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On paratopies of orthogonal systems of ternary quasigroups. I

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Abstract. A paratopy of an orthogonal system $\Sigma = \{A_1, A_2, \dots, A_n\}$ of n-ary quasigroups, defined on a nonempty set Q, is a mapping $\theta : Q^n \mapsto Q^n$ such that $\Sigma\theta = \Sigma$, where $\Sigma\theta = \{A_1\theta, A_2\theta, \dots, A_n\theta\}$. The paratopies of the orthogonal systems, consisting of two binary quasigroups and two binary selectors, have been described by Belousov in [1]. He proved that there exist 9 such systems, admitting at least one non-trivial paratopy and that the existence of paratopies implies (in many cases) the parastrophic-orthogonality of a quasigroup from Σ. A generalization of this result (ternary case) is considered in the present paper. We prove that there exist 153 orthogonal systems, consisting of three ternary quasigroups and three ternary selectors, which admit at least one non-trivial paratopy. The existence of paratopies implies (in many cases) some identities. One of them was considered earlier by T. Evans, who proved that it implies the self-orthogonality of the corresponding ternary quasigroup. The present paper contains the first part of our investigation. We give the necessary and sufficient conditions when a triple θ, consisting of three ternary quasigroup operations or of a ternary selector and two ternary quasigroup operations, defines a paratopy of Σ.

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Let Q be a nonempty set and let n be a positive integer. We will use below (x_1^n) to denote the n-tuple $(x_1, x_2, \dots, x_n) \in Q^n$ and $i = \overline{1,n}$ for "$i = 1, 2, \dots, n$". An n-ary groupoid (Q, A) is called an n-ary quasigroup if in the equality $A(x_1, x_2, \dots, x_n) = x_{n+1}$ any element of the set $\{x_1, x_2, \dots, x_{n+1}\}$ is uniquely determined by the remaining n elements. If (Q, A) is an n-ary quasigroup and $\sigma \in S_n$, then the operation ${}^{\sigma}A$ defined by the equivalence: ${}^{\sigma}A(x_{\sigma 1}, x_{\sigma 2}, \dots, x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1, x_2, \dots, x_n) = x_{n+1}$, for every $x_1, x_2, \dots, x_n, x_{n+1} \in Q$, is called a σ-parastrophe (or, simply, a parastrophe) of (Q, A). Following [3], we will denote the transposition $(i, n+1)$, where $i \in \{1, 2, \dots n\}$, by π_i, so ${}^{(i,n+1)}A = {}^{\pi_i} A$. A σ-parastrophe of an n-ary quasigroup (Q, A) is called a principal parastrophe if $\sigma(n+1) = n+1$. The n-ary operations $A_1, A_2, \dots, A_n$, defined on Q, are called orthogonal if, for every $a_1, a_2, \dots, a_n \in Q$, the system of equations $\{A_i(x_1, x_2, \dots, x_n) = a_i\}_{i=\overline{1,n}}$ has a unique solution [4]. A system of n-ary operations $A_1, A_2, \dots, A_s$, defined on a set Q, where $s \geq n$, is called orthogonal if every n operations of this system are orthogonal. For every mapping $\theta : Q^n \to Q^n$ there exist, and are unique, n n-ary operations $A_1, A_2, \dots, A_n$, defined on Q, such that

$\theta((x_1^n)) = (A_1(x_1^n), A_2(x_1^n), \ldots, A_n(x_1^n))$, for every $(x_1^n) \in Q^n$. Moreover, the mapping θ is a bijection if and only if the operations $A_1, A_2, \ldots, A_n$ are orthogonal [4, 6]. The operations $E_1, E_2, \ldots, E_n$, defined on Q, where $E_i(x_1, x_2, \ldots, x_n) = x_i$, for every $x_1, x_2, \ldots, x_n \in Q$, are called the n-ary selectors on Q. Remark that a system of n-ary operations $\{A_1, A_2, \ldots, A_s\}$, defined on Q, where $s \geq 1$, is called strongly orthogonal if the system $\{A_1, A_2, \ldots, A_s, E_1, E_2, \ldots, E_n\}$ is orthogonal [4]. An n-ary operation A is a quasigroup operation if and only if the system $\{A, E_1, E_2, \ldots, E_n\}$ is orthogonal, consequently in a strongly orthogonal system all non-selectors are quasigroup operations. n-Ary quasigroups for which there exist n orthogonal parastrophes (principal parastrophes) are called parastrophic-orthogonal (resp. self-orthogonal) quasigroups. It is known that quasigroups with minimal identities are parastrophic orthogonal [2, 7–9]. If $\Sigma = \{A_1, A_2, \ldots, A_n\}$ is an orthogonal system, then we will denote the system $\{A_1\theta, A_2\theta, \ldots, A_n\theta\}$ by $\Sigma\theta$. A bijection $\theta : Q^n \to Q^n$ is called a paratopy of the system Σ if $\Sigma\theta = \Sigma$.

Let $\Sigma = \{F, E, A, B\}$ be an orthogonal system, where A and B are binary quasigroups defined on a nonempty set Q, F and E are the binary selectors on Q: $F(x, y) = x, E(x, y) = y, \forall x, y \in Q$, and let $\theta : Q^2 \to Q^2$, $\theta = (C, D)$, be a mapping, where C and D are binary operations on Q and $\theta(x, y) = (C(x, y), D(x, y))$, for every $x, y \in Q$. If θ is a paratopy of Σ, i.e. if $\Sigma\theta = \Sigma$, then $\Sigma = \Sigma\theta = \{A\theta, B\theta, F\theta, E\theta\} = \{A\theta, B\theta, C, D\}$, so $C, D \in \Sigma$. Belousov described in [1] all orthogonal systems, consisting of two binary quasigroups and two binary selectors, which have at least one paratopy. Necessary and sufficient conditions when a pair of operations of $\Sigma = \{F, E, A, B\}$ is a paratopy of Σ are given in the following theorem.

Theorem 1 [1]. *Let $\Sigma = \{F, E, A, B\}$ be an orthogonal system of binary operations, defined on a nonempty set Q, where F and E are the binary selectors. Then:*
1. $\theta = (F, E)$ is a paratopy of Σ;
2. $\theta = (E, F)$ is a paratopy of Σ if and only if $B =^s A$, where $s = (12)$;
3. $\theta = (F, A)$ is a paratopy of Σ if and only if $B =^r A$, where $r = (23)$, and (Q, A) satisfies the identity $A(x, A(x, A(x, y))) = y$;
4. $\theta = (A, F)$ is a paratopy of Σ if and only if one of the following conditions holds:
a. $B =^{rl} A$, where $l = (13)$, and (Q, A) satisfies the identity $A(A(y, x), A(x, y)) = x$,
b. $A =^{lr} B(F, B)$ and (Q, B) satisfies the identity ${}^rB({}^rB({}^rB(x, y), y), y) = x$;
5. $\theta = (E, A)$ is a paratopy of Σ if and only if one of the following conditions holds:
a. $B =^{lr} A$ and (Q, A) satisfies the identity $A(A(x, y), x) = A(y, A(x, y))$,
b. $A =^{rl} B(B, E)$ and (Q, B) satisfies the identity ${}^rB({}^rB({}^rB(x, y), y), y) = x$;
6. $\theta = (A, E)$ is a paratopy of Σ if and only if $B =^l A$ and (Q, A) satisfies the identity $A(x, A(x, A(x, y))) = y$;
7. $\theta = (A, B)$ is a paratopy of Σ if and only if one of the following conditions holds:
a. $B =^s A$ and (Q, A) satisfies the identity $A(A(y, x), A(x, y)) = x$,
b. $B =^{rl} A(F, A)$.

A generalization of this result for the ternary case is given in the present paper. Let A_1, A_2, A_3 be ternary quasigroups defined on a nonempty set Q and let E_1, E_2, E_3 be the ternary selectors: $E_i(x_1, x_2, x_3) = x_i, \forall x_1, x_2, x_3 \in Q, i = \overline{1,3}$. We consider the orthogonal system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ and denote the set $\{A_1\theta, A_2\theta, A_3\theta, E_1\theta, E_2\theta, E_3\theta\}$ by $\Sigma\theta$. Let $\theta : Q^3 \to Q^3$, $\theta = (B_1, B_2, B_3)$, be a mapping, where B_1, B_2, B_3 are ternary operations on Q and $\theta(x_1^3) = (B_1(x_1^3), B_2(x_1^3), B_3(x_1^3))$, for every $(x_1^3) \in Q^3$. If θ is a paratopy of Σ, then $\Sigma = \Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_1\theta, E_2\theta, E_3\theta\} = \{A_1\theta, A_2\theta, A_3\theta, B_1, B_2, B_3\}$, so $\{B_1, B_2, B_3\} \subset \Sigma$, i.e. all paratopies of Σ are triples of operations from Σ. We study the necessary and sufficient conditions when a triple of operations of Σ defines a paratopy of Σ. So as the ternary selectors E_1, E_2, E_3 are fixed, we consider the tuples containing all possible distributions of the ternary selectors in their positions. We prove that, similarly to the binary case, a triple of operations of Σ defines a paratopy of Σ if and only if two quasigroup operations of Σ can be expressed by the third quasigroup operation (using parastrophy and/or superposition) and, in most of cases, the corresponding quasigroup satisfies an identity. Moreover, some of the obtained identities involve the self-orthogonality of the corresponding ternary quasigroup or of its binary retracts.

The present paper includes the first part of our investigation. We consider the triples with three ternary quasigroup operations of Σ and those with a ternary selector and two ternary quasigroup operations (there are 9 possible cases for the triples containing a selector, as E_i occurs in each of three positions and $i = 1, 2, 3$). We prove that there exist 48 orthogonal systems consisting of three ternary quasigroups and ternary selectors E_1, E_2, E_3, that admit at least one paratopy, which components are three ternary quasigroup operations or a ternary selector and two ternary quasigroup operations.

Lemma 1. *The triple (A_1, A_2, A_3) of ternary quasigroups, defined on a nonempty set Q, is a paratopy of the orthogonal system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ if and only if one of the following statements holds:*
1. $A_2 =^{(132)} A_1,\ A_3 =^{(123)} A_1$ *and* $A_1(A_1, {}^{(132)}A_1, {}^{(123)}A_1) = E_2$;
2. $A_2 =^{(132)} A_1,\ A_3 =^{(123)} A_1$ *and* $A_1(A_1, {}^{(132)}A_1, {}^{(123)}A_1) = E_3$;
3. $A_1 =^{(12)} A_2,\ A_3 =^{\pi_3} A_2({}^{(12)}A_2, A_2, E_1)$ *and* $A_3 =^{(12)} A_3$;
4. $A_2 =^{(23)} A_3,\ A_1 =^{\pi_1} A_3(E_2, {}^{(23)}A_3, A_3)$ *and* $A_1 =^{(23)} A_1$;
5. $A_3 =^{(13)} A_1,\ A_2 =^{\pi_2} A_1(A_1, E_3, {}^{(13)}A_1)$ *and* $A_2 =^{(13)} A_2$;
6. $A_3 =^{\pi_3} A_1(A_1, A_2, E_1) =^{\pi_3} A_2(A_1, A_2, E_2)$.

Proof. Let $\theta = (A_1, A_2, A_3)$ be a paratopy of the system Σ. Using $E_1\theta = A_1, E_2\theta = A_2$ and $E_3\theta = A_3$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, A_3\}$, hence $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_1, E_2, E_3\}$, i.e. there are six possible cases.

1. If $A_1\theta = E_2, A_2\theta = E_3, A_3\theta = E_1$, then $\theta^2 = (E_2, E_3, E_1)$, $\theta^3 = (A_2, A_3, A_1)$, $\theta^4 = (E_3, E_1, E_2)$, $\theta^5 = (A_3, A_1, A_2)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_2$ it follows $A_1\theta^4 = A_3$, i.e. $A_1(E_3, E_1, E_2) = A_3$, so

$$A_3 =^{(123)} A_1. \tag{1.1}$$

Also $A_1\theta = E_2$ implies $A_1\theta^2 = A_2$ and $A_1(E_2, E_3, E_1) = A_2$, hence

$$A_2 =^{(132)} A_1. \tag{1.2}$$

Using (1.1) and (1.2) in $A_1\theta = E_2$, we get

$$A_1(A_1,^{(132)}A_1,^{(123)}A_1) = E_2. \tag{1.3}$$

Conversely, if (1.1), (1.2) and (1.3) hold, then using (1.1) and (1.2) in (1.3), we obtain $A_1\theta = E_2$, so $A_1(A_1(x_1^3), A_2(x_1^3), A_3(x_1^3)) = E_2(x_1^3)$, $\forall x_1, x_2, x_3 \in Q$, which implies $^{(132)}A_1(A_3(x_1^3), A_1(x_1^3), A_2(x_1^3)) = E_2(x_1^3)$. Using (1.2) in the last equality, we get $A_2(^{(123)}A_1(x_1^3),^{(123)}A_2(x_1^3),^{(123)}A_3(x_1^3)) = E_2(x_1^3)$, hence

$$A_2(A_1(x_3, x_1, x_2), A_2(x_3, x_1, x_2), A_3(x_3, x_1, x_2)) = E_3(x_3, x_1, x_2),$$

i.e. $A_2\theta = E_3$. From $A_1\theta = E_2$ it follows $A_1(A_1(x_1^3),\ A_2(x_1^3),\ A_3(x_1^3)) = E_2(x_1^3)$, which implies $^{(123)}A_1(A_2(x_1^3),\ A_3(x_1^3),\ A_1(x_1^3)) = E_2(x_1^3)$. Using (1.1) in the last equality, we get $A_3(^{(132)}A_1(x_1^3),^{(132)}A_2(x_1^3),^{(132)}A_3(x_1^3)) = E_2(x_1^3)$ hence, for $\forall x_1, x_2, x_3 \in Q$, $A_3(A_1(x_2, x_3, x_1), A_2(x_2, x_3, x_1), A_3(x_2, x_3, x_1)) = E_1(x_2, x_3, x_1)$, i.e. $A_3\theta = E_1$.

