BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(80), 2016, Pages 70–77
ISSN 1024–7696

On isotopies of some classes of Moufang loops

A. Grishkov, M. Rasskazova, L. Sabinina

Abstract. In this note we discuss questions concerning loop isotopes of automorphic Moufang loops and related questions.

Mathematics subject classification: 20N05.
Keywords and phrases: LF-quasigroups, Moufang loops, automorphic loops, isotopy.

1 Introduction

Isotopies between various kinds of F-quasigroups and Moufang loops are just one aspect of the rich and fruitful field of investigations of Valentin Danilovich Belousov to whose memory this note is dedicated. We will report about some of these results and questions which have been of interest for recent research, particularly on automorphic Moufang loops. Whereas that are algebraic things we would like to mention the importance of this matter to homogenous spaces if one applies these results to smooth quasigroups ([18,21,22] and more). Section 3 and Section 4 contain essential parts of the article. In Section 3 we speak of the existence of an isotopy between quasigroups and loops from different varieties. In Section 4 we treat the problem of isotopical invariance of some varieties of loops.

2 Preliminaries and definitions

In this section we will give definitions and facts from the theory of quasigroups and loops which are necessary to make this note self-contained.

Let us consider a set with a binary operation $(Q, \cdot)$. If it does not lead to misunderstandings we will often write xy instead of $x \cdot y$.

The mapping $L_a : Q \to Q,\ L_a x = ax$ is called a *left multiplication*, analogously the mapping $R_a : Q \to Q,\ R_a y = ya$ is called a *right multiplication* for all elements $a \in Q$. With these notations $(Q, \cdot)$ is called a *quasigroup* if all mappings $L_a,\ R_b, a, b \in Q$ are bijections. Thus one can define $x\backslash y = L_x^{-1} y$ and $y/x = R_x^{-1} y$ for $x, y \in Q$.

If additionally there exists a neutral element $1 \in Q$ such that $L_1 = Id_Q = R_1$, then the quasigroup $(Q, \cdot, 1)$ is called a *loop*.

In these notes we will focus on the following varieties of quasigroups and loops:

Left distributive (LD)-quasigroups are defined by

$$x(yz) = (xy)(xz).$$

Right distributive (RD)-quasigroups are defined by the mirror identity of the LD-identity:

$$(zy)x = (zx)(yx).$$

Distributive quasigroups are left and right distributive at the same time.

V. D. Belousov has studied generalizations of left distributive, right distributive and distributive quasigroups, namely:

Left F (LF)-quasigroups:

$$x(yz) = (xy)(x\backslash x \cdot z).$$

Right F (RF)-quasigroups:

$$(zy)x = (z \cdot x/x)(yx).$$

F-quasigroups are left and right F-quasigroups at the same time.

Diassociative loops are defined in the following way: every pair of elements $x, y \in Q$ generates a subgroup of Q. It is known that no finite number of identities defines the variety of diassociative loops.

Moufang loops are defined by any of the following equivalent identities:

$$((xy)x)z = x(y(xz)),$$

$$((xy)z)y = x(y(zy)),$$

$$(xy)(zx) = (x(yz))x.$$

It is known that Moufang loops are diassociative. It is also known that a quasigroup in which any of the Moufang identities holds is a Moufang loop.

Commutative Moufang loops (CML) can be defined by one identity:

$$x^2(yz) = (xy)(xz).$$

We will use the concept of an *isotopy* for quasigroups and loops.

Let $(Q, \cdot)$ be a quasigroup. Let $T = (\alpha, \beta, \gamma)$ be a triple of permutations of the set Q. Then the quasigroup $(Q, \circ_T)$ is called isotopic to $(Q, \cdot)$ where

$$x \circ_T y = \gamma^{-1}(\alpha(x) \cdot \beta(y)).$$

Obviously isotopy is an equivalence relation. A *principal isotope* of $(Q, \cdot)$ is given by the triple $T = (R_f^{-1}, L_g^{-1}, Id)$ for $f, g \in Q$.
It is easy to see that every principal isotope of a quasigroup is a loop with the neutral element gf.

