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Triality and Universal Multiplication Groups of Moufang Loops

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Abstract. We investigate the triality status of combinatorial and universal multiplication groups of various classes of Moufang loops. We also investigate whether some of these are, qua Doro, the largest and smallest groups with triality associated with a given Moufang loop.

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1 Moufang Loops and Universal Multiplication Groups

A *loop* is a set with a single binary operation such that in $x \cdot y = z$, knowledge of any two of x, y, and z specifies the third uniquely, and with a unique two-sided identity element, denoted by 1. A *Moufang loop* is a loop satisfying the identity $x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z$. We use the notation x^{-1} to denote the unique 2-sided inverse of x.

The *commutant*, $\mathrm{C}(M)$, of a Moufang loop M is the set of those elements which commute with each element in the loop. That is, $\mathrm{C}(M) = \{c : \forall x \in M, cx = xc\}$; it is a subloop. Define the *left nucleus* of a Moufang loop, M, by $\mathrm{N}_\lambda(M) = \{a : a \cdot (x \cdot y) = (a \cdot x) \cdot y, \forall x, y \in M\}$. The *middle nucleus*, $\mathrm{N}_\mu(M)$, and the *right nucleus*, $\mathrm{N}_\rho(M)$, are defined analogously. The *nucleus*, $\mathrm{N}(M)$, is then defined by $\mathrm{N}(M) = \mathrm{N}_\lambda(M) \cap \mathrm{N}_\mu(M) \cap \mathrm{N}_\rho(M)$. (In fact, each of these four subsets coincides with the other three [2].) $\mathrm{N}(M)$ is a normal subloop of M [2]. The *center*, $\mathrm{Z}(M)$, of M is defined as $\mathrm{Z}(M) = \mathrm{C}(M) \cap \mathrm{N}(M)$; it is a normal subloop.

We use the standard notation for the right and left translations: $xR(y) = yL(x) = x \cdot y$. The *(combinatorial) multiplication group*, $\mathrm{Mlt}(M)$, of a loop M is the subgroup of the group of all bijections on M generated by right and left translations. Clearly, $\mathrm{Mlt}(M)$ acts as a permutation group on M.

Let M be a Moufang loop, and let $\mathbf{M}$ be an arbitrary variety of Moufang loops containing M. We also use $\mathbf{M}$ to denote the category whose objects are the Moufang loops in $\mathbf{M}$ and whose morphisms are loop homomorphisms. As an algebraic category, $\mathbf{M}$ is complete and co-complete [6, 13.12, 13.14]. In $\mathbf{M}$, form the coproduct of M with $< x >$, the free $\mathbf{M}$-algebra on one generator. Denote this coproduct by $M[x]$ (the variety, $\mathbf{M}$, though not explicitly noted in our coproduct notation, will be clear from context). Since M may be indentified with its image in $M[x]$ [8, p. 33], we can consider the subgroup of $\mathrm{Mlt}(M[x])$ generated by right and left multiplications by elements of M. This subgroup is the *universal multiplication group*, $U(M;\mathbf{M})$, of M in $\mathbf{M}$.

The assignment of $U(M;\mathbf{M})$ to M gives a functor from the category $\mathbf{M}$ to the category $\mathbf{GP}$ of all groups [8, p. 34]. Note that $U(M;\mathbf{M})$ is "variety dependent" in the sense that for a given Moufang loop M and two varieties $\mathbf{M}_1$ and $\mathbf{M}_2$ containing M, it is not necessarily the case that $U(M;\mathbf{M}_1) \cong U(M;\mathbf{M}_2)$ [8, p. 36]. But if $\mathbf{M}_1 \subseteq \mathbf{M}_2$, then there is a natural group epimorphism $F : U(M;\mathbf{M}_2) \to U(M;\mathbf{M}_1)$ [8, p. 55]. This can be summarized informally as "the smaller the variety, the smaller the universal multiplication group."

For any variety, $\mathbf{M}$ of Moufang loops containing M, there is a natural group epimorphism $F : U(M;\mathbf{M}) \to \mathrm{Mlt}M$ [8, p. 55]. This can summarized informally as "a universal multiplication group can be no smaller than the combinatorial multiplication group."