2. Let $A_1\theta = E_3, A_2\theta = E_1, A_3\theta = E_2$, then $\theta^2 = (E_3, E_1, E_2)$, $\theta^3 = (A_3, A_1, A_2)$, $\theta^4 = (E_2, E_3, E_1)$, $\theta^5 = (A_2, A_3, A_1)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_3$ it follows $A_1\theta^2 = A_3$, i.e. $A_1(E_3, E_1, E_2) = A_3$, so

$$A_3 =^{(123)} A_1. \tag{1.4}$$

Also $A_1\theta = E_3$ implies $A_1\theta^4 = A_2$, i.e. $A_1(E_2, E_3, E_1) = A_2$, hence

$$A_2 =^{(132)} A_1. \tag{1.5}$$

Using (1.4) and (1.5) in $A_1\theta = E_3$, we get

$$A_1(A_1,^{(132)}A_1,^{(123)}A_1) = E_3. \tag{1.6}$$

Conversely, if (1.4), (1.5) and (1.6) hold, then using (1.4) and (1.5) in (1.6), we obtain $A_1\theta = E_3$. Now, the equality $A_1\theta = E_3$ implies $A_1(A_1(x_1^3), A_2(x_1^3), A_3(x_1^3)) = E_3(x_1^3)$, so $^{(123)}A_1(A_2(x_1^3), A_3(x_1^3), A_1(x_1^3)) = E_3(x_1^3)$. Using (1.4) in the last equality, we get $A_3(^{(132)}A_1(x_1^3),^{(132)}A_2(x_1^3),^{(132)}A_3(x_1^3)) = E_3(x_1^3)$, hence

$$A_3(A_1(x_2, x_3, x_1), A_2(x_2, x_3, x_1), A_3(x_2, x_3, x_1)) = E_2(x_2, x_3, x_1),$$

$\forall x_1, x_2, x_3 \in Q$, i.e. $A_3\theta = E_2$. From $A_1\theta = E_3$ follows $A_1(A_1(x_1^3), A_2(x_1^3), A_3(x_1^3)) = E_3(x_1^3)$, which implies $^{(132)}A_1(A_3(x_1^3), A_1(x_1^3), A_2(x_1^3)) = E_2(x_1^3)$. Using (1.5) in the last equality, we get $A_2(^{(123)}A_1(x_1^3),^{(123)}A_2(x_1^3),^{(123)}A_3(x_1^3)) = E_3(x_1^3)$, hence

$$A_2(A_1(x_3, x_1, x_2), A_2(x_3, x_1, x_2), A_3(x_3, x_1, x_2)) = E_1(x_3, x_1, x_2),$$

$\forall x_1, x_2, x_3 \in Q$, i.e. $A_2\theta = E_1$.

3. Let $A_1\theta = E_2$, $A_2\theta = E_1$, $A_3\theta = E_3$, then $\theta^2 = (E_2, E_1, E_3)$, $\theta^3 = (A_2, A_1, A_3)$, $\theta^4 = \varepsilon$. From $A_2\theta = E_1$ it follows $A_2\theta^2 = A_1$, i.e. $A_2(E_2, E_1, E_3) = A_1$, so

$$A_1 =^{(12)} A_2. \tag{1.7}$$

Also $A_2\theta = E_1$ and (1.7) imply

$$A_3 =^{\pi_3} A_2(^{(12)}A_2, A_2, E_1). \tag{1.8}$$

From $A_2\theta = E_3$ it follows $A_3\theta^2 = A_3$, i.e. $A_3(E_2, E_1, E_3) = A_3$, so

$$A_3 =^{(12)} A_3. \tag{1.9}$$

Conversely, if (1.7), (1.8) and (1.9) hold, then from (1.7) and (1.8) it follows $A_2\theta = E_1$, so $A_2(A_1(x_1^3), A_2(x_1^3), A_3(x_1^3)) = E_1(x_1^3)$, and ${}^{(12)}A_2(A_2(x_1^3), A_1(x_1^3), A_3(x_1^3)) = E_1(x_1^3)$. Using (1.7) in the last equality, we get

$A_1({}^{(12)}A_2(x_2, x_1, x_3), {}^{(12)}A_1(x_2, x_1, x_3), {}^{(12)}A_3(x_2, x_1, x_3)) = E_2(x_2, x_1, x_3)$,

hence $A_1(A_1(x_2, x_1, x_3), A_2(x_2, x_1, x_3), A_3(x_2, x_1, x_3)) = E_2(x_2, x_1, x_3)$, $\forall x_1, x_2, x_3 \in Q$, i.e. $A_1\theta = E_2$. Also, from $A_2\theta = E_1$ it follows $A_2(A_1\theta, A_2\theta, A_3\theta) = A_1$, so $A_2(E_2, E_1, A_3\theta) = A_1$. Using (1.7) in the last equality, we get ${}^{(12)}A_1(E_2, E_1, A_3\theta) = A_1$, hence $A_1(E_1, E_2, A_3\theta) = A_1$, which implies $A_3\theta = E_3$.

4. Let $A_1\theta = E_1, A_2\theta = E_3, A_3\theta = E_2$, then $\theta^2 = (E_1, E_3, E_2)$, $\theta^3 = (A_1, A_3, A_2)$, $\theta^4 = \varepsilon$. From $A_3\theta = E_2$ it follows $A_3\theta^2 = A_2$, i.e. $A_3(E_1, E_3, E_2) = A_2$, so

$$A_2 =^{(23)} A_3. \tag{1.10}$$

Also $A_3\theta = E_2$ and (1.10) imply

$$A_1 =^{\pi_1} A_3(E_2, {}^{(23)}A_3, A_3). \tag{1.11}$$

From $A_1\theta = E_1$ it follows $A_1\theta^2 = A_1$, i.e. $A_1(E_1, E_3, E_2) = A_1$, so

$$A_1 =^{(23)} A_1. \tag{1.12}$$

Conversely, if (1.10), (1.11) and (1.12) hold, then from (1.10) and (1.11) it follows $A_3\theta = E_2$. Now, the equality $A_3\theta = E_2$ implies $A_3(A_1(x_1^3), A_2(x_1^3), A_3(x_1^3)) = E_2(x_1^3)$, so ${}^{(23)}A_3(A_1(x_1^3), A_3(x_1^3), A_2(x_1^3)) = E_2(x_1^3)$. Using (1.10) in the last equality, we get $A_2({}^{(23)}A_1(x_1, x_3, x_2), {}^{(23)}A_3(x_1, x_3, x_2), {}^{(23)}A_2(x_1, x_3, x_2)) = E_3(x_1, x_3, x_2)$, hence $A_2(A_1(x_1, x_3, x_2), A_2(x_1, x_3, x_2), A_3(x_1, x_3, x_2)) = E_3(x_1, x_3, x_2)$, $\forall x_1, x_2, x_3 \in Q$, i.e. $A_2\theta = E_3$. Also from $A_3\theta = E_2$ it follows $A_3(A_1\theta, A_2\theta, A_3\theta) = A_2$, so $A_3(A_1\theta, E_3, E_2) = A_1$. Using (1.10) in the last equality, we get ${}^{(23)}A_2(A_1\theta, E_3, E_2) = A_1$, hence $A_2(A_1\theta, E_2, E_3) = A_2$, which implies $A_1\theta = E_1$.

5. Let $A_1\theta = E_3, A_2\theta = E_2, A_3\theta = E_1$, then $\theta^2 = (E_3, E_2, E_1)$, $\theta^3 = (A_3, A_2, A_1)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_3$ it follows $A_1\theta^2 = A_3$, i.e. $A_1(E_3, E_2, E_1) = A_3$, so

$$A_3 =^{(13)} A_1. \tag{1.13}$$

Also $A_1\theta = E_3$ and (1.13) imply

$$A_2 =^{\pi_2} A_1(A_1, E_3, {}^{(13)}A_1). \tag{1.14}$$

From $A_2\theta = E_2$ it follows $A_2\theta^2 = A_2$ i.e. $A_2(E_3, E_2, E_1) = A_2$, hence

$$A_2 =^{(13)} A_2. \tag{1.15}$$

Conversely, if (1.13), (1.14) and (1.15) hold, then $A_1\theta = E_3$. Now using $A_1\theta = E_3$, we get $A_1(A_1(x_1^3), A_2(x_1^3), A_3(x_1^3)) = E_3(x_1^3)$, so ${}^{(13)}A_1(A_3(x_1^3), A_2(x_1^3), A_1(x_1^3)) = E_3(x_1^3)$. Using (1.13) in the last equality, we get

$$A_3({}^{(13)}A_3(x_3,x_2,x_1),{}^{(13)}A_2(x_3,x_2,x_1),{}^{(13)}A_1(x_3,x_2,x_1)) = E_1(x_3,x_2,x_1),$$
for $\forall x_1,x_2,x_3 \in Q$, hence
$$A_3(A_1(x_3,x_2,x_1),A_2(x_3,x_2,x_1),A_3(x_3,x_2,x_1)) = E_1(x_3,x_2,x_1),$$
i. e. $A_3\theta = E_1$. Also from $A_1\theta = E_3$ it follows $A_1(A_1\theta,A_2\theta,A_3\theta) = A_3$, so $A_1(E_3,A_2\theta,E_1) = A_3$. Using (1.13) in the last equality, we get ${}^{(13)}A_3(E_3,A_2\theta,E_1) = A_3$, hence $A_3(E_1,A_2\theta,E_3) = A_3$, which implies $A_2\theta = E_2$.

6. Let $A_1\theta = E_1, A_2\theta = E_2, A_3\theta = E_3$, then $\theta^2 = \varepsilon$. From $A_1\theta = E_1$ it follows

$$A_3 ={}^{\pi_3} A_1(A_1,A_2,E_1), \tag{1.16}$$

and from $A_2\theta = E_2$ it follows

$$A_3 ={}^{\pi_3} A_2(A_1,A_2,E_2). \tag{1.17}$$

Conversely, if (1.16) and (1.17) hold, then from (1.16) we get $A_1\theta = E_1$, and (1.17) implies $A_2\theta = E_2$. Now (1.16) implies $A_3\theta ={}^{\pi_3} A_1(E_1,E_2,A_1)$, so $A_3\theta = E_3$. □

Remark. It was proved by T. Evans in [5] that, if a ternary quasigroup (Q,A) satisfies the identity $A(A,{}^{(132)}A,{}^{(123)}A) = E_i$, for some $i \in \{1,2,3\}$, then the triple of principal parastrophes $\{A,{}^{(132)}A,{}^{(123)}A\}$ is orthogonal, so (Q,A) is self-orthogonal of type $(\varepsilon,(132),(123))$, where ε is the unit of the symmetric group S_3.

Lemma 2. *The triple (E_1,A_1,A_2) is a paratopy of the system $\Sigma = \{A_1,A_2,A_3,E_1,E_2,E_3\}$ if and only if one of the following statements holds:*
1. $A_2 ={}^{(23)} A_1$, $A_3(E_1,A_1,{}^{(23)}A_1) = A_3$ *and* $A_1(E_1,A_1,{}^{(23)}A_1) = E_3$;
2. $A_2 ={}^{\pi_3} A_1(E_1,A_1,E_2)$, $A_3(E_1,A_1,{}^{\pi_3}A_1(E_1,A_1,E_2)) = A_3$;
3. $A_1 ={}^{\pi_2} A_2(E_1,E_2,A_2)$, $A_3 ={}^{\pi_3} A_2(E_1,A_2,E_3)$ *and* $A_2(E_1,E_3,{}^{\pi_2}A_2(E_1,E_2,A_2)) = {}^{\pi_3}A_2(E_1,A_2,E_3)$;
4. $A_2 ={}^{\pi_3} A_1(E_1,A_1,E_3)$, $A_3 ={}^{\pi_2} A_1(E_1,E_2,A_1)$ *and* $A_1(E_1,{}^{\pi_3}A_1(E_1,A_1,E_3),E_2) = {}^{\pi_2}A_1(E_1,E_2,A_1)$.

Proof. Let the tuple (E_1,A_1,A_2) be a paratopy of the system Σ. Using $E_1\theta = E_1, E_2\theta = A_1, E_3\theta = A_2$, we obtain $\Sigma = \Sigma\theta = \{A_1\theta,A_2\theta,A_3\theta,A_1,A_2,E_1\}$, i. e. $\{A_1\theta,A_2\theta,A_3\theta\} = \{E_2,E_3,A_3\}$.

1. If $A_1\theta = E_3, A_2\theta = E_2, A_3\theta = A_3$, then $\theta^2 = (E_1,E_3,E_2)$, $\theta^3 = (E_1,A_2,A_1)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_3$ it follows $A_1\theta^2 = A_2$, so

$$A_2 ={}^{(23)} A_1. \tag{2.1}$$

Using (2.1) in $A_3\theta = A_3$, we get

$$A_3(E_1,A_1,{}^{(23)}A_1) = A_3. \tag{2.2}$$

From $A_1\theta = E_3$ and (2.1) it follows

$$A_1(E_1,A_1,{}^{(23)}A_1) = E_3. \tag{2.3}$$

Conversely, if (2.1), (2.2) and (2.3) hold, then (2.1) and (2.3) imply

$$A_1(E_1,A_1,A_2) = E_3, \tag{2.4}$$

so $A_1\theta = E_3$. From (2.1) it follows

$$E_2 =^{\pi_3} A_1(E_1, E_3, A_2). \tag{2.5}$$

The equality (2.4) implies $A_2 =^{\pi_3} A_1(E_1, A_1, E_3)$, hence $A_2\theta =^{\pi_3} A_1(E_1, E_3, A_2)$. Using (2.5) in the last equality, we obtain $A_2\theta = E_2$. From (2.2) and (2.1) it follows $A_3\theta = A_3$.

2. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = E_2$, then $\theta^2 = (E_1, E_3, A_3)$, $\theta^3 = (E_1, A_2, E_2)$, $\theta^4 = (E_1, A_3, A_1)$, $\theta^5 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_2 =^{\pi_3} A_1(E_1, A_1, E_3). \tag{2.6}$$

Also $A_1\theta = E_3$ implies $A_1\theta^4 = E_2$, i.e. $A_1(E_1, A_3, A_1) = E_2$, so

$$A_3 =^{\pi_2} A_1(E_1, E_2, A_1). \tag{2.7}$$

Analogously, from $A_1\theta = E_3$ it follows $A_1\theta^3 = A_3$, i.e. $A_1(E_1, A_2, E_2) = A_3$. Using (2.6) and (2.7) in the last equality, we get

$$A_1(E_1, {}^{\pi_3}A_1(E_1, A_1, E_3), E_2) =^{\pi_2} A_1(E_1, E_2, A_1). \tag{2.8}$$

Conversely, if (2.6), (2.7) and (2.8) hold, then (2.6) implies $A_1\theta = E_3$. From (2.7) follows $A_3\theta =^{\pi_2} A_1(E_1, A_1, E_3)$, so $A_3\theta = E_2$. Using (2.6) and (2.7) in (2.8), we obtain $A_1(E_1, A_2, E_2) = A_3$, so $A_2 =^{\pi_2} A_1(E_1, A_3, E_2), \Rightarrow A_2\theta =^{\pi_2} A_1(E_1, E_2, A_1)$. Using (2.7) in the last equality, we get $A_2\theta = A_3$.

3. If $A_1\theta = A_3, A_2\theta = E_2, A_3\theta = E_3$, then $\theta^2 = (E_1, A_3, E_2)$, $\theta^3 = (E_1, E_3, A_1)$, $\theta^4 = (E_1, A_2, A_3)$, $\theta^5 = \varepsilon$. From $A_2\theta = E_2$ it follows

$$A_1 =^{\pi_2} A_2(E_1, E_2, A_2). \tag{2.9}$$

Also $A_2\theta = E_2$ implies $A_2\theta^4 = E_3$, i.e. $A_2(E_1, A_2, A_3) = E_3$, so

$$A_3 =^{\pi_3} A_2(E_1, A_2, E_3). \tag{2.10}$$

Analogously, $A_2\theta = E_2$ implies $A_2\theta^3 = A_3$, i.e. $A_2(E_1, E_3, A_1) = A_3$. Using (2.9) and (2.10) in the last equality, we get

$$A_2(E_1, E_3, {}^{\pi_2}A_2(E_1, E_2, A_2)) =^{\pi_3} A_2(E_1, A_2, E_3). \tag{2.11}$$

Conversely, if (2.9), (2.10) and (2.11) hold, then (2.9) implies $A_2\theta = E_2$ and from (2.10) it follows $A_3\theta =^{\pi_3} A_2(E_1, E_2, A_2)$, so $A_3\theta = E_3$. Using (2.9) and (2.10) in (2.11), we obtain $A_2(E_1, E_3, A_1) = A_3$, so $A_1 =^{\pi_3} A_2(E_1, E_3, A_3)$, which implies $A_1\theta =^{\pi_3} A_2(E_1, A_2, E_3)$. Using (2.10) in the last equality, we get $A_1\theta = A_3$.

4. If $A_1\theta = E_2, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = \varepsilon$. From $A_1\theta = E_2$ it follows

$$A_2 =^{\pi_3} A_1(E_1, A_1, E_2). \tag{2.12}$$

Using (2.12) in $A_3\theta = A_3$, we get

$$A_3(E_1, A_1, {}^{\pi_3}A_1(E_1, A_1, E_2)) = A_3. \tag{2.13}$$

Conversely, if (2.12) and (2.13) hold, then from (2.12) it follows $A_1\theta = E_2$ and $A_2\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, so $A_2\theta = E_3$. Using (2.12) in (2.13), we obtain $A_3\theta = A_3$.

5. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = E_2$, then $\theta^2 = (E_1, A_3, E_3)$, $\theta^3 = (E_1, E_2, A_2)$. From $A_3\theta = E_2$ it follows $A_3\theta^3 = A_3$, i.e. $A_3(E_1, E_2, A_2) = A_3(E_1, E_2, E_3)$, so $A_2 = E_3$, which is a contradiction as A_2 is a quasigroup operation.

6. If $A_1\theta = E_2, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_1, E_2, A_3)$. From $A_1\theta = E_2$ it follows $A_1\theta^2 = A_1$, i.e. $A_1(E_1, E_2, A_3) = A_1(E_1, E_2, E_3)$, so $A_3 = E_3$, which is a contradiction as A_3 is a quasigroup operation. □

Corollary 1. *If a ternary quasigroup (Q, A) satisfies the identity*

$$A(E_1(x_1^3), A(x_1^3), {}^{(23)}A(x_1^3)) = E_3(x_1^3), \tag{2.14}$$

then, for $\forall a \in Q$, its 1-retract $B(x, y) = A(a, x, y)$ is self-orthogonal.

Proof. Let (Q, A) be a ternary quasigroup which satisfies the identity (2.14). Replacing $x_2 \mapsto x_3, x_3 \mapsto x_2$ in (2.14), we obtain:

$$A(E_1(x_1, x_3, x_2), A(x_1, x_3, x_2), {}^{(23)}A(x_1, x_3, x_2)) = E_3(x_1, x_3, x_2),$$

which implies $A(E_1(x_1^3), {}^{(23)}A(x_1^3), A(x_1^3)) = E_2(x_1^3)$, so

$$^{(23)}A(E_1, A, {}^{(23)}A) = E_2. \tag{2.15}$$

Taking $x_1 = a$, where $a \in Q$, in (2.14) and (2.15), and using the 1-retract $B(x, y) = A(a, x, y)$, we get $B(B, {}^{(12)}B) = E$ and ${}^{(12)}B(B, {}^{(12)}B) = F$, respectively, hence $B \perp {}^{(12)}B$, i.e. B is self-orthogonal. □

Lemma 3. *The triple (A_1, E_1, A_2) is a paratopy of the system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ if and only if one of the following statements holds:*
1. $A_2 = {}^{\pi_3}A_1(A_1, E_1, E_2)$, $A_3 = {}^{\pi_3}A_1(E_2, A_1, E_1)$;
2. $A_1 = {}^{\pi_2}A_3(E_3, A_3, E_2)$, $A_2 = {}^{\pi_1}A_3(A_3, E_3, E_1)$ *and*
$A_3({}^{\pi_2}A_3(E_3, A_3, E_2), E_1, {}^{\pi_1}A_3(A_3, E_3, E_1)) = A_3$;
3. $A_2 = {}^{\pi_3}A_1(A_1, E_1, E_3)$, $A_3 = {}^{\pi_2}A_1(E_2, E_1, A_1)$ *and*
$A_1({}^{\pi_2}A_1(E_2, E_1, A_1), {}^{\pi_3}A_1(A_1, E_1, E_3), E_1) = E_2$;
4. $A_1 = {}^{\pi_3}A_2(A_2, E_3, {}^{\pi_3}A_2(E_2, A_2, E_3))$, $A_3 = {}^{\pi_3}A_2(E_2, A_2, E_3)$ *and*
$A_2({}^{\pi_3}A_2(A_2, E_3, {}^{\pi_3}A_2(E_2, A_2, E_3)), E_1, A_2) = E_2$;
5. $A_1 = {}^{\pi_1}A_2(E_3, E_1, A_2)$, $A_3 = {}^{\pi_2}A_2(E_2, E_3, A_2)$ *and*
$A_2({}^{\pi_2}A_2(E_2, E_3, A_2), {}^{\pi_1}A_2(E_3, E_1, A_2), E_3) = A_2$.

Proof. Let the tuple (A_1, E_1, A_2) be a paratopy of the system Σ. As $E_1\theta = A_1, E_2\theta = E_1, E_3\theta = A_2$, we obtain $\Sigma = \Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, E_1, A_2\}$, i.e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_2, E_3, A_3\}$.

1. If $A_1\theta = E_2, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_2, A_1, A_3), \theta^3 = \varepsilon$. The equality $A_1\theta = E_2$, i.e. $A_1(A_1, E_1, A_2) = E_2$, implies

$$A_2 = {}^{\pi_3}A_1(A_1, E_1, E_2). \tag{3.1}$$

Also, from $A_1\theta = E_2$, we get $A_1\theta^2 = E_1$, i.e. $A_1(E_2, A_1, A_3) = E_1$, so

$$A_3 =^{\pi_3} A_1(E_2, A_1, E_1). \tag{3.2}$$

Conversely, if the equalities (3.1) and (3.2) hold, then (3.1) implies $A_1(A_1, E_1, A_2) = E_2$, hence $A_1\theta = E_2$. Also, from (3.1) it follows $A_2\theta =^{\pi_3} A_1(E_2, A_1, E_1)$ so, using (3.2) in the last equality, we get $A_2\theta = A_3$. From the equality (3.2) it follows $A_3\theta =^{\pi_3} A_1(E_1, E_2, A_1)$, hence $A_3\theta = E_3$.

2. If $A_1\theta = E_3, A_2\theta = E_2, A_3\theta = A_3$, then $\theta^2 = (E_3, A_1, E_2)$, $\theta^3 = (A_2, E_3, E_1)$. From $A_3\theta = A_3$ it follows $A_3\theta^2 = A_3$, i.e. $A_3(E_3, A_1, E_2) = A_3$, so

$$A_1 =^{\pi_2} A_3(E_3, A_3, E_2). \tag{3.3}$$

Also, from $A_3\theta = A_3$ it follows $A_3\theta^3 = A_3$, hence

$$A_2 =^{\pi_1} A_3(A_3, E_3, E_1). \tag{3.4}$$

Using (3.3) and (3.4) in $A_3\theta = A_3$, we obtain

$$A_3({}^{\pi_2}A_3(E_3, A_3, E_2), E_1, {}^{\pi_1}A_3(A_3, E_3, E_1)) = A_3. \tag{3.5}$$

Conversely, if (3.3), (3.4) and (3.5) hold, then using (3.3) and (3.4) in (3.5) we get

$$A_3(A_1, E_1, A_2) = A_3, \tag{3.6}$$

i.e. $A_3\theta = A_3$. From (3.4) it follows $A_3(A_2, E_3, E_1) = A_3$, so

$$E_3 =^{\pi_2} A_3(A_2, A_3, E_1). \tag{3.7}$$

As (3.3) implies $A_1\theta =^{\pi_2} A_3(A_2, A_3, E_1)$, using (3.7) in the last equality we get $A_1\theta = E_3$. From (3.3) it follows $A_3(E_3, A_1, E_2) = A_3$, so

$$E_2 =^{\pi_3} A_3(E_3, A_1, A_3). \tag{3.8}$$

From (3.6) it follows $A_2 =^{\pi_3} A_3(A_1, E_1, A_3), \Rightarrow A_2\theta =^{\pi_3} A_3(E_3, A_1, A_3)$. Using (3.8) in the last equality, we get $A_2\theta = E_2$.

3. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = E_2$, then $\theta^2 = (E_3, A_1, A_3)$, $\theta^3 = (A_2, E_3, E_2)$, $\theta^4 = (A_3, A_2, E_1)$, $\theta^5 = (E_2, A_3, A_1)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_2 =^{\pi_3} A_1(A_1, E_1, E_3). \tag{3.9}$$

Also, $A_1\theta = E_3$ implies $A_1\theta^5 = E_3\theta^4 \Rightarrow A_1(E_2, A_3, A_1) = E_1$, so

$$A_3 =^{\pi_2} A_1(E_2, E_1, A_1). \tag{3.10}$$

The equality $A_1\theta = E_3$ implies $A_1\theta^4 = E_3\theta^3, \Rightarrow A_1(A_3, A_2, E_1) = E_2$. Using (3.9) and (3.10) in the last equality, we obtain

$$A_1({}^{\pi_2}A_1(E_2, E_1, A_1), {}^{\pi_3}A_1(A_1, E_1, E_3), E_1) = E_2. \tag{3.11}$$

Conversely, if (3.9), (3.10) and (3.11) hold, then from (3.9) we have $A_1(A_1, E_1, A_2) = E_3$, so $A_1\theta = E_3$. From (3.10) it follows $A_3\theta =^{\pi_2} A_1(E_1, A_1, E_3)$, i.e. $A_3\theta = E_2$. Using (3.9) and (3.10) in (3.11) we obtain $A_1(A_3, A_2, E_1) = E_2$, so $A_2 = {}^{\pi_2}A_1(A_3, E_2, E_1), \Rightarrow A_2\theta =^{\pi_2} A_1(E_2, E_1, A_1)$ and, using (3.10), we get $A_2\theta = A_3$.

4. If $A_1\theta = A_3, A_2\theta = E_2, A_3\theta = E_3$, then $\theta^2 = (A_3, A_1, E_2)$, $\theta^3 = (E_3, A_3, E_1)$ $\theta^4 = (A_2, E_3, A_1)$, $\theta^5 = (E_2, A_2, A_3)$, $\theta^6 = \varepsilon$. From $A_2\theta = E_2$ it follows $A_2\theta^5 = E_2\theta^4$, i.e. $A_2(E_2, A_2, A_3) = E_3$, so

$$A_3 =^{\pi_3} A_2(E_2, A_2, E_3). \tag{3.12}$$

Also $A_2\theta = E_2$ implies $A_2\theta^4 = E_2\theta^3$, i.e. $A_2(A_2, E_3, A_1)$, so $A_1 =^{\pi_3} A_2(A_2, E_3, A_3)$ and, using (3.12), we obtain

$$A_1 =^{\pi_3} A_2(A_2, E_3, {}^{\pi_3} A_2(E_2, A_2, E_3)). \tag{3.13}$$

Now, using (3.12) and (3.13) in the equality $A_2\theta = E_2$, we have

$$A_2({}^{\pi_3}A_2(A_2, E_3, {}^{\pi_3} A_2(E_2, A_2, E_3)), E_1, A_2) = E_2. \tag{3.14}$$

Conversely, if (3.12), (3.13) and (3.14) hold, then using (3.13) in (3.14), we get $A_2(A_1, E_1, A_2) = E_2$, which implies $A_2\theta = E_2$. From (3.12) it follows $A_3\theta = {}^{\pi_3}A_2(E_1, E_2, A_2) = E_3$. Using (3.12) in (3.13) we obtain $A_1 =^{\pi_3} A_2(A_2, E_3, A_3)$, which implies the equality $A_1\theta =^{\pi_3} A_2(E_2, A_2, E_3)$. So, using (3.12) in the last equality, we have $A_1\theta = A_3$.

5. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = E_2$, then $\theta^2 = (A_3, A_1, E_3)$, $\theta^3 = (E_2, A_3, A_2)$, $\theta^4 = \varepsilon$. From the equality $A_2\theta = E_3$ we get

$$A_1 =^{\pi_1} A_2(E_3, E_1, A_2). \tag{3.15}$$

Also, from $A_2\theta = E_3$ it follows $A_2\theta^3 = E_3\theta^2$, i.e. $A_2(E_2, A_3, A_2) = E_3$, so

$$A_3 =^{\pi_2} A_2(E_2, E_3, A_2). \tag{3.16}$$

Moreover, $A_2\theta = E_3$ implies $A_2\theta^2 = A_2$, i.e. $A_2(A_3, A_1, E_3) = A_2$. Using (3.15) and (3.16) in the last equality we get

$$A_2({}^{\pi_2}A_2(E_2, E_3, A_2), {}^{\pi_1} A_2(E_3, E_1, A_2), E_3) = A_2. \tag{3.17}$$

Conversely, let (3.15), (3.16) and (3.17) hold. Then from (3.15) it follows $A_2\theta = E_3$ and (3.16) implies $A_3\theta =^{\pi_2} A_2(E_1, A_2, E_3) = E_2$. Using (3.15) and (3.16) in (3.17) we get $A_2(A_3, A_1, E_3) = A_2$, which implies $A_1 =^{\pi_2} A_2(A_3, A_2, E_3)$, so $A_1\theta = {}^{\pi_2}A_2(E_2, E_3, A_2)$. Using (3.16) in the last equality, we get $A_1\theta = A_3$.

6. If $A_1\theta = E_2, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = (E_2, A_1, E_3)$, $\theta^3 = (E_1, E_2, A_2)$ and $A_1\theta^3 = E_2\theta^2$, so $A_1(E_1, E_2, A_2) = A_1$, i.e. $A_1(E_1, E_2, A_2) = A_1(E_1, E_2, E_3)$, hence $A_2 = E_3$, which is a contradiction as A_2 is a quasigroup operation. □

Lemma 4. *The triple (A_1, A_2, E_1) is a paratopy of the system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ if and only if one of the following statements holds:*

1. $A_1 =^{\pi_3} A_3(E_2, E_3, A_3)$, $A_2 =^{\pi_1} A_3(A_3, E_1, E_2)$ *and*
$A_3(^{\pi_3}A_3(E_2, E_3, A_3),^{\pi_1} A_3(A_3, E_1, E_2), E_1) = A_3$*;*
2. $A_2 =^{\pi_2} A_1(A_1, E_2, E_1)$, $A_3 =^{\pi_2} A_1(E_2,^{\pi_2} A_1(A_1, E_2, E_1), A_1)$ *and*
$A_1(E_3, A_1,^{\pi_2} A_1(E_2,^{\pi_2} A_1(A_1, E_2, E_1), A_1)) = E_1$;
3. $A_2 =^{\pi_2} A_1(A_1, E_3, E_1)$, $A_3 =^{\pi_2} A_1(E_3, E_1, A_1)$;
4. $A_1 =^{\pi_1} A_2(E_2, A_2, E_1)$, $A_3 =^{\pi_3} A_2(E_3, A_2, E_2)$ *and*
$A_2(^{\pi_3}A_2(E_3, A_2, E_2), E_2,^{\pi_1} A_2(E_2, A_2, E_1)) = A_2$;
5. $A_1 =^{\pi_1} A_2(E_3, A_2, E_1)$, $A_3 =^{\pi_2} A_2(E_3, E_2, A_2)$ *and*
$A_2(A_2,^{\pi_1} A_2(E_3, A_2, E_1), E_2) =^{\pi_2} A_2(E_3, E_2, A_2)$.

Proof. Let the tuple (A_1, A_2, E_1) be a paratopy of the system Σ. As $E_1\theta = A_1, E_2\theta = A_2$, $E_3\theta = E_1$, we obtain $\Sigma = \Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, E_1\}$, i.e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_2, E_3, A_3\}$.

1. If $A_1\theta = E_2, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = (E_2, E_3, A_1)$, $\theta^3 = (A_2, E_1, E_2)$, $\theta^4 = (E_3, A_1, A_2)$, $\theta^5 = \varepsilon$. As $A_3\theta = A_3, \Rightarrow A_3\theta^2 = A_3, \Rightarrow A_3(E_2, E_3, A_1) = A_3$, we get

$$A_1 =^{\pi_3} A_3(E_2, E_3, A_3). \tag{4.1}$$

Also the equality $A_3\theta = A_3$ implies $A_3\theta^3 = A_3$, i.e. $A_3(A_2, E_1, E_2) = A_3$, hence

$$A_2 =^{\pi_1} A_3(A_3, E_1, E_2). \tag{4.2}$$

Using (4.1) and (4.2) in $A_3\theta = A_3$ we obtain

$$A_3(^{\pi_3}A_3(E_2, E_3, A_3),^{\pi_1} A_3(A_3, E_1, E_2), E_1) = A_3. \tag{4.3}$$

Conversely, if (4.1), (4.2) and (4.3) hold then, using (4.1) and (4.2) in (4.3), we get $A_3\theta = A_3$. From (4.2) it follows $A_3(A_2, E_1, E_2) = A_3$, so

$$^{\pi_3}A_3(A_2, E_1, A_3) = E_2. \tag{4.4}$$

From (4.1) we obtain $A_1\theta =^{\pi_3} A_3(A_2, E_1, A_3)$, and using (4.4) in the last equality we get $A_1\theta = E_2$. Also from (4.1) it follows $A_3(E_2, E_3, A_1) = A_3$, so

$$^{\pi_2}A_3(E_2, A_3, A_1) = E_3. \tag{4.5}$$

Using (4.1) and (4.2) in (4.3) we get $A_3(A_1, A_2, E_1) = A_3$, so $A_2 =^{\pi_2} A_3(A_1, A_3, E_1)$, which implies $A_2\theta =^{\pi_2} A_3(E_2, A_3, A_1)$ and, using (4.5), we obtain $A_2\theta = E_3$.

2. If $A_1\theta = E_2, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_2, A_3, A_1)$, $\theta^3 = (A_2, E_3, E_2)$, $\theta^4 = (A_3, E_1, A_2)$, $\theta^5 = (E_3, A_1, A_3)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_2$ it follows

$$A_2 =^{\pi_2} A_1(A_1, E_2, E_1). \tag{4.6}$$

Also, $A_1\theta = E_2$ implies $A_1\theta^2 = A_2$, i.e. $A_1(E_2, A_3, A_1) = A_2$, so $A_3 = {}^{\pi_2}A_1(E_2, A_2, A_1)$. Using (4.6) in the last equality we obtain

$$A_3 =^{\pi_2} A_1(E_2,^{\pi_2} A_1(A_1, E_2, E_1), A_1). \tag{4.7}$$

From $A_1\theta = E_2$ we also get $A_1\theta^5 = E_1$, i.e. $A_1(E_3, A_1, A_3) = E_1$ and, using (4.7),

$$A_1(E_3, A_1, {}^{\pi_2}A_1(E_2, {}^{\pi_2}A_1(A_1, E_2, E_1), A_1)) = E_1. \tag{4.8}$$

Conversely, if (4.6), (4.7) and (4.8) hold, then from (4.6) we get $A_1\theta = E_2$. Also, from (4.6) it follows $A_2\theta =^{\pi_2} A_1(E_2, A_2, A_1)$. Using (4.6) and (4.7), from the last equality we obtain $A_2\theta = A_3$. Using (4.7) in (4.8), we have $A_1(E_3, A_1, A_3) = E_1$, so $A_3 =^{\pi_3} A_1(E_3, A_1, E_1) \Rightarrow A_3\theta =^{\pi_3} A_1(E_1, E_2, A_1)$, hence $A_3\theta = E_3$.

3. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = E_2$, then $\theta^2 = (E_3, A_3, A_1), \theta^3 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_2 =^{\pi_2} A_1(A_1, E_3, E_1). \tag{4.9}$$

Also $A_1\theta = E_3$ implies $A_1\theta^2 = E_1$, i.e. $A_1(E_3, A_3, A_1) = E_1$, hence

$$A_3 =^{\pi_2} A_1(E_3, E_1, A_1). \tag{4.10}$$

Conversely, let (4.9) and (4.10) hold. Then the equality (4.9) implies $A_1\theta = E_3$ and $A_2\theta =^{\pi_2} A_1(E_3, E_1, A_1)$. Using (4.10) in the last equality, we obtain $A_2\theta = A_3$. From (4.10) it follows $A_3\theta =^{\pi_2} A_1(E_1, A_1, E_3)$, so $A_3\theta = E_2$.

4. If $A_1\theta = A_3, A_2\theta = E_2, A_3\theta = E_3$, then $\theta^2 = (A_3, E_2, A_1)$, $\theta^3 = (E_3, A_2, A_3)$, $\theta^4 = \varepsilon$. From $A_2\theta = E_2$ it follows

$$A_1 =^{\pi_1} A_2(E_2, A_2, E_1). \tag{4.11}$$

Also $A_2\theta = E_2, \Rightarrow A_2\theta^3 = E_2, \Rightarrow A_2(E_3, A_2, A_3) = E_2$, so

$$A_3 =^{\pi_3} A_2(E_3, A_2, E_2). \tag{4.12}$$

From $A_2\theta = E_2$, we have $A_2\theta^2 = A_2$, i.e. $A_2(A_3, E_2, A_1) = A_2$. Using (4.11) and (4.12) in the last equality we get

$$A_2({}^{\pi_3}A_2(E_3, A_2, E_2), E_2, {}^{\pi_1}A_2(E_2, A_2, E_1)) = A_2. \tag{4.13}$$

Conversely, if (4.11), (4.12) and (4.13) hold, then from (4.11) it follows $A_2\theta = E_2$, and (4.12) implies $A_3\theta =^{\pi_3} A_2(E_1, E_2, A_2)$, so $A_3\theta = E_3$. Using (4.11) and (4.12) in (4.13), we obtain: $A_2(A_3, E_2, A_1) = A_2, \Rightarrow A_1 =^{\pi_3} A_2(A_3, E_2, A_2)$, hence the equality $A_1\theta =^{\pi_3} A_2(E_3, A_2, E_2)$ holds. Now, using (4.12) in the last equality, we have $A_1\theta = A_3$.

5. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = E_2$, then $\theta^2 = (A_3, E_3, A_1)$, $\theta^3 = (E_2, E_1, A_3)$, $\theta^4 = (A_2, A_1, E_2)$, $\theta^5 = (E_3, A_3, A_2)$, $\theta^6 = \varepsilon$. From $A_2\theta = E_3$ it follows

$$A_1 =^{\pi_1} A_2(E_3, A_2, E_1). \tag{4.14}$$

Also, from $A_2\theta = E_3$ we get $A_2\theta^5 = E_2$, i.e. $A_2(E_3, A_3, A_2) = E_2$, so

$$A_3 =^{\pi_2} A_2(E_3, E_2, A_2). \tag{4.15}$$

Once again $A_2\theta = E_3$ implies $A_2\theta^4 = A_3$, i.e. $A_2(A_2, A_1, E_2) = A_3$. Using (4.14) and (4.15) in the last equality, we obtain

$$A_2(A_2, {}^{\pi_1}A_2(E_3, A_2, E_1), E_2) = {}^{\pi_2}A_2(E_3, E_2, A_2). \tag{4.16}$$

Conversely, if (4.14), (4.15) and (4.16) hold, then from (4.14) it follows $A_2\theta = E_3$, and (4.15) implies $A_3\theta = {}^{\pi_2}A_2(E_1, A_2, E_3)$, so $A_3\theta = E_2$. Using (4.14) and (4.15) in (4.16), we get $A_2(A_2, A_1, E_2) = A_3$, hence $A_1 = {}^{\pi_2}A_2(A_2, A_3, E_2)$, which implies $A_1\theta = {}^{\pi_2}A_2(E_3, E_2, A_2)$. Using (4.15) in the last equality, we get $A_1\theta = A_3$.

6. If $A_1\theta = E_3, A_2\theta = E_2, A_3\theta = A_3$, then $\theta^2 = (E_3, E_2, A_1)$, $\theta^3 = (E_1, A_2, E_3)$. The equality $A_3\theta = A_3$ implies $A_3\theta^3 = A_3$, so $A_3(E_1, A_2, E_3) = A_3(E_1, E_2, E_3) \Rightarrow A_2 = E_2$, which is a contradiction as A_2 is a quasigroup operation. □

Lemma 5. *The triple (E_2, A_1, A_2) is a paratopy of the system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ if and only if one of the following statements holds:*
1. $A_2 = {}^{\pi_3}A_1(E_2, A_1, E_1)$, $A_3 = {}^{\pi_3}A_1(A_1, E_1, E_2)$;
2. $A_1 = {}^{\pi_1}A_3(A_3, E_3, E_1)$, $A_2 = {}^{\pi_2}A_3(E_3, A_3, E_2)$ *and*
$A_3(E_2, {}^{\pi_1}A_3(A_3, E_3, E_1), {}^{\pi_2}A_3(E_3, A_3, E_2)) = A_3$;
3. $A_2 = {}^{\pi_3}A_1(E_2, A_1, E_3)$, $A_3 = {}^{\pi_1}A_1(E_2, E_1, A_1)$ *and*
$A_1({}^{\pi_3}A_1(E_2, A_1, E_3), {}^{\pi_1}A_1(E_2, E_1, A_1), E_2) = E_1$;
4. $A_3 = {}^{\pi_3}A_2(A_2, E_1, E_3)$, $A_1 = {}^{\pi_3}A_2(E_3, A_2, {}^{\pi_3}A_2(A_2, E_1, E_3))$ *and*
$A_2(E_2, {}^{\pi_3}A_2(E_3, A_2, {}^{\pi_3}A_2(A_2, E_1, E_3)), A_2) = E_1$;
5. $A_1 = {}^{\pi_2}A_2(E_2, E_3, A_2)$, $A_3 = {}^{\pi_1}A_2(E_3, E_1, A_2)$ *and*
$A_2({}^{\pi_2}A_2(E_2, E_3, A_2), {}^{\pi_1}A_2(E_3, E_1, A_2), E_3) = A_2$.

Proof. Let the tuple (E_2, A_1, A_2) be a paratopy of the system Σ. As $E_1\theta = E_2, E_2\theta = A_1, E_3\theta = A_2$, we obtain $\Sigma = \Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, E_2\}$, i.e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_1, E_3, A_3\}$.

