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Central and medial quasigroups of small order

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Abstract. We enumerate central and medial quasigroups of order less than 128 up to isomorphism, with the exception of those quasigroups that are isotopic to $C_4 \times C_2^4$, C_2^6, C_3^4 or C_5^3. We give an explicit formula for the number of quasigroups that are affine over a finite cyclic group.

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This paper was written on the occasion of the 90th anniversary of Valentin Danilovich Belousov's birthday. Prof. Belousov pioneered enumerative results for quasigroups in his book "Fundametals of the theory of quasigroups and loops" and his work has been a frequent source of inspiration for the Prague algebraic school.

1 Introduction

Given an abelian group $(G,+)$, automorphisms φ, ψ of $(G,+)$, and an element $c \in G$, define a new operation $*$ on G by

$$x * y = \varphi(x) + \psi(y) + c.$$

The resulting quasigroup $(G,*)$ is said to be *affine over* $(G,+)$, and it will be denoted by $\mathcal{Q}(G,+,\varphi,\psi,c)$. Quasigroups that are affine over an abelian group are called *central quasigroups* or *T-quasigroups*. We will use the terms "quasigroup affine over an abelian group" and "central quasigroup" interchangeably. Central quasigroups are precisely the abelian quasigroups in the sense of universal algebra [15].

A quasigroup $(Q,\cdot)$ is called *medial* if it satisfies the medial law

$$(x \cdot y) \cdot (u \cdot v) = (x \cdot u) \cdot (y \cdot v).$$

Medial quasigroups are also known as *entropic* quasigroups. The fundamental Toyoda-Bruck theorem [13, Theorem 3.1] states that, up to isomorphism, medial quasigroups are precisely central quasigroups $\mathcal{Q}(G,+,\varphi,\psi,c)$ with commuting automorphisms φ, ψ.

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The classification of central (or medial) quasigroups up to isotopy is trivial in the sense that it coincides with the classification of abelian groups up to isomorphism. Indeed:

- If $(G,*) = \mathcal{Q}(G,+,\varphi,\psi,c)$ is a central quasigroup then $(G,*)$ is isotopic to $(G,+)$ via the isotopism $(x \mapsto \varphi(x),\, x \mapsto \psi(x)+c,\, x \mapsto x)$.
- If two central quasigroups $Q_i = \mathcal{Q}(G_i,+_i,\varphi_i,\psi_i,c_i)$ are isotopic then the underlying groups $(G_i,+_i)$ are isotopic. But isotopic groups are necessarily isomorphic, cf. [10, Proposition 1.4].

Classifying and enumerating central and medial quasigroups up to isomorphism is nontrivial, however, and that is the topic of the present paper.

There are not many results in the literature concerning enumeration and classification of central and medial quasigroups.

Simple idempotent medial quasigroups were classified by Smith in [9, Theorem 6.1]. Sokhatsky and Syvakivskij [12] classified n-ary quasigroups affine over cyclic groups and obtained a formula for the number of those of prime order. Kirnasovsky [5] carried out a computer enumeration of central quasigroups up to order 15, and obtained more classification results in his PhD thesis [6]. Idempotent medial quasigroups of order p^k, $k \leq 4$, were classified by Hou [4, Table 1].

At the time of writing this paper, the On-line Encyclopedia of Integer Sequences [8] gives the number of medial quasigroups of order ≤ 8 up to isomorphism as the sequence A226193, and there appears to be no entry for the number of central quasigroups up to isomorphism.

Drápal [1] and Sokhatsky [11] obtained a general isomorphism theorem for quasigroups isotopic to groups, cf. [1, Theorem 2.10] and [11, Corollary 28], and for central quasigroups in particular, cf. [1, Theorem 3.2], or its restatement, Theorem 2.5. Drápal applied the machinery to calculate isomorphism classes of quasigroups of order 4 (by hand), and Kirnasovsky used Sokhatsky's theory for the calculations mentioned above. In the present paper, we use a similar approach to obtain stronger enumeration results, taking advantages of the computer system `GAP` [2].