2 Groups with Triality

If M is a Moufang loop, there exists an involutary automorphism, σ on $\mathrm{Mlt}M$, defined on generators by $R(x)^\sigma = L(x^{-1})$ and $L(x)^\sigma = R(x^{-1})$ [4]. If $\mathrm{N}(M) = 1$, Glauberman [4] showed that there exists an automorphism ρ on $\mathrm{Mlt}M$, defined on generators by $L(x)^\rho = R(x), R(x)^\rho = P(x)$ and $P(x)^\rho = L(x)$. Here and throughout, $P(x) = R(x^{-1})L(x^{-1})$; and so note that $P(x)R(x)L(x) = 1$. Also, clearly $\rho^3 = 1$. So it is easy to see that if both σ and ρ are nontrivial, then together they generate S_3.

Inspired by Glauberman, Doro [3] defined a *group with triality* to be a group, G with two automorphisms, σ and ρ, such that $\sigma^2 = 1$, $\rho^3 = 1$, $< \sigma, \rho >= S_3$ and satisfying the identity $g^{-1}g^\sigma g^{-\rho} g^{\sigma\rho} g^{-\rho^2} g^{\rho\sigma} = 1$, $\forall g \in G$ (this identity is a kind of encoding of $P(x)R(x)L(x) = 1$, the details are in [3]). Groups with triality were crucial in Liebeck's classification of all finite simple Moufang loops [5].

Given a group with triality, G, Doro [3] constructs a Moufang loop, M so that $\mathrm{Mlt}M$ is a homomorphic image of G. Conversely, given a Moufang loop, M, Doro constructs a group with triality, G, such that the construction in the previous sentence yields M, and such that $\mathrm{Mlt}M$ is a homomorphic image of G. Note that for a given Moufang loop, M, there may be more than one group with triality which gives M via Doro's construction. But, for a given M, Doro [3] shows that there is a largest group with triality, denoted by $\mathrm{G}(M)$, that gives M, in the sense that any other group with triality that gives M is itself a homomorphic image of $\mathrm{G}(M)$. Doro [3] also shows that there is a smallest group with triality, denoted by $\mathrm{G}_0(M)$, that gives M, in the sense that $\mathrm{G}_0(M)$ is a homomorphic image of any other group with triality that gives M. And Doro shows that, given any group with triality, G, if M is the Moufang loop constructed from G, then $\mathrm{Mlt}M$ is a homomorphic image of G. Thus, given any group with triality, G, with associated Moufang loop M, there is a sequence of group epimoprhisms, from $\mathrm{G}(M)$ to G to $\mathrm{G}_0(M)$ to $\mathrm{Mlt}M$.

Given a Moufang loop, M, to determine whether any of its multiplication groups is with triality, it suffices to determine which, if any, of these groups admit the automorphism ρ. This means that if we define ρ on generators (i.e., on the $R(x)$'s and $L(x)$'s, as above), we must decide if ρ extends to the entire group. Thus, it suffices to determine if $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n) = 1$ implies that $Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho = 1$ (here, each $Q_i(x_i)$ is either $R(x_i)$ or $L(x_i)$). This task is greatly simplified by the following result from Glauberman [4]: if $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n) = 1$ then $Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho = R(c)$ for some $c \in \mathrm{N}(M)$. Thus, the multiplication group admits ρ, and hence is with triality, precisely if this nuclear element c equals 1. We use this fact freely in the balance of the paper.

3 Results

In this section, M always represents a Moufang loop (perhaps with more structure, as noted in those instances). The first five results in this section focus on Moufang loop multiplication groups with triality. In [7], the triality status of $\mathrm{Mlt}M$ is established for all M except for those of the following form:

$$1 < \mathrm{N}(M) \leq \mathrm{C}(M) < M \text{ and } \mathrm{C}(M)^3 = 1.$$

We note that the center of a loop and the center of its combinatorial multiplication group are isomorphic via the mapping $z \mapsto R(z)$ [1]. Thus, we use $\mathrm{Z}(M)$ and $\mathrm{Z}(\mathrm{Mlt}M)$ interchangeably, as in the next theorem.