1. If $A_1\theta = E_1, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (A_1, E_1, A_3)$, $\theta^3 = \varepsilon$. From $A_1\theta = E_1$ it follows

$$A_2 = {}^{\pi_3}A_1(E_2, A_1, E_1). \tag{5.1}$$

Also $A_1\theta = E_1$ implies $A_1\theta^2 = E_2$, i.e. $A_1(A_1, E_1, A_3) = E_2$, so

$$A_3 = {}^{\pi_3}A_1(A_1, E_1, E_2). \tag{5.2}$$

Conversely, if (5.1) and (5.2) hold, then from (5.1) it follows $A_1\theta = E_1$. Moreover, (5.1) implies $A_2\theta = {}^{\pi_3}A_1(A_1, E_1, E_2)$ and, using (5.2) in the last equality, we get $A_2\theta = A_3$. From (5.2) it follows $A_3\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, therefore $A_3\theta = E_3$.

2. If $A_1\theta = E_3, A_2\theta = E_1, A_3\theta = A_3$, then $\theta^2 = (A_1, E_3, E_1)$, $\theta^3 = (E_3, A_2, E_2)$, $\theta^4 = (A_2, E_1, A_1)$, $\theta^5 = \varepsilon$. As $A_3\theta = A_3, \Rightarrow A_3\theta^2 = A_3, \Rightarrow A_3(A_1, E_3, E_1) = A_3$, we get

$$A_1 = {}^{\pi_1}A_3(A_3, E_3, E_1). \tag{5.3}$$

Also, $A_3\theta = A_3$ implies $A_3\theta^3 = A_3$, i.e. $A_3(E_3, A_2, E_2) = A_3$, hence

$$A_2 = {}^{\pi_2}A_3(E_3, A_3, E_2). \tag{5.4}$$

Using (5.3) and (5.4) in $A_3\theta = A_3$, we have

$$A_3(E_2, {}^{\pi_1}A_3(A_3, E_3, E_1), {}^{\pi_2}A_3(E_3, A_3, E_2)) = A_3. \tag{5.5}$$

Conversely, if (5.3), (5.4) and (5.5) hold then, using (5.3) and (5.4) in (5.5), we get

$$A_3(E_2, A_1, A_2) = A_3, \tag{5.6}$$

so $A_3\theta = A_3$. From (5.4) it follows $A_3(E_3, A_2, E_2) = A_3$, i.e.

$$E_3 = {}^{\pi_1}A_3(A_3, A_2, E_2). \tag{5.7}$$

From (5.3) it follows $A_1\theta = {}^{\pi_1}A_3(A_3, A_2, E_2)$. Using (5.7) in the last equality we obtain $A_1\theta = E_3$. From (5.3) we get $A_3(A_1, E_3, E_1) = A_3$, hence

$$E_1 = {}^{\pi_3}A_3(A_1, E_3, A_3). \tag{5.8}$$

The equality (5.6) implies $A_2 = {}^{\pi_3}A_3(E_2, A_1, A_3) \Rightarrow A_2\theta = {}^{\pi_3}A_3(A_1, E_3, A_3)$. Using (5.8) in the last equality, we get $A_2\theta = E_1$.

3. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = E_1$, then $\theta^2 = (A_1, E_3, A_3)$, $\theta^3 = (E_3, A_2, E_1)$, $\theta^4 = (A_2, A_3, E_2)$, $\theta^5 = (A_3, E_1, A_1)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_2 = {}^{\pi_3}A_1(E_2, A_1, E_3). \tag{5.9}$$

Also $A_1\theta = E_3$ implies $A_1\theta^5 = E_2$, i. e. $A_1(A_3, E_1, A_1) = E_2$, hence

$$A_3 = {}^{\pi_1}A_1(E_2, E_1, A_1). \tag{5.10}$$

Moreover, from $A_1\theta = E_3$ we get $A_1\theta^4 = E_1$, i.e. $A_1(A_2, A_3, E_2) = E_1$. Using (5.9) and (5.10) in the last equality, we obtain

$$A_1({}^{\pi_3}A_1(E_2, A_1, E_3), {}^{\pi_1}A_1(E_2, E_1, A_1), E_2) = E_1. \tag{5.11}$$

Conversely, if (5.9), (5.10) and (5.11) hold, then from (5.9) it follows $A_1\theta = E_3$ and (5.10) implies $A_3\theta = {}^{\pi_1}A_1(A_1, E_2, E_3)$, so $A_3\theta = E_1$. Using (5.9) and (5.10) in (5.11) we obtain $A_1(A_2, A_3, E_2) = E_1$, which implies $A_2 = {}^{\pi_1}A_1(E_1, A_3, E_2)$, so $A_2\theta = {}^{\pi_1}A_1(E_2, E_1, A_1)$. Using (5.10) in the last equality, we get $A_2\theta = A_3$.

4. If $A_1\theta = A_3, A_2\theta = E_1, A_3\theta = E_3$, then $\theta^2 = (A_1, A_3, E_1)$, $\theta^3 = (A_3, E_3, E_2)$, $\theta^4 = (E_3, A_2, A_1)$, $\theta^5 = (A_2, E_1, A_3)$, $\theta^6 = \varepsilon$. From $A_2\theta = E_1$ it follows $A_2\theta^5 = E_3$, i. e. $A_2(A_2, E_1, A_3) = E_3$, so

$$A_3 = {}^{\pi_3}A_2(A_2, E_1, E_3). \tag{5.12}$$

Also, $A_2\theta = E_1$ implies $A_2\theta^4 = A_3$, i.e. $A_2(E_3, A_2, A_1) = A_3$, so $A_1 = {}^{\pi_3}A_2(E_3, A_2, A_3)$. Using (5.12) in the last equality we get

$$A_1 = {}^{\pi_3}A_2(E_3, A_2, {}^{\pi_3}A_2(A_2, E_1, E_3)). \tag{5.13}$$

Using (5.13) in $A_2\theta = E_1$, we obtain

$$A_2(E_2, {}^{\pi_3}A_2(E_3, A_2, {}^{\pi_3}A_2(A_2, E_1, E_3)), A_2) = E_1. \tag{5.14}$$

Conversely, if (5.12), (5.13) and (5.14) hold, then using (5.13) in (5.14), we get $A_2\theta = E_1$. From (5.12) it follows $A_3\theta = {}^{\pi_3}A_2(E_1, E_2, A_2)$, so $A_3\theta = E_3$. Using (5.12) in (5.13), we obtain $A_1 = {}^{\pi_3}A_2(E_3, A_2, A_3)$, which implies $A_1\theta = {}^{\pi_3}A_2(A_2, E_1, E_3)$. Using (5.12) in the last equality, we get $A_1\theta = A_3$.

5. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = E_1$, then $\theta^2 = (A_1, A_3, E_3)$, $\theta^3 = (A_3, E_1, A_2)$, $\theta^4 = \varepsilon$. From $A_2\theta = E_3$ it follows

$$A_1 = {}^{\pi_2}A_2(E_2, E_3, A_2). \tag{5.15}$$

Also $A_2\theta = E_3$ implies $A_2\theta^3 = E_3$, i. e. $A_2(A_3, E_1, A_2) = E_3$, hence

$$A_3 = {}^{\pi_1}A_2(E_3, E_1, A_2). \tag{5.16}$$

From $A_2\theta = E_3$ it follows $A_2\theta^2 = A_2$, i.e. $A_2(A_1, A_3, E_3) = A_2$. Using (5.15) and (5.16) in the last equality, we obtain

$$A_2({}^{\pi_2}A_2(E_2, E_3, A_2), {}^{\pi_1}A_2(E_3, E_1, A_2), E_3) = A_2. \tag{5.17}$$

Conversely, if (5.15), (5.16) and (5.17) hold, then from (5.15) it follows $A_2\theta = E_3$ and from (5.16) we get $A_3\theta = {}^{\pi_1}A_2(A_2, E_2, E_3)$, so $A_3\theta = E_1$. Using (5.15) and (5.16) in (5.17), we obtain $A_2(A_1, A_3, E_3) = A_2$, hence $A_1 = {}^{\pi_1}A_2(A_2, A_3, E_3)$, which implies $A_1\theta = {}^{\pi_1}A_2(E_3, E_1, A_2)$ and, using (5.16) in the last equality, we get $A_1\theta = A_3$.

6. If $A_1\theta = E_1, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = (A_1, E_1, E_3)$, $\theta^3 = (E_1, E_2, A_2)$. From $A_1\theta = E_1$ it follows $A_1\theta^3 = E_1\theta^2$, so $A_1(E_1, E_2, A_2) = A_1(E_1, E_2, E_3)$, hence $A_2 = E_3$, which is a contradiction as A_2 is a quasigroup operation. □

Lemma 6. *The triple* (A_1, E_2, A_2) *is a paratopy of the system* $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ *if and only if one of the following statements holds:*
1. $A_2 = {}^{\pi_3}A_1(A_1, E_2, E_1)$, $A_3(A_1, E_2, {}^{\pi_3}A_1(A_1, E_2, E_1)) = A_3$;
2. $A_2 = {}^{(13)}A_1$, $A_3(A_1, E_2, {}^{(13)}A_1) = A_3$ *and* $A_1(A_1, E_2, {}^{(13)}A_1) = E_3$;
3. $A_2 = {}^{\pi_3}A_1(A_1, E_2, E_3)$, $A_3 = {}^{\pi_1}A_1(E_1, E_2, A_1)$ *and*
$A_1({}^{\pi_3}A_1(A_1, E_2, E_3), E_2, E_1) = {}^{\pi_1}A_1(E_1, E_2, A_1)$;
4. $A_1 = {}^{\pi_1}A_2(E_1, E_2, A_2)$, $A_3 = {}^{\pi_3}A_2(A_2, E_2, E_3)$ *and*
$A_2(E_3, E_2, {}^{\pi_1}A_2(E_1, E_2, A_2)) = {}^{\pi_3}A_2(A_2, E_2, E_3)$.

Proof. Let the tuple (A_1, E_2, A_2) be a paratopy of the system Σ. Using $E_1\theta = A_1, E_2\theta = E_2, E_3\theta = A_2$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, E_2\}$, i. e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_1, E_3, A_3\}$.

1. If $A_1\theta = E_1, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = \varepsilon$. From $A_1\theta = E_1$ it follows

$$A_2 = {}^{\pi_3}A_1(A_1, E_2, E_1), \tag{6.1}$$

so, using (6.1) in $A_3\theta = A_3$, we get

$$A_3(A_1, E_2, {}^{\pi_3}A_1(A_1, E_2, E_1)) = A_3. \tag{6.2}$$

Conversely, if (6.1) and (6.2) hold, then from (6.1) it follows $A_1\theta = E_1$ and $A_2\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, so $A_2\theta = E_3$. Using (6.1) in (6.2), we obtain $A_3\theta = A_3$.

2. If $A_1\theta = E_3, A_2\theta = E_1, A_3\theta = A_3$, then $\theta^2 = (E_1, E_3, E_2)$, $\theta^3 = (A_2, E_2, A_1)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_3$ it follows $A_1\theta^2 = A_2$, i. e. $A_1(E_3, E_2, E_1) = A_2$, so

$$A_2 =^{(13)} A_1. \tag{6.3}$$

Using (6.3) in $A_3\theta = A_3$, we get

$$A_3(A_1, E_2, {}^{(13)}A_1) = A_3. \tag{6.4}$$

Now, $A_1\theta = E_3$ and (6.3) imply

$$A_1(A_1, E_2, {}^{(13)}A_1) = E_3. \tag{6.5}$$

Conversely, if (6.3), (6.4) and (6.5) hold, then (6.3) and (6.5) imply

$$A_1(A_1, E_2, A_2) = E_3, \tag{6.6}$$

so $A_1\theta = E_3$, and from (6.3) it follows $A_2 = A_1(E_3, E_2, E_1)$, hence

$$E_1 =^{\pi_3} A_1(E_3, E_2, A_2). \tag{6.7}$$

The equality (6.6) implies $A_2 =^{\pi_3} A_1(A_1, E_2, E_3)$, hence $A_2\theta =^{\pi_3} A_1(E_3, E_2, A_2)$ and, using (6.7), we obtain $A_2\theta = E_1$. Now, using (6.3) in (6.4), we get $A_3\theta = A_3$.

3. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = E_1$, then $\theta^2 = (E_3, E_2, A_3)$, $\theta^3 = (A_2, E_2, E_1)$, $\theta^4 = (A_3, E_2, A_1)$, $\theta^5 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_2 =^{\pi_3} A_1(A_1, E_2, E_3). \tag{6.8}$$

Also $A_1\theta = E_3$ implies $A_1\theta^4 = E_1$, i. e. $A_1(A_3, E_2, A_1) = E_1$, so

$$A_3 =^{\pi_1} A_1(E_1, E_2, A_1). \tag{6.9}$$

Using once again $A_1\theta = E_3$ we have $A_1\theta^3 = A_3$, i. e. $A_1(A_2, E_2, E_1) = A_3$. Using (6.8) and (6.9) in the last equality, we get

$$A_1({}^{\pi_3}A_1(A_1, E_2, E_3), E_2, E_1) =^{\pi_1} A_1(E_1, E_2, A_1). \tag{6.10}$$

Conversely, if (6.8), (6.9) and (6.10) hold, then (6.8) implies $A_1\theta = E_3$. From (6.9) it follows $A_3\theta =^{\pi_1} A_1(A_1, E_2, E_3)$, so $A_3\theta = E_1$. Using (6.8) and (6.9) in (6.10), we obtain $A_1(A_2, E_2, E_1) = A_3$, so $A_2 =^{\pi_1} A_1(A_3, E_2, E_1), \Rightarrow A_2\theta =^{\pi_1} A_1(E_1, E_2, A_1)$ and, using (6.9), we get $A_2\theta = A_3$.