We refer the reader to [10] for general theory of quasigroups, to [1] for a more extensive list of references on central quasigroups, to [11] for results on quasigroups isotopic to groups, to [13] for results on quasigroups affine over various kinds of loops, and to [14, 15] for a broader context on affine representation of general algebraic structures. The article [3] gives a gentle introduction into automorphism groups of finite abelian groups and points to original sources on that topic.

The paper is organized as follows.

In Section 2, we formulate an isomorphism theorem for central quasigroups, Theorem 2.4, which is less general than [1, Theorem 2.10] or [11, Corollary 28], and equivalent to but less technical than [1, Theorem 3.2]. We also present the enumeration algorithm in detail.

In Section 3, we establish our own version of [12, Theorem 2] and [1, Theorem 3.5] for cyclic p-groups, Theorem 3.1, providing an explicit formula for the number of isomorphism classes. We were informed that the same result was obtained by Kirnasovsky in his unpublished PhD thesis [6]. Since the automorphism groups of cyclic groups are commutative, Theorem 3.1 also yields the number of medial quasigroups up to isomorphism over finite cyclic groups, and of prime order in particular.

Finally, the results of the enumeration are presented in the Appendix.

2 Isomorphism theorem and enumeration algorithm

2.1 Elementary properties of the counting functions cq and mq

For an abelian group G, let $cq(G)$ (resp. $mq(G)$) denote the number of all central (resp. medial) quasigroups over G up to isomorphism. For $n \geq 1$, let $cq(n)$ (resp. $mq(n)$) denote the number of all central (resp. medial) quasigroups of order n up to isomorphism.

Let us establish two fundamental properties of the counting functions.

First, by the remarks in the introduction,

$$cq(n) = \sum_{|G|=n} cq(G) \quad \text{and} \quad mq(n) = \sum_{|G|=n} mq(G),$$

where the summations run over all abelian groups of order n up to isomorphism.

Second, Proposition 2.1 shows that the classification of central and medial quasigroups can be reduced to prime power orders. As far as enumeration is concerned, Proposition 2.1 implies that the functions cq, $mq : \mathbb{N}^+ \to \mathbb{N}^+$ are multiplicative in the number-theoretic sense.

Proposition 2.1. *Let $G = H \times K$ be an abelian group such that* $\gcd(|H|, |K|) = 1$*. Up to isomorphism, any quasigroup affine over G can be expressed in a unique way as a direct product of a quasigroup affine over H and a quasigroup affine over K. In particular,*

$$cq(G) = cq(H) \cdot cq(K) \quad \textit{and} \quad mq(G) = mq(H) \cdot mq(K).$$

Proof. Any automorphism of G decomposes uniquely as a direct product of an automorphism of H and an automorphism of K, cf. [3, Lemma 2.1]. The rest is easy. □

2.2 The isomorphism problem for central quasigroups

Let us now consider the isomorphism problem for quasigroups affine over a fixed abelian group $(G, +)$.

Consider any group A. (Later we will take $A = \mathrm{Aut}(G, +)$.) Then A acts on itself by conjugation, and A also acts on $A \times A$ by a simultaneous conjugation in both coordinates, i.e., $(\alpha, \beta)^\gamma = (\alpha^\gamma, \beta^\gamma)$.

Lemma 2.2. *Let A be a group. Let X be a complete set of orbit representatives of the conjugation action of A on itself. For $\xi \in X$, let Y_ξ be a complete set of orbit representatives of the conjugation action of the centralizer $C_A(\xi)$ on A. Then*

$$\{(\xi, \upsilon) : \xi \in X,\ \upsilon \in Y_\xi\}$$

is a complete set of orbit representatives of the conjugation action of A on $A \times A$.