The mappings L_a and R_a for all $a \in Q$ generate Mlt(Q), the *multiplication group* of Q. If a quasigroup Q is isotopic to a loop L, then the multiplication groups Mlt(Q) and Mlt(L) are isomorphic.

For a quasigroup $(Q,\cdot)$ we will speak of the group $\mathrm{Inn}(Q)$, the *inner mapping group* of $(Q,\cdot)$, generated by the following three types of generators:

$$\ell_{x,y} = L_{xy}^{-1} \circ L_x \circ L_y, \qquad r_{x,y} = R_{xy}^{-1} \circ R_y \circ R_x,$$

$$T_x = L_x^{-1} \circ R_x$$

for all $x, y \in Q$. If $(Q,\cdot,1)$ is a loop, the group $\mathrm{Inn}(Q)$ is just the stabilizer of the neutral element 1, that is

$$\mathrm{Inn}(Q) = \{\phi \in \mathrm{Mlt(Q)} \mid \phi(1) = 1\}.$$

We will call a quasigroup Q *left automorphic* or an A_l-quasigroup if the mappings $\ell_{x,y}$ are automorphisms of Q for all $x, y \in Q$, analogously *right automorphic* or an A_r-quasigroup if the mappings $r_{x,y}$ are automorphisms of Q for all $x, y \in Q$ and T-*automorphic* or an A_T-quasigroup if the mappings T_x are automorphisms of Q for all $x \in Q$.

A quasigroup Q is called *automorphic* or an A-quasigroup if $\mathrm{Inn}(Q)$ is a subgroup of the group of all automorphisms of Q.

A subquasigroup of a given quasigroup Q is *normal* if and only if it is invariant under the group $\mathrm{Inn}(Q)$. In the case of loops a *normal* subloop is just the kernel of a loop homomorphism.

In any automorphic quasigroup all characteristic subquasigroups are normal. In a quasigroup Q we define the *commutator subquasigroup* $[Q,Q]$ as the normal closure of the subquasigroup generated by all elements of the form $(yx)\backslash(xy) = [x,y]$, where $x, y \in Q$ and the *associator subquasigroup* (Q,Q,Q) as the normal closure of the subquasigroup generated by all elements of the form $(x(yz))\backslash((xy)z) = (x,y,z)$ for all $x, y, z \in Q$.

For all $a, b \in Q$ the subgroups

$$N_l(Q) = \{x \in Q \mid (x,a,b) = 1\}, \quad N_r(Q) = \{y \in Q \mid (a,b,y) = 1\},$$

$$N_m(Q) = \{z \in Q \mid (a,z,b) = 1\}$$

are called the *left nucleus*, the *right nucleus* and the *middle nucleus* respectively and $N(Q) = N_l(Q) \cap N_r(Q) \cap N_m(Q)$ is called just the *nucleus* of Q.
The subgroup $C(Q) = \{z \in N(Q) \mid xz = zx \text{ for all } x \in Q\}$ is called the *center* of the quasigroup Q. In a Moufang loop M all nuclei coincide and the nucleus and the center are normal subgroups. Moreover the set $K(M) = \{x \in M : xy = yx \text{ for all } y \in M\}$ is a normal subloop which is called the *commutant* of M (see [11]).

3 Some important isotopies

For the variety of groups the following fact is true: if a group is isotopic to a loop, then it is isomorphic to this isotope. It is known that any isotope of a Moufang loop is a Moufang loop. In other words, the variety of Moufang loops is isotopically invariant.