4. If $A_1\theta = A_3, A_2\theta = E_1, A_3\theta = E_3$, then $\theta^2 = (A_3, E_2, E_1)$, $\theta^3 = (E_3, E_2, A_1)$, $\theta^4 = (A_2, E_2, A_3)$, $\theta^5 = \varepsilon$. From $A_2\theta = E_1$ it follows

$$A_1 =^{\pi_1} A_2(E_1, E_2, A_2), \tag{6.11}$$

which implies $A_2\theta^4 = E_3$, i. e. $A_2(A_2, E_2, A_3) = E_3$, so

$$A_3 =^{\pi_3} A_2(A_2, E_2, E_3). \tag{6.12}$$

Analogously, $A_2\theta = E_1$ implies $A_2\theta^3 = A_3$, i. e. $A_2(E_3, E_2, A_1) = A_3$. Using (6.11) and (6.12) in the last equality, we get

$$A_2(E_3, E_2, {}^{\pi_1}A_2(E_1, E_2, A_2)) =^{\pi_3} A_2(A_2, E_2, E_3). \tag{6.13}$$

Conversely, if (6.11), (6.12) and (6.13) hold, then (6.11) implies $A_2\theta = E_1$ and from (6.12) it follows $A_3\theta =^{\pi_3} A_2(E_1, E_2, A_2)$, so $A_3\theta = E_3$. Using (6.11) and (6.12) in (6.13), we obtain $A_2(E_3, E_2, A_1) = A_3$, so $A_1 =^{\pi_3} A_2(E_3, E_2, A_3)$, which implies $A_1\theta =^{\pi_3} A_2(A_2, E_2, E_3)$. Using (6.12) in the last equality, we get $A_1\theta = A_3$.

5. If $A_1\theta = E_1, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_1, E_2, A_3)$. From $A_1\theta = E_1$ it follows $A_1\theta^2 = A_1$, i. e. $A_1(E_1, E_2, A_3) = A_1(E_1, E_2, E_3)$, so $A_3 = E_3$, which is a contradiction as A_3 is a quasigroup operation.

6. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = E_1$, then $\theta^2 = (A_3, E_2, E_3)$, $\theta^3 = (E_1, E_2, A_2)$. From $A_3\theta = E_1$ it follows $A_3\theta^3 = A_3$, i. e. $A_3(E_1, E_2, A_2) = A_3(E_1, E_2, E_3)$, so $A_2 = E_3$, which is a contradiction as A_2 is a quasigroup operation. □

Corollary 2. *If a ternary quasigroup (Q, A) satisfies the identity $A(A, E_2, {}^{(13)}A) = E_3$ then, for $\forall a \in Q$, its 2-retract $B(x, y) = A(x, a, y)$ is self-orthogonal.*

The proof is analogous to that of Corollary 1.

Lemma 7. *The triple (A_1, A_2, E_2) is a paratopy of the system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ if and only if one of the following statements holds:*
1. $A_2 =^{\pi_2} A_1(A_1, E_1, E_2)$, $A_3 =^{\pi_3} A_1(A_1, E_3, E_1)$ *and*
$A_1(E_1, {}^{\pi_3}A_1(A_1, E_3, E_1), {}^{\pi_2}A_1(A_1, E_1, E_2)) = A_1$;
2. $A_1 =^{\pi_1} A_2(E_1, A_2, E_2)$, $A_3 =^{\pi_1} A_2({}^{\pi_1}A_2(E_1, A_2, E_2), E_1, A_2)$ *and*
$A_2(A_2, E_3, {}^{\pi_1}A_2({}^{\pi_1}A_2(E_1, A_2, E_2), E_1, A_2))) = E_2$;
3. $A_1 =^{\pi_1} A_2(E_3, A_2, E_2)$, $A_3 =^{\pi_1} A_2(E_2, E_3, A_2)$;
4. $A_1 =^{\pi_2} A_3(E_2, A_3, E_1)$, $A_2 =^{\pi_3} A_3(E_3, E_1, A_3)$ *and*
$A_3({}^{\pi_2}A_3(E_2, A_3, E_1), {}^{\pi_3}A_3(E_3, E_1, A_3), E_2) = A_3$;
5. $A_2 =^{\pi_2} A_1(A_1, E_3, E_2)$, $A_3 =^{\pi_1} A_1(E_1, E_3, A_1)$ *and*
$A_1({}^{\pi_2}A_1(A_1, E_3, E_2), A_1, E_1) =^{\pi_1} A_1(E_1, E_3, A_1)$.

Proof. Let the tuple (A_1, A_2, E_2) be a paratopy of the system Σ. As $E_1\theta = A_1, E_2\theta = A_2, E_3\theta = E_2$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, E_2\}$, i. e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_1, E_3, A_3\}$.

1. If $A_1\theta = E_1, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_1, A_3, A_2)$, $\theta^3 = (A_1, E_3, A_3)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_1$ it follows

$$A_2 =^{\pi_2} A_1(A_1, E_1, E_2). \tag{7.1}$$

Also $A_1\theta = E_1$ implies $A_2\theta^3 = E_1$, i. e. $A_1(A_1, E_3, A_3) = E_1$, so

$$A_3 =^{\pi_3} A_1(A_1, E_3, E_1). \tag{7.2}$$

The equality $A_1\theta = E_1$ also implies $A_1\theta^2 = A_1$, i.e. $A_1(E_1, A_3, A_2) = A_1$. Using (7.1) and (7.2) in the last equality we get

$$A_1(E_1, {}^{\pi_3}A_1(A_1, E_3, E_1), {}^{\pi_2}A_1(A_1, E_1, E_2)) = A_1. \tag{7.3}$$

Conversely, if (7.1), (7.2) and (7.3) hold, then from (7.1) it follows $A_1\theta = E_1$. The equality (7.2) implies $A_3\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, so $A_3\theta = E_3$. Using (7.1) and (7.2) in (7.3), we obtain $A_1(E_1, A_3, A_2) = A_1$, which implies $A_2 = {}^{\pi_3}A_1(E_1, A_3, A_1)$, hence $A_2\theta = {}^{\pi_3}A_1(A_1, E_3, E_1)$. Using (7.2) in the last equality, we get $A_2\theta = A_3$.

2. If $A_1\theta = A_3, A_2\theta = E_1, A_3\theta = E_3$, then $\theta^2 = (A_3, E_1, A_2)$, $\theta^3 = (E_3, A_1, E_1)$, $\theta^4 = (E_2, A_3, A_1)$, $\theta^5 = (A_2, E_3, A_3)$, $\theta^6 = \varepsilon$. From $A_2\theta = E_1$ it follows

$$A_1 = {}^{\pi_1}A_2(E_1, A_2, E_2). \tag{7.4}$$

Also $A_2\theta = E_1$ implies $A_2\theta^2 = A_1, \Rightarrow A_2(A_3, E_1, A_2) = A_1, \Rightarrow A_3 = {}^{\pi_1}A_2(A_1, E_1, A_2)$. Using (7.4) in the last equality we obtain

$$A_3 = {}^{\pi_1}A_2({}^{\pi_1}A_2(E_1, A_2, E_2), E_1, A_2). \tag{7.5}$$

The equality $A_2\theta = E_1$ also implies $A_2\theta^5 = E_2$, i.e. $A_2(A_2, E_3, A_3) = E_2$. Now using (7.5) in the last equality we get

$$A_2(A_2, E_3, {}^{\pi_1}A_2({}^{\pi_1}A_2(E_1, A_2, E_2), E_1, A_2)) = E_2. \tag{7.6}$$

Conversely, if (7.4), (7.5) and (7.6) hold, then from (7.4) we get $A_2\theta = E_1$. Also, from (7.4) it follows $A_1\theta = {}^{\pi_1}A_2(A_1, E_1, A_2)$. Using (7.4) and (7.5) in the last equality we obtain $A_1\theta = A_3$. Using (7.4) and (7.5) in (7.6), we have $A_2(A_2, E_3, A_3) = E_2$, so $A_3 = {}^{\pi_3}A_2(A_2, E_3, E_2) \Rightarrow A_3\theta = {}^{\pi_3}A_2(E_1, E_2, A_2)$, hence $A_3\theta = E_3$.

3. If $A_1\theta = A_3$, $A_2\theta = E_3$, $A_3\theta = E_1$ then $\theta^2 = (A_3, E_3, A_2)$, $\theta^3 = \varepsilon$. From $A_2\theta = E_3$ it follows

$$A_1 = {}^{\pi_1}A_2(E_3, A_2, E_2). \tag{7.7}$$

Also $A_2\theta = E_3$ implies $A_2\theta^2 = E_2$, i.e. $A_2(A_3, E_3, A_2) = E_2$, hence

$$A_3 = {}^{\pi_1}A_2(E_2, E_3, A_2). \tag{7.8}$$

Conversely, if (7.7) and (7.8) hold, then from (7.7) follows $A_2\theta = E_3$ and $A_1\theta = {}^{\pi_1}A_2(E_2, E_3, A_2)$. Using (7.8) in the last equality we obtain $A_1\theta = A_3$. The equality (7.8) implies also $A_3\theta = {}^{\pi_1}A_2(A_2, E_2, E_3)$, so $A_3\theta = E_1$.

4. If $A_1\theta = E_3, A_2\theta = E_1, A_3\theta = A_3$, then $\theta^2 = (E_3, E_1, A_2)$, $\theta^3 = (E_2, A_1, E_1)$, $\theta^4 = (A_2, E_3, A_1)$, $\theta^5 = \varepsilon$. From $A_3\theta = A_3$ it follows $A_3\theta^2 = A_3$, i.e. $A_3(E_3, E_1, A_2) = A_3$, so

$$A_2 = {}^{\pi_3}A_3(E_3, E_1, A_3). \tag{7.9}$$

Also the equality $A_3\theta = A_3$ implies $A_3\theta^3 = A_3$, i.e. $A_3(E_2, A_1, E_1) = A_3$, hence

$$A_1 = {}^{\pi_2}A_3(E_2, A_3, E_1). \tag{7.10}$$

Using (7.9) and (7.10) in $A_3\theta = A_3$ we obtain

$$A_3({}^{\pi_2}A_3(E_2, A_3, E_1), {}^{\pi_3}A_3(E_3, E_1, A_3), E_2) = A_3. \quad (7.11)$$

Conversely, if (7.9), (7.10) and (7.11) hold, then using (7.9) and (7.10) in (7.11) we get

$$A_3(A_1, A_2, E_2) = A_3, \quad (7.12)$$

i.e. $A_3\theta = A_3$. From (7.10) it follows $A_3(E_2, A_1, E_1) = A_3$, so

$$^{\pi_3}A_3(E_2, A_1, A_3) = E_1. \quad (7.13)$$

From (7.9) we obtain $A_2\theta ={}^{\pi_3} A_3(E_2, A_1, A_3)$ so, using (7.13) in the last equality, we get $A_2\theta = E_1$. Also from (7.9) it follows $A_3(E_3, E_1, A_2) = A_3$, so

$$^{\pi_1}A_3(A_3, E_1, A_2) = E_3. \quad (7.14)$$

From (7.12) we get $A_1 ={}^{\pi_1} A_3(A_3, A_2, E_2)$, which implies $A_1\theta ={}^{\pi_1} A_3(A_3, E_1, A_2)$. Using (7.14) in the last equality we obtain $A_1\theta = E_3$.

5. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = E_1$, then $\theta^2 = (E_3, A_3, A_2)$, $\theta^3 = (E_2, E_1, A_3)$, $\theta^4 = (A_2, A_1, E_1)$, $\theta^5 = (A_3, E_3, A_1)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_2 ={}^{\pi_2} A_1(A_1, E_3, E_2). \quad (7.15)$$

Also, from $A_1\theta = E_3$ we get $A_1\theta^5 = E_1$, i.e. $A_1(A_3, E_3, A_1) = E_1$, so

$$A_3 ={}^{\pi_1} A_1(E_1, E_3, A_1). \quad (7.16)$$

The equality $A_1\theta = E_3$ also implies $A_1\theta^4 = A_3$, i.e. $A_1(A_2, A_1, E_1) = A_3$. Using (7.15) and (7.16) in the last equality, we obtain

$$A_1({}^{\pi_2}A_1(A_1, E_3, E_2), A_1, E_1) ={}^{\pi_1} A_1(E_1, E_3, A_1). \quad (7.17)$$

Conversely, if (7.15), (7.16) and (7.17) hold, then from (7.15) it follows $A_1\theta = E_3$. The equality (7.16) implies $A_3\theta ={}^{\pi_1} A_1(A_1, E_2, E_3)$ so $A_3\theta = E_1$. Using (7.15) and (7.16) in (7.17), we get $A_1(A_2, A_1, E_1) = A_3$, hence $A_2 ={}^{\pi_1} A_1(A_3, A_1, E_1)$, which implies $A_2\theta ={}^{\pi_1} A_1(E_1, E_3, A_1)$. Using (7.16) in the last equality, we get $A_2\theta = A_3$.

6. If $A_1\theta = E_1, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = (E_1, E_3, A_2)$, $\theta^3 = (A_1, E_2, E_3)$. The equality $A_3\theta = A_3$ implies $A_3\theta^3 = A_3$, so $A_3(A_1, E_2, E_3) = A_3(E_1, E_2, E_3) \Rightarrow A_1 = E_1$, which is a contradiction as A_1 is a quasigroup operation. □

Lemma 8. *The triple* (E_3, A_1, A_2) *is a paratopy of the system* $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ *if and only if one of the following statements holds:*
1. $A_1 ={}^{\pi_2} A_2(E_3, E_1, A_2)$, $A_3 ={}^{\pi_2} A_2(A_2, E_3, E_1)$;
2. $A_2 ={}^{\pi_1} A_3(A_3, E_1, E_2)$, $A_1 ={}^{\pi_3} A_3(E_2, E_3, A_3)$ *and*
$A_3(E_3, {}^{\pi_3}A_3(E_2, E_3, A_3), {}^{\pi_1}A_3(A_3, E_1, E_2)) = A_3$;
3. $A_1 ={}^{\pi_2} A_2(E_3, E_2, A_2)$, $A_3 ={}^{\pi_1} A_2(E_3, A_2, E_1)$ *and*
$A_2({}^{\pi_2}A_2(E_3, E_2, A_2), E_3, {}^{\pi_1}A_2(E_3, A_2, E_1)) = E_1$;

4. $A_2 =^{\pi_3} A_1(E_3, A_1, E_1)$, $A_3 =^{\pi_2} A_1(A_1, E_2, E_1)$ *and*
$A_1(E_2,^{\pi_3} A_1(E_3, A_1, E_1), A_1) =^{\pi_2} A_1(A_1, E_2, E_1)$;
5. $A_2 =^{\pi_3} A_1(E_3, A_1, E_2)$, $A_3 =^{\pi_1} A_1(E_2, A_1, E_1)$ *and*
$A_1(^{\pi_3}A_1(E_3, A_1, E_2), E_2,^{\pi_1} A_1(E_2, A_1, E_1)) = A_1$.