Proof. For every $(\alpha, \beta) \in A \times A$ there is a unique $\xi \in X$ and some $\gamma \in A$ such that (α, β) and (ξ, γ) are in the same orbit. For a fixed $\xi \in X$ and some β, $\gamma \in A$, we have (ξ, β) in the same orbit as (ξ, γ) if and only if there is $\delta \in C_A(\xi)$ such that $\beta^\delta = \gamma$. ☐

Lemma 2.3. *Let $(G, +)$ be an abelian group, $A = \mathrm{Aut}(G, +)$ and α, $\beta \in A$. Then $C_A(\alpha) \cap C_A(\beta)$ acts naturally on $G/\mathrm{Im}(1 - \alpha - \beta)$.*

Proof. Let $U = \mathrm{Im}(1 - \alpha - \beta)$. It suffices to show that for every $\gamma \in C_A(\alpha) \cap C_A(\beta)$ the mapping $u + U \mapsto \gamma(u) + U$ is well-defined. Now, if $u + U = v + U$ then $u = v + w - \alpha(w) - \beta(w)$ for some $w \in G$ and we have $\gamma(u) = \gamma(v) + \gamma(w) - \gamma\alpha(w) - \gamma\beta(w) = \gamma(v) + \gamma(w) - \alpha\gamma(w) - \beta\gamma(w) = \gamma(v) + (1 - \alpha - \beta)(\gamma(w)) \in \gamma(v) + U$. ☐

We will now state a theorem that solves the isomorphism problem for central and medial quasigroups over $(G, +)$. Instead of showing how it follows from the more general [1, Theorem 2.10], we show that it is equivalent to [1, Theorem 3.2], which we restate as Theorem 2.5 here.

Theorem 2.4 (Isomorphism problem for central quasigroups)**.** *Let $(G, +)$ be an abelian group, let φ_1, ψ_1, φ_2, $\psi_2 \in \mathrm{Aut}(G, +)$, and let c_1, $c_2 \in G$. Then the following statements are equivalent:*

(i) the central quasigroups $\mathcal{Q}(G, +, \varphi_1, \psi_1, c_1)$ and $\mathcal{Q}(G, +, \varphi_2, \psi_2, c_2)$ are isomorphic;

(ii) there is an automorphism γ of $(G, +)$ and an element $u \in \mathrm{Im}(1 - \varphi_1 - \psi_1)$ such that

$$\varphi_2 = \gamma\varphi_1\gamma^{-1}, \quad \psi_2 = \gamma\psi_1\gamma^{-1}, \quad c_2 = \gamma(c_1 + u).$$

Theorem 2.5 ([1, Theorem 3.2])**.** *Let $(G, +)$ be an abelian group and denote $A = \mathrm{Aut}(G, +)$. The isomorphism classes of central quasigroups (resp. medial quasigroups) over $(G, +)$ are in one-to-one correspondence with the elements of the set*

$$\{(\varphi, \psi, c) : \varphi \in X,\ \psi \in Y_\varphi,\ c \in G_{\varphi,\psi}\},$$

where

- *X is a complete set of orbit representatives of the conjugation action of A on itself;*
- *Y_φ is a complete set of orbit representatives of the conjugation action of $C_A(\varphi)$ on A (resp. on $C_A(\varphi)$), for every $\varphi \in X$;*
- *$G_{\varphi,\psi}$ is a complete set of orbit representatives of the natural action of $C_A(\varphi) \cap C_A(\psi)$ on $G/\mathrm{Im}(1-\varphi-\psi)$.*

Here is a proof of the equivalence of Theorems 2.4 and 2.5: By Lemma 2.2, we can assume that we are investigating the equivalence of two triples (φ, ψ, c_1) and (φ, ψ, c_2) for some $\varphi \in X$, $\psi \in Y_\varphi$ and c_1, $c_2 \in G$. Let $U = \mathrm{Im}(1-\varphi-\psi)$. The following conditions are then equivalent for any $\gamma \in \mathrm{Aut}(G,+)$, using Lemma 2.3: $c_2 = \gamma(c_1 + u)$ for some $u \in U$, $c_2 \in \gamma(c_1 + U) = \gamma(c_1) + U$, $c_2 + U = \gamma(c_1) + U = \gamma(c_1 + U)$. This finishes the proof.