V. D. Belousov [1,2] proved that any distributive quasigroup Q is isotopic to some commutative Moufang loop. He also gave a sufficient condition when a commutative Moufang loop M is an isotope of some distributive quasigroup. This is the case if the mapping $x \to x^2$ is a bijection in M. One can show a bit more:

Proposition 1. *Any commutative Moufang loop M is an isotope of some F-quasigroup.*

Proof. Let us consider a loop M such that the identity

$$x^2(yz) = (xy)(xz)$$

holds. The isotope defined by $x \circ y = x^{-1}y$ is an F-quasigroup. (See [20].) □

Belousov was interested in isotopes of LF- and F-quasigroups. In 2007 Kepka, Kinyon, Phillips [6], solving a Belousov problem, showed that any F-quasigroup is isotopic to some automorphic Moufang loop M, such that M is the product of the nucleus $N(M)$ and the commutant $K(M)$.

For LF-quasigroups Belousov has proved that loop isotopes of LF-quasigroups belong to the class of so-called left M-loops, and showed that a loop isotope of any left M-loop is a left M-loop again. Note that an identity which defines an M-loop contains a mapping $\phi : M \to M$ (see [3]).

Developing further the ideas of Belousov, V. Shcherbacov studied the isotopy of an LF-quasigroup. He used Belousov's result that for an LF-quasigroup Q the mapping $e : Q \to Q$ defined by $e(x) = x\backslash x$ is an endomorphism (see [3]). The main statement of his paper [23] is the following. For an LF- quasigroup Q there exists an isotopic loop which is a product of a normal subgroup N and a subloop S such that S is isotopic to an LD-quasigroup. He claimed that this product is a direct product.

At the same time in [18] in a geometric context a different isotope was constructed, using another result of Belousov. He had observed that every LF-quasigroup is isotopic to an LF-quasigroup with a left neutral element ([3] and see also [21]). This fact allows the use of kernels in the homomorphism theory. It was shown that an LF-quasigroup Q is isotopic to a semidirect product of a normal factor N_1 which is a nuclear extension of a group by a group and a loop S_1 which is an isotope of an LD-quasigroup. Inspired by Shcherbacov's paper, in [19] the authors showed that in the above construction N_1 is just a group. The authors realized that N_1 is the left nucleus of the LF-quasigroup Q. Thus the endomorphism e restricted to $e(Q)$ is an automorphism.

However, they could not verify that S_1 is normal. From an example of Gagola [9] one learns that in the finite case it is not always possible to get an isotope which is a direct product. It should be mentioned that in the geometric situation one treats smooth quasigroups. Isotopy of smooth quasigroups is important since isotopic smooth quasigroups define equivalent geometries. For this case the question of the existence of a direct decomposition is still open.

4 Automorphic Moufang loops, a theory in progress

The concept of automorphic loops goes back to a paper of R. Bruck and L. Paige [5]. In this paper they observed that diassociative automorphic loops have similar properties as Moufang loops, but the conjecture that these two varieties of loops are identical was proved only many years later in [7]. In their paper Kinyon, Kunen and Phillips wrote: "We also do not know whether every loop isotope of a Moufang A-loop is a (Moufang) A-loop." We will return to this question later. The variety of left automorphic Moufang loops is defined by Moufang identity and by the additional identity

$$([x, y], z, t) = 1.$$

Note that the variety of left automorphic Moufang loops coincides with the variety of right automorphic Moufang loops. The variety of T-automorphic Moufang loops is defined by the Moufang identity and by the additional identity (see [4])

$$(x^3, y, z) = 1.$$

The papers [7] and [6] gave a strong impulse to study the variety of automorphic loops and in particular the variety of automorphic Moufang loops. For instance in the paper [17] it was shown that in the variety of automorphic Moufang loops the Restricted Burnside Problem has a positive answer. Some particular properties of finite automorphic Moufang loops like the consequences of Sylow's theorems for Moufang loops were treated in [12].