Proof. Let the tuple (E_3, A_1, A_2) be a paratopy of the system Σ. As $E_1\theta = E_3, E_2\theta = A_1, E_3\theta = A_2$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, E_3\}$, i.e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_1, E_2, A_3\}$.

1. If $A_1\theta = A_3, A_2\theta = E_1, A_3\theta = E_2$, then $\theta^2 = (A_2, A_3, E_1)$, $\theta^3 = \varepsilon$. From $A_2\theta = E_1$ it follows

$$A_1 =^{\pi_2} A_2(E_3, E_1, A_2). \tag{8.1}$$

Also $A_2\theta = E_1$ implies $A_2\theta^2 = E_3$, i.e. $A_2(A_2, A_3, E_1) = E_3$, so

$$A_3 =^{\pi_2} A_2(A_2, E_3, E_1). \tag{8.2}$$

Conversely, if (8.1) and (8.2) hold, then from (8.1) it follows $A_2\theta = E_1$. Moreover, (8.1) implies $A_1\theta =^{\pi_2} A_2(A_2, E_3, E_1)$, using (8.2) in the last equality, we get $A_1\theta = A_3$. From (8.2) it follows $A_3\theta =^{\pi_2} A_2(E_1, A_2, E_3)$, therefore $A_3\theta = E_2$.

2. If $A_1\theta = E_1, A_2\theta = E_2, A_3\theta = A_3$ then $\theta^2 = (A_2, E_1, E_2)$, $\theta^3 = (E_2, E_3, A_1)$, $\theta^4 = (A_1, A_2, E_1)$, $\theta^5 = \varepsilon$. From $A_3\theta = A_3$ it follows $A_3\theta^2 = A_3$, i.e. $A_3(A_2, E_1, E_2) = A_3$, so

$$A_2 =^{\pi_1} A_3(A_3, E_1, E_2). \tag{8.3}$$

Also, $A_3\theta = A_3$ implies $A_3\theta^3 = A_3$, i.e. $A_3(E_2, E_3, A_1) = A_3$, hence

$$A_1 =^{\pi_3} A_3(E_2, E_3, A_3). \tag{8.4}$$

Using (8.3) and (8.4) in $A_3\theta = A_3$, we get

$$A_3(E_3,^{\pi_3} A_3(E_2, E_3, A_3),^{\pi_1} A_3(A_3, E_1, E_2)) = A_3. \tag{8.5}$$

Conversely, if (8.3), (8.4) and (8.5) hold, then using (8.3) and (8.4) in (8.5), we obtain

$$A_3(E_3, A_1, A_2) = A_3, \tag{8.6}$$

i.e. $A_3\theta = A_3$. From (8.4) it follows $A_3(E_2, E_3, A_1) = A_3$, so

$$E_2 =^{\pi_1} A_3(A_3, E_3, A_1). \tag{8.7}$$

From (8.3) it follows $A_2\theta =^{\pi_1} A_3(A_3, E_3, A_1)$. Using (8.7) in the last equality we obtain $A_2\theta = E_2$. From (8.3) we get $A_3(A_2, E_1, E_2) = A_3$, then

$$E_1 =^{\pi_2} A_3(A_2, A_3, E_2). \tag{8.8}$$

The equality (8.6) implies $A_1 =^{\pi_2} A_3(E_3, A_3, A_2)$, hence $A_1\theta =^{\pi_2} A_3(A_2, A_3, E_2)$. Using (8.8) in the last equality, we get $A_1\theta = E_1$.

3. If $A_1\theta = A_3, A_2\theta = E_2, A_3\theta = E_1$, then $\theta^2 = (A_2, A_3, E_2)$, $\theta^3 = (E_2, E_1, A_1)$, $\theta^4 = (A_1, E_3, A_3)$, $\theta^5 = (A_3, A_2, E_1)$, $\theta^6 = \varepsilon$. From $A_2\theta = E_2$ it follows

$$A_1 =^{\pi_2} A_2(E_3, E_2, A_2). \tag{8.9}$$

Also $A_2\theta = E_2$ implies $A_2\theta^5 = E_3$, i.e. $A_2(A_3, A_2, E_1) = E_3$, hence

$$A_3 =^{\pi_1} A_2(E_3, A_2, E_1). \tag{8.10}$$

Moreover, from $A_2\theta = E_2$ we get $A_2\theta^4 = E_1$, i.e. $A_2(A_1, E_3, A_3) = E_1$. Using (8.9) and (8.10) in the last equality, we obtain

$$A_2({}^{\pi_2}A_2(E_3, E_2, A_2), E_3, {}^{\pi_1} A_2(E_3, A_2, E_1)) = E_1. \tag{8.11}$$

Conversely, if (8.9), (8.10) and (8.11) hold, then from (8.9) it follows $A_2\theta = E_2$ and (8.10) implies $A_3\theta =^{\pi_1} A_2(A_2, E_2, E_3)$, so $A_3\theta = E_1$. Using (8.9) and (8.10) in (8.11) we obtain $A_2(A_1, E_3, A_3) = E_1$, which implies $A_1 =^{\pi_1} A_2(E_1, E_3, A_3)$, so $A_1\theta =^{\pi_1} A_2(E_3, A_2, E_1)$. Using (8.10) in the last equality we get $A_1\theta = A_3$.

4. If $A_1\theta = E_1, A_2\theta = A_3, A_3\theta = E_2$, then $\theta^2 = (A_2, E_1, A_3)$, $\theta^3 = (A_3, E_3, E_2)$, $\theta^4 = (E_2, A_2, A_1)$, $\theta^5 = (A_1, A_3, E_1)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_1$ it follows

$$A_2 =^{\pi_3} A_1(E_3, A_1, E_1). \tag{8.12}$$

Also $A_1\theta = E_1$ implies $A_1\theta^5 = E_2$, i. e. $A_1(A_1, A_3, E_1) = E_2$, so

$$A_3 =^{\pi_2} A_1(A_1, E_2, E_1). \tag{8.13}$$

Analogously, $A_1\theta = E_1$ implies $A_1\theta^4 = A_3$, i.e. $A_1(E_2, A_2, A_1) = A_3$. Using (8.12) and (8.13) in the last equality we get

$$A_1(E_2, {}^{\pi_3} A_1(E_3, A_1, E_1), A_1) =^{\pi_2} A_1(A_1, E_2, E_1). \tag{8.14}$$

Conversely, if (8.12), (8.13) and (8.14) hold, then (8.12) implies $A_1\theta = E_1$. From (8.13) it follows $A_3\theta =^{\pi_2} A_1(E_1, A_1, E_3)$, so $A_3\theta = E_2$. Using (8.12) and (8.13) in (8.14), we obtain $A_1(E_2, A_2, A_1) = A_3$, which implies $A_2 =^{\pi_2} A_1(E_2, A_3, A_1)$, so $A_2\theta =^{\pi_2} A_1(A_1, E_2, E_1)$. Finally, using (8.13) in the last equality, we get $A_2\theta = A_3$.

5. If $A_1\theta = E_2, A_2\theta = A_3, A_3\theta = E_1$, then $\theta^2 = (A_2, E_2, A_3)$, $\theta^3 = (A_3, A_1, E_1)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_2$ it follows

$$A_2 =^{\pi_3} A_1(E_3, A_1, E_2). \tag{8.15}$$

Also $A_1\theta = E_2$ implies $A_1\theta^3 = E_2$, i.e. $A_1(A_3, A_1, E_1) = E_2$, hence

$$A_3 =^{\pi_1} A_1(E_2, A_1, E_1). \tag{8.16}$$

From $A_1\theta = E_2$ it follows $A_1\theta^2 = A_1$, i. e. $A_1(A_2, E_2, A_3) = A_1$. Using (8.15) and (8.16) in the last equality, we obtain

$$A_1({}^{\pi_3}A_1(E_3, A_1, E_2), E_2, {}^{\pi_1} A_1(E_2, A_1, E_1)) = A_1. \tag{8.17}$$

Conversely, if (8.15), (8.16) and (8.17), then from (8.15) it follows $A_1\theta = E_2$. From (8.16) we get $A_3\theta =^{\pi_1} A_1(A_1, E_2, E_3)$, so $A_3\theta = E_1$. Using (8.15) and (8.16) in (8.17), we obtain $A_1(A_2, E_2, A_3) = A_1$, hence $A_2 =^{\pi_1} A_1(A_1, E_2, A_3)$, which implies $A_2\theta =^{\pi_1} A_1(E_2, A_1, E_1)$ and using (8.16) in the last equality, we get $A_2\theta = A_3$.

6. If $A_1\theta = E_2, A_2\theta = E_1, A_3\theta = A_3$, then $\theta^2 = (A_2, E_2, E_1)$, $\theta^3 = (E_1, A_1, E_3)$. From $A_2\theta = E_1$ it follows $A_2\theta^3 = E_1\theta^2$, so $A_2(E_1, A_1, E_3) = A_2(E_1, E_2, E_3)$, hence $A_1 = E_2$, which is a contradiction as A_1 is a quasigroup operation. □

Lemma 9. *The triple (A_1, E_3, A_2) is a paratopy of the system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ if and only if one of the following statements holds:*
1. $A_1 =^{\pi_1} A_2(E_2, E_3, A_2)$, $A_3 =^{\pi_1} A_2(E_3, A_2, E_2)$;
2. $A_2 =^{\pi_3} A_1(A_1, E_3, E_1)$, $A_3 =^{\pi_2} A_1(A_1, E_1, E_2)$ *and*
$A_1(E_1, {}^{\pi_3}A_1(A_1, E_3, E_1), {}^{\pi_2}A_1(A_1, E_1, E_2)) = A_1$;
3. $A_2 =^{\pi_2} A_3(E_2, A_3, E_1)$, $A_1 =^{\pi_3} A_3(E_3, E_1, A_3)$ *and*
$A_3({}^{\pi_3}A_3(E_3, E_1, A_3), E_3, {}^{\pi_2}A_3(E_2, A_3, E_1)) = A_3$;
4. $A_2 =^{\pi_3} A_1(A_1, E_3, E_2)$, $A_3 =^{\pi_1} A_1(E_1, A_1, E_2)$ *and*
$A_1({}^{\pi_3}A_1(A_1, E_3, E_2), E_1, A_1) =^{\pi_1} A_1(E_1, A_1, E_2)$;
5. $A_1 =^{\pi_1} A_2(E_1, E_3, A_2)$, $A_3 =^{\pi_2} A_2(A_2, E_3, E_2)$ *and*
$A_2(E_3, {}^{\pi_1}A_2(E_1, E_3, A_2), {}^{\pi_2}A_2(A_2, E_3, E_2)) = E_2$.

Proof. Let the tuple (A_1, E_3, A_2) be a paratopy of the system Σ. As $E_1\theta = A_1, E_2\theta = E_3, E_3\theta = A_2$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, E_3\}$, i.e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_1, E_2, A_3\}$.

1. If $A_1\theta = A_3, A_2\theta = E_2, A_3\theta = E_1$, then $\theta^2 = (A_3, A_2, E_2)$, $\theta^3 = \varepsilon$. From $A_2\theta = E_2$ it follows

$$A_1 =^{\pi_1} A_2(E_2, E_3, A_2). \tag{9.1}$$

Also $A_2\theta = E_2$ implies $A_2\theta^2 = E_3$, i.e. $A_2(A_3, A_2, E_2) = E_3$, so

$$A_3 =^{\pi_1} A_2(E_3, A_2, E_2). \tag{9.2}$$

Conversely, if (9.1) and (9.2) hold, then from (9.1) it follows $A_2\theta = E_2$. Moreover, (9.1) implies $A_1\theta =^{\pi_1} A_2(E_3, A_2, E_2)$, using (9.2) in the last equality, we get $A_1\theta = A_3$. From (9.2) it follows $A_3\theta =^{\pi_1} A_2(A_2, E_2, E_3)$, therefore $A_3\theta = E_1$.

2. If $A_1\theta = E_1, A_2\theta = A_3, A_3\theta = E_2$, then $\theta^2 = (E_1, A_2, A_3)$, $\theta^3 = (A_1, A_3, E_2)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_1$ it follows

$$A_2 =^{\pi_3} A_1(A_1, E_3, E_1). \tag{9.3}$$

Also, $A_1\theta = E_1$ implies $A_1\theta^3 = E_1$, i. e. $A_1(A_1, A_3, E_2) = E_1$, hence

$$A_3 =^{\pi_2} A_1(A_1, E_1, E_2). \tag{9.4}$$

Moreover, from $A_1\theta = E_1$ it follows $A_1\theta^2 = A_1$, i. e. $A_1(E_1, A_2, A_3) = A_1$. Using (9.3) and (9.4) in the last equality, we get

$$A_1(E_1, {}^{\pi_3}A_1(A_1, E_3, E_1), {}^{\pi_2}A_1(A_1, E_1, E_2)) = A_1. \tag{9.5}$$

Conversely, if (9.3), (9.4) and (9.5) hold, then from (9.3) it follows $A_1\theta = E_1$ and (9.4) implies $A_3\theta =^{\pi_2} A_1(E_1, A_1, E_3)$, therefore $A_3\theta = E_2$. Using (9.3) and (9.4) in (9.5), we obtain $A_1(E_1, A_2, A_3) = A_1$, so $A_2 =^{\pi_2} A_1(E_1, A_1, A_3)$, which implies $A_2\theta =^{\pi_2} A_1(A_1, E_1, E_2)$. Using (9.4) in the last equality, we get $A_2\theta = A_3$.