Theorem 3.1. *If* $\mathrm{N}(M) \leq \mathrm{C}(M)$ *then* $\mathrm{Mlt}M/\mathrm{Z}(M)$ *is with triality.*

Proof. Elements in $\mathrm{Mlt}M/\mathrm{Z}(M)$ have the form $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n)\mathrm{Z}(M)$. So, if $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n)\mathrm{Z}(M) = 1\mathrm{Z}(M)$, then there exists an element z in the center, such that $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n) = R(z)$. Rearranging gives $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n)R(z^{-1}) = 1$. Thus, applying ρ we have $Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho R(z^{-1})^\rho = R(c)$, for some element c in the center. Rearranging gives $Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho = R(z^{-2}c)$, where obviously $z^{-2}c$ is in the center. Thus,

$$Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho \mathrm{Z}(M) = 1\mathrm{Z}(M).$$

Hence ρ is well defined on $\mathrm{Mlt}M/\mathrm{Z}(M)$, and so $\mathrm{Mlt}M/\mathrm{Z}(M)$ is with triality. □

Theorem 3.2. *If* $\mathrm{Mlt}M$ *is a group with triality, then so too is* $U(M;\boldsymbol{M})$*, where* $\boldsymbol{M}$ *is any variety of Moufang loops containing* M*.*

Proof. Define ρ on the generators of $U(M;\mathbf{M})$. We show that ρ extends to all of $U(M;\mathbf{M})$. Assume $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n) = 1$ in $U(M;\mathbf{M})$. Then $Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho = R(c)$ for some $c \in \mathrm{N}(M[x])$. But $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n) = 1$ in $\mathrm{Mlt}M$, also. And since $\mathrm{Mlt}M$ is with triality, we have $Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho = 1$ in $\mathrm{Mlt}M$. This means that $c = 1 \cdot c = 1Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho = 1$. Hence, ρ is well defined, and so $U(M;\mathbf{M})$ is a group with triality. □

Before proving our next theorem we need two technical lemmas. (For the balance of the paper, if G is a group with triality, we let S be the subset of elements of G fixed by ρ, and we let I be the subset of elements fixed by σ.)

Lemma 3.3. *If* G *is a group with triality and D is the S_3 group of triality automorphisms acting on* G, *then* $\mathrm{C}_G(GD) \cong (I \cap S \cap \mathrm{Z}(\mathrm{G}(M)))$.

Proof.

$$\begin{aligned} \mathrm{C}_G(GD) &= \{(g,1) : \forall h \in G, \forall \theta \in D, (g,1)(h,\theta) = (h,\theta)(g,1)\} \\ &= \{(g,1) : \forall h \in G, \forall \theta \in D, (gh,\theta) = (hg^{\theta^{-1}},\theta)\}. \end{aligned}$$

Taking $h = 1$ and $\theta = \sigma$ ($\theta = \rho^2$, $\theta = 1$, respectively) yields

$$\mathrm{C}_G(GD) \subset I \quad (\subset S, G), \text{respectively}).$$

The converse is now trivial. □

Lemma 3.4. *If M is a Moufang loop such that* $\mathrm{Mlt}M$ *is with triality, then* $\mathrm{C}_{\mathrm{Mlt}M}(\mathrm{Mlt}MD) = 1$.

Proof. From the proof of the preceeding lemma, $\mathrm{C}_{\mathrm{Mlt}M}(\mathrm{Mlt}MD) = I \cap S \cap \mathrm{Z}(\mathrm{Mlt}M)$. Now, if $h \in \mathrm{Z}(\mathrm{Mlt}M)$, then $h = R(c)$ for some $c \in \mathrm{Z}(M)$ [1, Thm. 11]. But h is also in I, so $c = 1$, and hence $h = 1$. Thus, $I \cap S \cap \mathrm{Z}(\mathrm{Mlt}M) = 1$. □

Our next theorem is a generalization of [3, Corollary 5]. It is offered here because the proof in [3] is incorrect.

Theorem 3.5. *If M is a Moufang loop such that* $\mathrm{Mlt}M$ *is with triality, then* $\mathrm{Mlt}M = \mathrm{G}_0(M)$.

Proof. $\mathrm{G}_0(M) \cong \mathrm{Mlt}M/\mathrm{C}_{\mathrm{Mlt}M}(\mathrm{Mlt}MD) \cong \mathrm{Mlt}M$. The first isomorphism is [3, Cor. 1, p. 384]. The second isomorphism is by the previous lemma. □

We turn our attention now to cyclic groups. We begin with a technical lemma.