Recently the structure of automorphic Moufang loops was described in the papers [13] and [14]. In [13] the structure of a free automorphic Moufang loop L_n of rank n was described. Take a free group F_n of rank n with a base $x_1, \dots, x_n$ and a free commutative Moufang loop C_n of the same rank with a base $y_1, \dots, y_n$. Then the elements $(x_1, y_1), \dots, (x_n, y_n)$ of the direct product $F_n \times C_n$ form a base of L_n. Using this approach one can see that for every Moufang loop M the factor loop $A = M/([M, M] \cap (M, M, M))$ is an automorphic Moufang loop. It is clear that for the loop A one has $[A, A] \cap (A, A, A) = 1$. But of course not every automorphic Moufang loop B has the property that $([B, B] \cap (B, B, B)) = 1$. In [14] we have shown that for every automorphic Moufang loop B there is a minimal automorphic Moufang loop B_1 and an epimorphism $\eta : B_1 \to B$ such that $([B_1, B_1] \cap (B_1, B_1, B_1)) = 1$.

In [7] results of Hala Pflugfelder from [15] are mentioned. There M_k-loops are introduced and the following results are proved. The variety $\mathcal{M}_k$ is defined by the generalized Moufang M_k-idenity:

$$x(yz)x^k = (xy)(zx^k)$$

for every $k \in \mathbb{N}$.

H. Pflugfelder proved that M_k-loops are isotopically invariant for every $k \in \mathbb{N}$ and that M_k-loops are Moufang loops. It was also proved that the variety $\mathcal{M}_k$ can be defined by the Moufang identity and by $(x^{k-1}, y, z) = 1$. It is easy to see that for Moufang loops the M_4-identity is equivalent to the property A_T. Thus we see that Moufang A_T-loops are isotopically invariant.

In fact using the same reasoning one can describe the isotopy of the class of code loops. Code loops are left (and right) automorphic Moufang loops, but not A_T-loops. For code loops the identity $(x^2, y, z) = 1$ holds. This means that code loops belong to the isotopically invariant variety $\mathcal{M}_3$. By [16, Theorem IV.4.11, page 105], a loop isotopic to a code loop C is isomorphic to C.

The variety of commutative Moufang loops is not isotopically invariant (see [16]). However, every isotope of a commutative Moufang loop satisfies the following identities:

$$(x^3, y, z) = 1, \quad [x^3, y] = 1,$$

$$[[x, y], y] = 1$$

(see [4, Lemma 5.1, page 122 and Lemma 5.7, page 128]).

Returning to the initial question of this section whether every isotope of any automomorphic Moufang loop is automorphic we are convinced that the answer is negative in general. We have in mind an isotope of a free commutative Moufang loop of exponent 3 and of rank 5. Such an isotope was used by G. Nagy in [8].

Gagola's result gives us sufficient conditions when every isotope of an automomorphic Moufang loop is automorphic. In [10] the following result was obtained. Let M be a Moufang loop, and let H be a subloops generated by all cubes of elements of M. If one of conditions

(a) $H = M$,

(b) $[M : H] = 3$

holds, then $Inn(M) = \langle T_x \rangle$.

Suppose now additionally that M is an automorphic Moufang loop. Then note that the case (a) is trivial since M is a group. In the case (b) one sees that $M/N(M) \cong \mathbb{Z}_3$ or $M = N(M)$, where $N(M)$ is the nucleus of M. Thus in both cases any isotope of M is automorphic.

Hence we have the

Conjecture. *Any isotope of an automorphic Moufang loop is automorphic if and only if the identity*

$$((x, y, z), u, v) = 1$$

holds.

Acknowledgements. The first author thanks FAPESP, CNPQ and Russian grant RFFI 13-01-00239. The second author thanks the Russian Foundation for Basic Research (grant no. 2014/319, project no. 258). The third author thanks FAPESP grant processo 2015/07245-4 for support.

References

[1] Belousov V. D. *The structure of distributive quasigroups.* Mat. Sb., 1960, **50(92)**, 267–298.

[2] Belousov V. D. *Foundations of the theory of quasigroups and loops.* Moscow, Nauka, 1967 (in Russian).

[3] Belousov V. D. *Elements of Quasigroups Theory: A Special Course.* Kishinev State University Press, Kishinev, 1981 (in Russian).