3. If $A_1\theta = E_2, A_2\theta = E_1, A_3\theta = A_3$, then $\theta^2 = (E_2, A_2, E_1)$, $\theta^3 = (E_3, E_1, A_1)$, $\theta^4 = (A_2, A_1, E_2)$, $\theta^5 = \varepsilon$. From $A_3\theta = A_3$ it follows $A_3\theta^2 = A_3$, i.e. $A_3(E_2, A_2, E_1) = A_3$, so

$$A_2 =^{\pi_2} A_3(E_2, A_3, E_1). \tag{9.6}$$

Also, $A_3\theta = A_3$ implies $A_3\theta^3 = A_3$, i.e. $A_3(E_3, E_1, A_1) = A_3$, hence

$$A_1 =^{\pi_3} A_3(E_3, E_1, A_3). \tag{9.7}$$

Using (9.6) and (9.7) in $A_3\theta = A_3$, we get

$$A_3({}^{\pi_3}A_3(E_3, E_1, A_3), E_3, {}^{\pi_2}A_3(E_2, A_3, E_1)) = A_3. \tag{9.8}$$

Conversely, if (9.6), (9.7) and (9.8) hold, then using (9.6) and (9.7) in (9.8), we obtain

$$A_3(A_1, E_3, A_2) = A_3, \tag{9.9}$$

therefore $A_3\theta = A_3$. From (9.7) it follows $A_3(E_3, E_1, A_1) = A_3$, so

$$E_1 =^{\pi_2} A_3(E_3, A_3, A_1). \tag{9.10}$$

From (9.6) it follows $A_2\theta =^{\pi_2} A_3(E_3, A_3, A_1)$. Using (9.10) in the last equality we obtain $A_2\theta = E_1$. From (9.6) we get $A_3(E_2, A_1, A_1) = A_3$, then

$$E_2 =^{\pi_1} A_3(A_3, A_2, E_1). \tag{9.11}$$

The equality (9.9) implies $A_1 =^{\pi_1} A_3(A_3, E_3, A_2)$, so $A_1\theta =^{\pi_1} A_3(A_3, A_2, E_1)$. Using (9.11) in the last equality, we get $A_1\theta = E_2$.

4. If $A_1\theta = E_2, A_2\theta = A_3, A_3\theta = E_1$, then $\theta^2 = (E_2, A_2, A_3)$, $\theta^3 = (E_3, A_3, E_1)$, $\theta^4 = (A_2, E_1, A_1)$, $\theta^5 = (A_3, A_1, E_2)$, $\theta^6 = \varepsilon$. From $A_1\theta = E_2$ it follows

$$A_2 =^{\pi_3} A_1(A_1, E_3, E_2). \tag{9.12}$$

Also $A_1\theta = E_2$ implies $A_1\theta^5 = E_1$, i. e. $A_2(A_3, A_1, E_2) = E_3$, hence

$$A_3 =^{\pi_1} A_1(E_1, A_1, E_2). \tag{9.13}$$

Moreover, from $A_1\theta = E_2$, we get $A_1\theta^4 = A_3$, i. e. $A_1(A_2, E_1, A_1) = A_3$. Using (9.12) and (9.13) in the last equality, we obtain

$$A_1({}^{\pi_3}A_1(A_1, E_3, E_2), E_1, A_1) =^{\pi_1} A_1(E_1, A_1, E_2). \tag{9.14}$$

Conversely, if (9.12), (9.13) and (9.14) hold, then from (9.12) it follows $A_1\theta = E_2$ and (9.13) implies $A_3\theta =^{\pi_1} A_1(A_1, E_2, E_3)$, so $A_3\theta = E_1$. Using (9.12) and (9.13) in (9.14) we obtain $A_1(A_2, E_1, A_1) = A_3$, which implies $A_2 = ^{\pi_1} A_1(A_3, E_1, A_1)$, so $A_1\theta = ^{\pi_1} A_1(E_1, A_1, E_2)$. Using (9.13) in the last equality we get $A_1\theta = A_3$.

5. If $A_1\theta = A_3, A_2\theta = E_1, A_3\theta = E_2$, then $\theta^2 = (A_3, A_2, E_1)$, $\theta^3 = (E_2, E_1, A_1)$, $\theta^4 = (E_3, A_1, A_3)$, $\theta^5 = (A_2, A_3, E_2)$, $\theta^6 = \varepsilon$. From $A_2\theta = E_1$ it follows

$$A_1 = ^{\pi_1} A_2(E_1, E_3, A_2). \tag{9.15}$$

Also $A_2\theta = E_1$ implies $A_2\theta^5 = E_3$, i. e. $A_2(A_2, A_3, E_2) = E_3$, so

$$A_3 = ^{\pi_2} A_2(A_2, E_3, E_2). \tag{9.16}$$

Analogously, $A_2\theta = E_1$ implies $A_2\theta^4 = E_2$, i. e. $A_2(E_3, A_1, A_3) = E_2$. Using (9.15) and (9.16) in the last equality we get

$$A_2(E_3, ^{\pi_1} A_2(E_1, E_3, A_2), ^{\pi_2} A_2(A_2, E_3, E_2)) = E_2. \tag{9.17}$$

Conversely, if (9.15), (9.16) and (9.17) hold, then (9.15) implies $A_2\theta = E_1$. From (9.16) it follows $A_3\theta = ^{\pi_2} A_2(E_1, A_2, E_3)$, so $A_3\theta = E_2$. Using (9.15) and (9.16) in (9.17), we obtain $A_2(E_3, A_1, A_3) = E_2$, which implies $A_1 = ^{\pi_2} A_2(E_3, E_2, A_3)$, so $A_1\theta = ^{\pi_2} A_2(A_2, E_3, E_2)$. Using (9.16) in the last equality, we get $A_1\theta = A_3$.

6. If $A_1\theta = E_1, A_2\theta = E_2, A_3\theta = A_3$, then $\theta^2 = (E_1, A_2, E_2)$, $\theta^3 = (A_1, E_2, E_3)$. From $A_2\theta = E_2$ it follows $A_2\theta^3 = E_2\theta^2$, so $A_2(A_1, E_2, E_3) = A_2(E_1, E_2, E_3)$, hence $A_1 = E_1$, which is a contradiction as A_1 is a quasigroup operation. □

Lemma 10. *The triple (A_1, A_2, E_3) is a paratopy of the system $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$ if and only if one of the following statements holds:*
1. $A_2 = ^{\pi_2} A_1(A_1, E_1, E_3)$, $A_3(A_1, ^{\pi_2} A_1(A_1, E_1, E_3), E_3) = A_3$;
2. $A_2 = ^{(12)} A_1$, $A_3(A_1, ^{(12)} A_1, E_3) = A_3$ *and* $A_1(A_1, ^{(13)} A_1, E_3) = E_2$;
3. $A_2 = ^{\pi_2} A_1(A_1, E_2, E_3)$, $A_3 = ^{\pi_1} A_1(E_1, A_1, E_3)$ *and*
$A_1(^{\pi_2} A_1(A_1, E_2, E_3), E_1, E_3) = ^{\pi_1} A_1(E_1, A_1, E_3)$;
4. $A_1 = ^{\pi_1} A_2(E_1, A_2, E_3)$, $A_3 = ^{\pi_2} A_2(A_2, E_2, E_3)$ *and*
$A_2(E_2, ^{\pi_1} A_2(E_1, A_2, E_3), E_3) = ^{\pi_2} A_2(A_2, E_2, E_3)$.

Proof. Let the tuple (A_1, A_2, E_3) be a paratopy of the system Σ. As $E_1\theta = A_1, E_2\theta = A_2, E_3\theta = E_3$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, A_1, A_2, E_3\}$, i.e. $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_1, E_2, A_3\}$.

1. If $A_1\theta = E_1, A_2\theta = E_2, A_3\theta = A_3$, then $\theta^2 = \varepsilon$. From $A_1\theta = E_1$ it follows

$$A_2 = ^{\pi_2} A_1(A_1, E_1, E_3). \tag{10.1}$$

Using (10.1) in $A_3\theta = A_3$, we get

$$A_3(A_1, ^{\pi_2} A_1(A_1, E_1, E_3), E_3) = A_3. \tag{10.2}$$

Conversely, if (10.1) and (10.2) hold, then from (10.1) it follows $A_1\theta = E_1$ and $A_2\theta = ^{\pi_2} A_1(E_1, A_1, E_3)$, so $A_2\theta = E_2$. Using (10.1) in (10.2), we obtain $A_3\theta = A_3$.

2. If $A_1\theta = E_2, A_2\theta = E_1, A_3\theta = A_3$, then $\theta^2 = (E_2, E_1, E_3)$, $\theta^3 = (A_2, A_1, E_3)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_2$ it follows $A_1\theta^2 = A_2$, so

$$A_2 =^{(12)} A_1. \tag{10.3}$$

Using (10.3) in $A_3\theta = A_3$, we get

$$A_3(A_1, ^{(12)}A_1, E_3) = A_3. \tag{10.4}$$

The equality $A_1\theta = E_2$ and (10.3) imply

$$A_1(A_1, ^{(12)}A_1, E_3) = E_2. \tag{10.5}$$

Conversely, if (10.3), (10.4) and (10.5) hold, then (10.3) and (10.5) imply

$$A_1(A_1, A_2, E_3) = E_2, \tag{10.6}$$

so $A_1\theta = E_2$. From (10.3) it follows

$$E_1 =^{\pi_2} A_1(E_2, A_2, E_3). \tag{10.7}$$

The equality (10.6) implies $A_2 =^{\pi_2} A_1(A_1, E_2, E_3)$, hence $A_2\theta =^{\pi_2} A_1(E_2, A_2, E_3)$. Using (10.7) in the last equality, we obtain $A_2\theta = E_1$. From (10.3) and (10.4) it follows $A_3\theta = A_3$.

3. If $A_1\theta = E_2, A_2\theta = A_3, A_3\theta = E_1$, then $\theta^2 = (E_2, A_3, E_3)$, $\theta^3 = (A_2, E_1, E_3)$, $\theta^4 = (A_3, A_1, E_3)$, $\theta^5 = \varepsilon$. From $A_1\theta = E_2$ it follows

$$A_2 =^{\pi_2} A_1(A_1, E_2, E_3). \tag{10.8}$$

Also $A_1\theta = E_2$ implies $A_1\theta^4 = E_1$, i. e. $A_1(A_3, A_1, E_3) = E_1$, so

$$A_3 =^{\pi_1} A_1(E_1, A_1, E_3). \tag{10.9}$$

In a similar way, $A_1\theta = E_2$ implies $A_1\theta^3 = A_3$, i. e. $A_1(A_2, E_1, E_3) = A_3$. Using (10.8) and (10.9) in the last equality, we get

$$A_1(^{\pi_2}A_1(A_1, E_2, E_3), E_1, E_3) =^{\pi_1} A_1(E_1, A_1, E_3). \tag{10.10}$$

Conversely, if (10.8), (10.9) and (10.10) hold, then (10.8) implies $A_1\theta = E_2$. From (10.9) it follows $A_3\theta =^{\pi_1} A_1(A_1, E_2, E_3)$, so $A_3\theta = E_1$. Using (10.8) and (10.9) in (10.10), we obtain $A_1(A_2, E_1, E_3) = A_3$, so $A_2 =^{\pi_1} A_1(A_3, E_1, E_3)$, which implies $A_2\theta =^{\pi_1} A_1(E_1, A_1, E_3)$. Using (10.9) in the last equality, we get $A_2\theta = A_3$.

4. If $A_1\theta = A_3, A_2\theta = E_1, A_3\theta = E_2$, then $\theta^2 = (A_3, E_1, E_3)$, $\theta^3 = (E_2, A_1, E_3)$, $\theta^4 = (A_2, A_3, E_3)$, $\theta^5 = \varepsilon$. From $A_2\theta = E_1$ it follows

$$A_1 =^{\pi_1} A_2(E_1, A_2, E_3). \tag{10.11}$$

Also $A_2\theta = E_1$ implies $A_2\theta^4 = E_2$, i.e. $A_2(A_2, A_3, E_3) = E_2$, so

$$A_3 =^{\pi_2} A_2(A_2, E_2, E_3). \tag{10.12}$$

Analogously, $A_2\theta = E_1$ implies $A_2\theta^3 = A_3$, i.e. $A_2(E_2, A_1, E_3) = A_3$. Using (10.11) and (10.12) in the last equality, we get

$$A_2(E_2, {}^{\pi_1}A_2(E_1, A_2, E_3), E_3) = {}^{\pi_2}A_2(A_2, E_2, E_3). \qquad (10.13)$$

Conversely, if (10.11), (10.12) and (10.13) hold, then (10.11) implies $A_2\theta = E_1$. From (10.12) it follows $A_3\theta = {}^{\pi_2}A_2(E_1, A_2, E_3)$, so $A_3\theta = E_2$. Using (10.11) and (10.12) in (10.13), we obtain $A_2(E_2, A_1, E_3) = A_3$, so $A_1 = {}^{\pi_2}A_2(E_2, A_3, E_3)$, which implies $A_1\theta = {}^{\pi_2}A_2(A_2, E_2, E_3)$. Using (10.12) in the last equality, we get $A_1\theta = A_3$.

5. If $A_1\theta = E_1, A_2\theta = A_3, A_3\theta = E_2$, then $\theta^2 = (E_1, A_3, E_3)$. From $A_1\theta = E_1$ it follows $A_1\theta^2 = A_1$, i.e. $A_1(E_1, A_3, E_3) = A_1(E_1, E_2, E_3)$, so $A_3 = E_2$, which is a contradiction as A_3 is a quasigroup operation.

6. If $A_1\theta = A_3, A_2\theta = E_2, A_3\theta = E_1$, then $\theta^2 = (A_3, E_2, E_3)$. From $A_2\theta = E_2$ it follows $A_2\theta^2 = A_2$, i.e. $A_2(A_3, E_2, E_3) = A_2(E_1, E_2, E_3)$, so $A_3 = E_1$, which is a contradiction as A_3 is a quasigroup operation. □

Corollary 3. *If a ternary quasigroup (Q, A) satisfies the identity $A(A, {}^{(12)}A, E_3) = E_2$ then, for $\forall a \in Q$, its 3-retract $B(x, y) = A(x, y, a)$ is self-orthogonal.*

The proof is analogous to the proof of Corollary 1.

Theorem 2. *There exist 48 orthogonal systems consisting of three ternary quasigroups and ternary selectors E_1, E_2, E_3, that admit at least one paratopy, which components are three ternary quasigroup operations or a ternary selector and two ternary quasigroup operations.* The proof follows from Lemmas 1–10.

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