Lemma 3.6. *If M is a cyclic group, then $M \cap \mathrm{C}(M[x]) = 1$. (Here, the coproduct $M[x]$ is any category of Moufang loops containing all groups.)*

Proof. M embeds in some group G so that $\mathrm{Z}(G) \cap M = 1$. (If M is infinite, take G free on two or more generators; if the order of M is n, take $G =< x, y : x^n = 1 >$.) Say $f : M \to G$ is such an embedding. Then, given $y \in M$, there is a $g \in G$ such that $f(y)g \neq gf(y)$. Let $h :< x > \to G$ be determined by $x \mapsto g$. Thus, there is a unique $F : M[x] \to G$ such that the coproduct diagram commutes. So, if $xy = yx$, then $f(y)g = f(y)h(x) = F(y)F(x) = F(yx) = F(xy) = F(x)F(y) = h(x)f(y) = gf(y)$. Hence, $yx \neq xy$ and $y \notin \mathrm{C}(M[x])$. And thus, $M \cap \mathrm{C}(M[x]) = 1$. □

We are now able to describe the universal multiplication groups of cyclic groups.

Theorem 3.7. *If M is a cyclic group and $\boldsymbol{M}$ is any variety of Moufang loops containing M and all groups, then $U(M;\boldsymbol{M}) \cong M \times M$ and $U(M;\boldsymbol{M})$ is with triality.*

Proof. Let $R(M) =< R(x) : x \in M >_{U(M;\mathbf{M})}$, i. e., the subgroup of $U(M;\mathbf{M})$ generated by the set of all right translations by elements in M. Similarly, let $L(M) =< L(x) : x \in M >_{U(M;\mathbf{M})}$. Since M is cyclic, $M[x]$ is generated by two elements, and so by Moufang's Theorem [2], $M[x]$ is a group. Thus, both $R(M)$ and $L(M)$ are normal in $U(M;\mathbf{M})$. Thus, by the preceding lemma, $R(M) \cap L(M) = 1$, and since $U(M;\mathbf{M}) =< R(M), L(M) >$, we have $U(M;\mathbf{M}) \cong R(M) \times L(M) \cong M \times M$.

Now, define ρ on $U(M;\mathbf{M})$ as follows: $(R(w)L(y))^\rho = R(w^{-1}y)L(y^{-1})$. Next, we compute,

$$\begin{aligned}
([R(w_1)L(y_1)][R(w_2)L(y_2)])^\rho &= (R(w_1w_2)L(y_1y_2))^\rho \\
&= R(w_2^{-1}w_1^{-1}y_1y_2)L(w_2^{-1}w_1^{-1}) \\
&= R(w_1^{-1}y_1w_2^{-1}y_2)L(w_2^{-1}w_1^{-1}) \\
&= R(w_1^{-1}y_1)L(w_1^{-1})R(w_2^{-1}y_2)L(w_2^{-1}) \\
&= [R(w_1)L(y_1)]^\rho[R(w_2)L(y_2)]^\rho.
\end{aligned}$$

Thus, ρ is a well-defined homomorphism, and so $U(M;\mathbf{M})$ is with triality. (As an alternate proof, note that the proof of the following theorem shows that $U(M;\mathbf{M}) \cong \mathrm{G}(M)$, and hence, is with triality.) □

We are also able to describe $\mathrm{G}(M)$, the largest group with triality associated with an arbitrary cyclic group, M.

Theorem 3.8. *If M is a cyclic group, then* $\mathrm{G}(M) \cong M \times M$.

Proof. Let $M =< a >$. Two trivial induction arguments show that for every pair of positive integers m and n, we have $R(a^m)R(a^n) = R(a^{m+n})$ and $L(a^m)L(a^n) = L(a^{m+n})$.

Next, we use induction on $m+n$ to show that $R(a^m)L(a^n) = L(a^n)R(a^m)$. The cases $m+n = 1$ and either $m = 0$ or $n = 0$ are both trivial. The nontrivial instance of $m+n = 2$ is proved by noting that $R(a)L(a) = R(a)P(1)L(a) = P(a^{-1}) = L(a)P(1)R(a) = L(a)R(a)$. So assume that the statement is true for all $m+n < k$. Now consider the case $m+n = k$:

$$\begin{aligned}
R(a^m)L(a^n) &= R(a)R(a^{m-1})L(a^{n-1})L(a) \\
&= R(a)L(a^{n-1})R(a^{m-1})L(a) \\
&= L(a^{n-1})R(a)L(a)R(a^{m-1}) \\
&= L(a^{n-1})L(a)R(a)R(a^{m-1}) \\
&= L(a^n)R(a^m).
\end{aligned}$$

Thus, we have shown that $R(a^m)L(a^n) = L(a^n)R(a^m)$, and hence, the following map is onto: $F : M \times M \to \mathrm{G}(M); (a^m, a^n) \mapsto R(a^m)L(a^n)$. By the computations above, F is a homomorphism. Finally, $U(M;\mathbf{M}) \cong M \times M$ is a homomorphic image of $\mathrm{G}(M)$, and so F is one-to-one. □

For a finite cyclic group M we have a complete description of $\mathrm{G}_0(M)$, the smallest group with triality associated with M.