2.3 The algorithm

Theorem 2.5 together with the results of Subsection 2.1 gives rise to the following algorithm that enumerates central and medial quasigroups of order n. In the algorithm we denote by `R(H,X)` a complete set of representatives of the (clear from context) action of `H` on `X`.

Algorithm 2.6.
Input: positive integer n
Output: $cq(n)$ and $mq(n)$

```
cqn := 0; mqn := 0;
for G in the set of abelian groups of order n up to isomorphism do
    cqG := 0; mqG := 0;
    A := automorphism group of G;
    for f in R(A,A) do
        for g in R(C_A(f),A) do
            for c in R( Intersection(C_A(f),C_A(g)), G/Im(1-f-g) ) do
                cqG := cqG + 1;
                if f*g=g*f then mqG := mqG + 1; fi;
            od;
        od;
    od;
    cqn := cqn + cqG; mqn := mqn + mqG;
od;
return cqn, mqn;
```

The algorithm was implemented in the `GAP` system [2] in a straightforward fashion, taking advantage of some functionality of the `LOOPS` [7] package. The code is available from the second author at `www.math.du.edu/~petr`.

In small situations it is possible to directly calculate the orbits of the conjugation action of $A = \mathrm{Aut}(G, +)$ on $A \times A$. For larger groups, it is safer (due to memory constraints) to work with one conjugacy class of A at a time, as in Algorithm 2.5.

Among the cases we managed to calculate, the elementary abelian group C_2^5 took the most effort, about 4 hours on a standard personal computer. It might not be difficult to calculate some of the missing entries for $mq(G)$. However, $cq(C_2^6)$, for instance, appears out of reach without further theoretical advances or more substantial computational resources.

The outcome of the calculation can be found in the Appendix.

3 Quasigroups affine over cyclic groups

Let G be a cyclic group. Since $\mathrm{Aut}(G)$ is commutative, every quasigroup affine over G is medial.

Theorem 3.1 ([6, p. 70]). *Let p be a prime and k a positive integer. Then*

$$cq(C_{p^k}) = mq(C_{p^k}) = p^{2k} + p^{2k-2} - p^{k-1} - \sum_{i=k-1}^{2k-1} p^i.$$

Proof. Let $G = C_{p^k}$ and $A = \mathrm{Aut}(G)$. We will identify A with the $p^k - p^{k-1}$ elements of $G^* = \{a \in G : p \nmid a\}$. We will follow Algorithm 2.6. Since A is commutative, the conjugation action is trivial and we have to consider every $(\varphi, \psi) \in A \times A$. For a fixed $(\varphi, \psi) \in A \times A$, we must consider a complete set of orbit representatives $G_{\varphi,\psi}$ of the action of $A = C_A(\varphi) \cap C_A(\psi)$ on $G/\mathrm{Im}(1 - \varphi - \psi)$. Now, $\mathrm{Im}(1 - \varphi - \psi)$ is equal to $p^i G$ if and only if $p^i \mid 1 - \varphi - \psi$ and $p^{i+1} \nmid 1 - \varphi - \psi$.

Case $i = 0$, i.e.,

$$\varphi + \psi \not\equiv 1 \pmod{p}.$$

In this case, we can take $G_{\varphi,\psi} = \{0\}$. How many such pairs (φ, ψ) exist? First, let us count those with $\varphi \equiv 1 \pmod{p}$. Then $\psi \in G^*$ can be chosen arbitrarily, hence we have $p^{k-1}(p^k - p^{k-1})$ such pairs. Next, let us count those with $\varphi \not\equiv 1 \pmod{p}$. Then $\psi \in G^*$ must satisfy $\psi \not\equiv 1 - \varphi \pmod{p}$, hence we have $(p^k - 2p^{k-1})(p^k - 2p^{k-1})$ such pairs. Since $|G_{\varphi,\psi}| = 1$, this case contributes to $cq(G)$ by

$$p^{k-1}(p^k - p^{k-1}) + (p^k - 2p^{k-1})^2.$$

Cases $i = 1, \ldots, k-1$, i.e.,

$$\varphi + \psi \equiv 1 \pmod{p^i} \quad \text{and} \quad \varphi + \psi \not\equiv 1 \pmod{p^{i+1}}.$$