[4] Bruck R. H. *A Survey of Binary Systems.* Springer-Verlag, 1966.

[5] Bruck R. H., Paige L. J. *Loops whose inner mappings are automorphisms.* Ann. of Math., 1956, (2) **63**, 308–323.

[6] Kepka T., Kinyon M., Phillips J. D. *The structure of F-quasigroups.* J. Algebra, 2007, **317**, No. 2, 435–461.

[7] Kinyon M., Kunen K., Phillips J. D. *Every diassociative A-loop is Moufang.* Proc. Amer. Math. Soc., 2002, **130**, No. 3, 619–624.

[8] Nagy G. P. *Burnside problems for Moufang and Bol loops of small exponent.* Acta Sci. Math. (Szeged), 2001, **67**, 687–696.

[9] Gagola S. M. *Nonsplitting F-quasigroups.* Comment. Math. Univ. Carolin., 2012, **53**, No. 3, 375–381.

[10] Gagola S. M. *When are inner mapping groups generated by conjugation maps?* Arch. Math. (Basel), 2013, **101**, No. 3, 207–212.

[11] Gagola S. M. *A Moufang's loop commutant.* Proc. Cambridge Phil. Soc., 2012, **152**, No. 2, 193–206.

[12] Grishkov A., Merlini Giuliani M. L., Rasskazova M., Sabinina L. *Half-isomorphisms of finite automorphic Moufang loops.* Comm. Algebra (in print).

[13] Grishkov A., Plaumann P., Sabinina L. *Structure of free automorphic Moufang loops.* Proc. Amer. Math. Soc., 2012, **140**, No. 7, 2209–2214.

[14] Grishkov A., Rasskazova M., Sabinina L. *On the structure of automorphic Moufang loops.* Publ. Math. Debrecen (in print).

[15] Orlik-Pflugfelder H. , *A special class of Moufang loops.* Proc. Amer. Math. Soc., 1970, **26**, 583-586.

[16] Pflugfelder H. *Quasigroups and Loops: Introduction.* Sigma Series in Pure Mathematics, **7**, Heldermann, Berlin, 1990.

[17] Plaumann P., Sabinina L. *On nuclearly nilpotent loops of finite exponent.* Comm. Algebra, 2008, **36**, No. 4, 1346–1353.

[18] Plaumann P., Sabinina L., Sbitneva L. *A decomposition of LF-quasigroups. Algebras, representations and applications*, Contemp. Math., **483**, Amer. Math. Soc., Providence, RI, 2009, 221–227.

[19] Plaumann P., Sabinina L., Sbitneva L. *Some Varieties of Quasigroups, Loops and their Parastrophes.* Communications in Mathematics and Applications, 2012, **3**, No. 1, 75–85.

[20] Sabinina L. *On smooth left square distributive quasigroup. Lecture Notes in Pure and Appl. Math.*, **211**, Dekker, New York, 2000, 345–348.

[21] Sabinin L. V., Sabinina L. *On the theory of left F-quasigroups.* Algebras Groups Geom., 1995, **12**, No. 2, 127–137.

[22] Sabinina L. *Trans-symmetric spaces.* Adv. Appl. Clifford Algebras, 1994, **4**, S1, 509–514.

[23] Shcherbacov V. A. *On the structure of left and right F-, SM-, and E-quasigroups,* J. Gen. Lie Theory Appl., 2009, **3 (3)**, 197–259.

Alexander Grishkov
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua de Matão 1010
São Paulo, 05508-090
Brazil
E-mail: *grishkov@ime.usp.br*

Received November 30, 2015

Marina Rasskazova
Omsk State Institute of Service
Pevtsova street 13
Omsk, 644099
Russia
E-mail: *marinarasskazova@yandex.ru*

Liudmila Sabinina
Universidad Autónoma del Estado de Morelos
Av. Universidad 1001
Cuernavaca, 62209
México
E-mail: *liudmila@uaem.mx*