Theorem 3.9. *If M is a finite cyclic group of order n then $\mathrm{G}_0(M) \cong M \times M$ if 3 does not divide n, and $\mathrm{G}_0(M) \cong (M \times M)/C_3$ if 3 divides n (cf. [3, Prop. 1]).*

Proof. Let $< x >= M$. Doro shows that $\mathrm{G}_0(M) \cong \mathrm{C}_{\mathrm{G}(M)}(\mathrm{G}(M)D)$ [3, p. 384]. Thus, by Lemma 3.3, $\mathrm{G}_0(M) \cong \mathrm{G}(M)/(I \cap S \cap \mathrm{Z}(\mathrm{G}(M)))$.

In $\mathrm{G}(M)$, $R(x^k)L(x^m)$ is in I if and only if $R(x^{k+m})L(x^{k+m}) = 1$. But since, as above, $\mathrm{G}(M)$ is really just $U(M; \mathbf{M})$ and since $U(M; \mathbf{Gp})$ is a homomorphic image of $U(M; \mathbf{M})$, the proof of [8, Thm 235] assures us that $|x|$ divides $m+k$. But clearly we are assuming that $|x|$ is greater than or equal to both m and k. Thus, $|x| = m+k$.

On the other hand, in $\mathrm{G}(M), R(x^k)L(x^m)$ is in S if and only if $|x|$ divides $3k$. So if 3 does not divide n (and note that $n = |x|$), we must have that $|x|$ divides k. And since $|x| = m + k$, this means that $m = 0$ and $n = k$. Thus, $R(x^k)L(x^m) = 1$ and so $I \cap S = 1$. Thus, $(I \cap S \cap \mathrm{Z}(\mathrm{G}(M))) = 1$. And hence, $\mathrm{G}_0(M) \cong \mathrm{G}(M)/(I \cap S \cap \mathrm{Z}(\mathrm{G}(M))) \cong \mathrm{G}(M) \cong M \times M$. This proves the first part of the theorem.

If 3 does divide n, say $n = 3s$, then it is easy to check that $I \cap S = \{1, R(x^s)L(x^{n-s}), R(x^{2s})L(x^{n-2s})\} = C_3$. And since $\mathrm{Z}(\mathrm{G}(M)) = \mathrm{G}(M)$ we have $I \cap S = (I \cap S \cap \mathrm{Z}(\mathrm{G}(M)))$. And hence, $\mathrm{G}_0(M) \cong \mathrm{G}(M)/(I \cap S \cap \mathrm{Z}(\mathrm{G}(M))) \cong (M \times M)/C_3$. □

And we can describe the smallest group with triality associated with the infinite cyclic group.

Theorem 3.10. *If M is the infinite cyclic group, then $\mathrm{G}_0(M) \cong M \times M$.*

Proof. In $\mathrm{G}(M), R(x^k)L(x^m)$ is in I if and only if $R(x^{k+m})L(x^{k+m}) = 1$. But as we have shown, $\mathrm{G}(M)$ is really just $U(M; \mathbf{M})$, and since $U(M; \mathbf{Gp})$ is a homomorphic image of $U(M; \mathbf{M})$, the proof of [8, Thm. 235] assures us that $|x|$ divides $m + k$. Thus $x = 1$ and hence, $I = 1$. Thus, $\mathrm{G}_0(M) = \mathrm{G}(M)/(I \cap S \cap \mathrm{Z}(\mathrm{G}(M))) \cong \mathrm{G}(M) \cong M \times M$. □

We have thus shown that if M is a finite cyclic group whose order is divisible by 3, then there are precisely two groups with triality giving rise to M, namely $M \times M$ and $(M \times M)/C_3$. If M is any other type of cyclic group (i.e., either infinite or of finite order coprime with 3), then there is precisely one group with triality giving rise to M, namely $M \times M$.

Next, in the corollary to the following theorem, we determine the triality status of the universal multiplication groups of finitely generated abelian groups.