In this case, we can take $G_{\varphi,\psi} = \{0, p^0, \ldots, p^{i-1}\}$. How many such pairs (φ, ψ) exist? For $\varphi \equiv 1 \pmod{p}$, any solution ψ to the congruence above is divisible by p, hence there is no such solution $\psi \in G^*$. For $\varphi \not\equiv 1 \pmod{p}$, we have precisely $p^{k-i} - p^{k-i-1}$

solutions to the conditions in G^*. Since $|G_{\varphi,\psi}| = i+1$, this case contributes to $cq(G)$ by

$$(p^k - 2p^{k-1})(p^{k-i} - p^{k-i-1})(i+1).$$

Case $i = k$, i. e.,

$$\varphi + \psi = 1.$$

In this case, we can take $G_{\varphi,\psi} = \{0, p^0, \dots, p^{k-1}\}$. How many such pairs (φ, ψ) exist? Since ψ is uniquely determined by φ and neither of φ, ψ shall be divisible by p, we have precisely $p^k - 2p^{k-1}$ such pairs. Since $|G_{\varphi,\psi}| = k+1$, this case contributes to $cq(G)$ by

$$(p^k - 2p^{k-1})(k+1).$$

Summarized, the cases $i = 1, \dots, k$ contribute to $cq(G)$ the total of

$$(p^k - 2p^{k-1})\left(\left(\sum_{i=1}^{k-1}(p^{k-i} - p^{k-i-1}) \cdot (i+1)\right) + (k+1)\right),$$

which, after rearrangement, gives

$$(p^k - 2p^{k-1})(2p^{k-1} + p^{k-2} + p^{k-3} + \cdots + p + 1).$$

The total sum is then

$$\begin{aligned} cq(G) &= p^{2k-1} - p^{2k-2} + (p^k - 2p^{k-1})((p^k - 2p^{k-1}) \\ &+ (2p^{k-1} + p^{k-2} + p^{k-3} + \cdots + p + 1)) \\ &= p^{2k-1} - p^{2k-2} + (p^k - 2p^{k-1})(p^k + p^{k-2} + p^{k-3} + \cdots + p + 1) \\ &= p^{2k} - p^{2k-1} - p^{2k-3} - \cdots - p^k - 2p^{k-1}, \end{aligned}$$

which can be expressed as in the statement of the theorem. □

Corollary 3.2. *For any* $k \geq 1$ *we have* $cq(C_{2^k}) = mq(C_{2^k}) = 2^{2k-2}$.

Corollary 3.3. *For any prime* p *we have* $cq(p) = mq(p) = p^2 - p - 1$.

Corollary 3.3 is a special case of [12, Corollary 2] for binary quasigroups.

As a counterpart to Theorem 3.1, we ask:

Problem 3.4. For a prime p and $k > 1$, find explicit formulas for $cq(C_p^k)$ and $mq(C_p^k)$.

Appendix: Central and medial quasigroups of order less than 128

The following table contains the results of our enumeration of central and medial quasigroups of order less than 128.

If a row in the table starts with n/k then: column "G" gives the catalog number n/k corresponding to the abelian group `SmallGroup(n,k)` of `GAP`; column "structure" gives a structural description of the group G from which a decomposition of G into p-primary components is readily seen and hence Proposition 2.1 can be routinely applied; column "$|A|$" gives the cardinality of the group $A = \mathrm{Aut}(G)$; column "$|X|$" gives the number of conjugacy classes of A; column "$|O|$" gives the number of orbits of the conjugation action of A on $A\times A$ (with action $(f,g)^h = (f^h, g^h)$), which is a lower bound on the number of quasigroups affine over G; column "cq" gives the number of quasigroups affine over G up to isomorphism; column "$|O_c|$" gives the number of orbits in O with a representative (f,g) such that $fg = gf$, which is a lower bound on the number of medial quasigroups over G; column "mq" gives the number of medial quasigroups over G up to isomorphism; and column "ref" gives a reference to a numbered result within this paper if the entries in the row follow from the cited result and possibly also from previously listed table entries.

