957, **9**, 515–525.

[15] Sabiduss G. *Graph multiplication.* Math. Z., 1960, **72**, 446–457.

[16] Shannon A., Atanassov K. T. *A first step to a theory of the intuitionistic fuzzy graphs,* Proc. of FUBEST, Sofia, 1994, 59–61.

[17] Shannon A., Atanassov K. T. *Intuitionistic fuzzy graphs from α-, β-, and (α, β)-levels.* Notes on Intuitionistic Fuzzy Sets, 1995, **1**, 32–35.

[18] Sunitha M. S., Vijaya Kumar A. *Complement of a fuzzy graph.* Indian J. Pure Appl. Math., 2002, **33**, 1451–1464.

[19] Zadeh L. A. *Fuzzy sets.* Inform. and Control, 1965, **8**, 338–353.

[20] Zhang W. R. *Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis.* Proc. IEEE Conference, 1994, 305–309.

[21] Zhang W. R. *Bipolar fuzzy sets.* Proc. of FUZZ-IEEE, 1998, 835–840.

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