Theorem 3.11. *If $A = \prod_{i \in I} A_i$ and if each $U(A_i; \boldsymbol{V})$ is a group with triality, then so too is $U(A; \boldsymbol{V})$ where $\boldsymbol{V}$ is any variety of Moufang loops containing each A_i.*

Proof. We will use vector notation, $\underline{x}$ to denote elements of A. So, $Q_1(\underline{x_1})Q_2(\underline{x_2}) \dots Q_n(\underline{x_n}) = 1$, implies that $Q_1(\underline{x_1})^\rho Q_2(\underline{x_2})^\rho \dots Q_n(\underline{x_n})^\rho = R(\underline{c})$ for some $\underline{c} \in \mathrm{N}(A)$.

Now, in each $U(A_i;\mathbf{V})$ we have $Q_1(x_1)Q_2(x_2)\dots Q_n(x_n) = 1$. But since each $U(A_i;\mathbf{V})$ is with triality, we have $Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho = 1$. Thus,

$$\begin{aligned} \underline{c} &= \underline{1}R(\underline{c}) \\ &= \underline{1}Q_1(\underline{x_1})^\rho Q_2(\underline{x_2})^\rho \dots Q_n(\underline{x_n})^\rho \\ &= \underline{1Q_1(x_1)^\rho Q_2(x_2)^\rho \dots Q_n(x_n)^\rho} \\ &= \underline{1}. \end{aligned}$$

Thus, A is a group with triality. □

Corollary 3.12. *If A is a finitely generated abelian group, then $U(A; \boldsymbol{V})$ is a group with triality.*

Finally, we offer two theorems about other classes of Moufang loops.

Theorem 3.13. *If M is a commutative Moufang loop of exponent three, if $\boldsymbol{V}$ is any variety of commutative Moufang loops of exponent three containing M, and if $\mathrm{Z}(U(M; \boldsymbol{V})) \cap I = 1$, then $U(M; \boldsymbol{V}) \cong \mathrm{Mlt}M$.*

Proof. $U(M;\mathbf{V})$ is a group with triality with $\rho = 1$. Let $G = U(M;\mathbf{V})$. By [3, Thm. 1] $G/\mathrm{core}_G(I) \cong \mathrm{Mlt}M$. Now, let k be in $\mathrm{core}_G(I)$. Thus, for every g in G, we have $k = g^{-1}g^\sigma k(g^{-1})^\sigma g$. Taking $g = R(x)$, for any x in M, we get $k = P(x^{-1})kP(x)$. But $M[x]$ is a commutative Moufang loop of exponent three, so $R(x) = L(x) = P(x)$. And since x was arbitrary, k is in $\mathrm{Z}(G)$. Thus, k is in $\mathrm{Z}(G) \cap I = 1$, and so $k = 1$. Thus, $\mathrm{core}_G(I) = 1$, and so $U(M;\mathbf{V}) \cong \mathrm{Mlt}M$. □

We note that if M is an infinitely generated free commutative Moufang loop of exponent three, and if $\mathbf{V}$ is any variety of commutative Moufang loops of exponent three containing M, then $\mathrm{Z}(U(M;\mathbf{V})) \cap I = 1$. Hence, we have the following corollary.

Corollary 3.14. *With M and $\boldsymbol{V}$ as in the previous theorem, $U(M; \boldsymbol{V}) \cong \mathrm{Mlt}M$.*

Theorem 3.15. *If M is a Moufang loop that is not commutative of exponent 2, then $M[x]$ is not commutative. (Here, the coproduct $M[x]$ is in the variety of all Moufang loops.)*

Proof. If M is not commutative, then there is nothing to show. So assume that M is commutative. Let $J : M \to M;\ x \mapsto x^{-1}$. Form the semidirect product $M < J >$. Select $y \in M$ such that $y^{-1} \neq y$. Let $1_M : M \to M < J >$; $y \mapsto (y, 1)$. Let $h :< x >\to M < J >$ be determined by sending $x \mapsto (1, J)$. Then there exists a unique $F : M[x] \to M < J >$ such that the coproduct diagram commutes.

Thus, if $yx = xy$ then $(y, J) = (y, 1)(1, J) = F(y)F(x) = F(yx) = F(xy) = F(x)F(y) = (1, J)(y, 1) = (y^{-1}, J)$, and thus, $y = y^{-1}$, a contradiction. Thus, $yx \neq xy$, and $M[x]$ is not commutative. □

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