If a row in the table starts with **n** then: column "G" gives the order n; column "cq" gives the number of central quasigroups of order n up to isomorphism; and column "mq" gives the number of medial quasigroups of order n up to isomorphism.

Entries that we were not able to establish are denoted by "?" or "**?**".

All entries corresponding to prime-power orders were explicitly calculated by Algorithm 2.6 although the cyclic cases follow from Theorem 3.1. Many of the entries corresponding to the remaining orders were also initially obtained by Algorithm 2.6 (to test the algorithm) but in the final version they were calculated directly from earlier entries using Proposition 2.1.

To reduce the number of transcription and arithmetical errors, the entries and the LaTeX source of the table were computer generated.

G	structure	$\|A\|$	$\|X\|$	$\|O\|$	cq	$\|O_c\|$	mq	ref
1/1	C_1	1	1	1	1	1	1	
1					**1**		**1**	
2/1	C_2	1	1	1	1	1	1	3.1
2					**1**		**1**	
3/1	C_3	2	2	4	5	4	5	3.1
3					**5**		**5**	
4/1	C_4	2	2	4	4	4	4	3.1
4/2	C_2^2	6	3	11	15	8	9	
4					**19**		**13**	
5/1	C_5	4	4	16	19	16	19	3.1
5					**19**		**19**	
6/2	$C_2\times C_3$	2	2	4	5	4	5	2.1
6					**5**		**5**	
7/1	C_7	6	6	36	41	36	41	3.1
7					**41**		**41**	

G	structure	$\|A\|$	$\|X\|$	$\|O\|$	cq	$\|O_c\|$	mq	ref
8/1	C_8	4	4	16	16	16	16	3.1
8/2	$C_4 \times C_2$	8	5	28	28	22	22	
8/5	C_2^3	168	6	197	341	32	35	
8					**385**		**73**	
9/1	C_9	6	6	36	48	36	48	3.1
9/2	C_3^2	48	8	136	183	56	68	
9					**231**		**116**	
10/2	$C_2 \times C_5$	4	4	16	19	16	19	2.1
10					**19**		**19**	
11/1	C_{11}	10	10	100	109	100	109	3.1
11					**109**		**109**	
12/2	$C_4 \times C_3$	4	4	16	20	16	20	2.1
12/5	$C_2^2 \times C_3$	12	6	44	75	32	45	2.1
12					**95**		**65**	
13/1	C_{13}	12	12	144	155	144	155	3.1
13					**155**		**155**	
14/2	$C_2 \times C_7$	6	6	36	41	36	41	2.1
14					**41**		**41**	
15/1	$C_3 \times C_5$	8	8	64	95	64	95	2.1
15					**95**		**95**	
16/1	C_{16}	8	8	64	64	64	64	3.1
16/2	C_4^2	96	14	400	624	168	188	
16/5	$C_8 \times C_2$	16	10	112	112	88	88	
16/10	$C_4 \times C_2^2$	192	13	564	820	146	150	
16/14	C_2^4	20160	14	20747	39767	160	179	
16					**41387**		**669**	
17/1	C_{17}	16	16	256	271	256	271	3.1
17					**271**		**271**	
18/2	$C_2 \times C_9$	6	6	36	48	36	48	2.1
18/5	$C_2 \times C_3^2$	48	8	136	183	56	68	2.1
18					**231**		**116**	
19/1	C_{19}	18	18	324	341	324	341	3.1
19					**341**		**341**	
20/2	$C_4 \times C_5$	8	8	64	76	64	76	2.1
20/5	$C_2^2 \times C_5$	24	12	176	285	128	171	2.1
20					**361**		**247**	
21/2	$C_3 \times C_7$	12	12	144	205	144	205	2.1
21					**205**		**205**	
22/2	$C_2 \times C_{11}$	10	10	100	109	100	109	2.1
22					**109**		**109**	
23/1	C_{23}	22	22	484	505	484	505	3.1
23					**505**		**505**	

G	structure	$\lvert A\rvert$	$\lvert X\rvert$	$\lvert O\rvert$	cq	$\lvert O_c\rvert$	mq	ref
24/2	$C_8 \times C_3$	8	8	64	80	64	80	2.1
24/9	$C_4 \times C_2 \times C_3$	16	10	112	140	88	110	2.1
24/15	$C_2^3 \times C_3$	336	12	788	1705	128	175	2.1
24					**1925**		**365**	
25/1	C_{25}	20	20	400	490	400	490	3.1
25/2	C_5^2	480	24	2336	2847	512	594	
25					**3337**		**1084**	
26/2	$C_2 \times C_{13}$	12	12	144	155	144	155	2.1
26					**155**		**155**	
27/1	C_{27}	18	18	324	441	324	441	3.1
27/2	$C_9 \times C_3$	108	20	864	1356	336	528	
27/5	C_3^3	11232	24	23236	34321	484	605	
27					**36118**		**1574**	
28/2	$C_4 \times C_7$	12	12	144	164	144	164	2.1
28/4	$C_2^2 \times C_7$	36	18	396	615	288	369	2.1
28					**779**		**533**	
29/1	C_{29}	28	28	784	811	784	811	3.1
29					**811**		**811**	
30/4	$C_2 \times C_3 \times C_5$	8	8	64	95	64	95	2.1
30					**95**		**95**	
31/1	C_{31}	30	30	900	929	900	929	3.1
31					**929**		**929**	
32/1	C_{32}	16	16	256	256	256	256	3.1
32/3	$C_8 \times C_4$	128	26	1216	1216	592	592	
32/16	$C_{16} \times C_2$	32	20	448	448	352	352	
32/21	$C_4^2 \times C_2$	1536	30	6224	9808	884	904	
32/36	$C_8 \times C_2^2$	384	26	2256	3280	584	600	
32/45	$C_4 \times C_2^3$	21504	30	48412	87580	804	834	
32/51	C_2^5	9999360	27	10024077	19721077	590	655	
32					**19823665**		**4193**	
33/1	$C_3 \times C_{11}$	20	20	400	545	400	545	2.1
33					**545**		**545**	
34/2	$C_2 \times C_{17}$	16	16	256	271	256	271	2.1
34					**271**		**271**	
35/1	$C_5 \times C_7$	24	24	576	779	576	779	2.1
35					**779**		**779**	
36/2	$C_4 \times C_9$	12	12	144	192	144	192	2.1
36/5	$C_2^2 \times C_9$	36	18	396	720	288	432	2.1
36/8	$C_4 \times C_3^2$	96	16	544	732	224	272	2.1
36/14	$C_2^2 \times C_3^2$	288	24	1496	2745	448	612	2.1
36					**4389**		**1508**	
37/1	C_{37}	36	36	1296	1331	1296	1331	3.1
37					**1331**		**1331**	

G	structure	$\lvert A\rvert$	$\lvert X\rvert$	$\lvert O\rvert$	cq	$\lvert O_c\rvert$	mq	ref
38/2	$C_2\times C_{19}$	18	18	324	341	324	341	2.1
38					**341**		**341**	
39/2	$C_3\times C_{13}$	24	24	576	775	576	775	2.1
39					**775**		**775**	
40/2	$C_8\times C_5$	16	16	256	304	256	304	2.1
40/9	$C_4\times C_2\times C_5$	32	20	448	532	352	418	2.1
40/14	$C_2^3\times C_5$	672	24	3152	6479	512	665	2.1
40					**7315**		**1387**	
41/1	C_{41}	40	40	1600	1639	1600	1639	3.1
41					**1639**		**1639**	
42/6	$C_2\times C_3\times C_7$	12	12	144	205	144	205	2.1
42					**205**		**205**	
43/1	C_{43}	42	42	1764	1805	1764	1805	3.1
43					**1805**		**1805**	
44/2	$C_4\times C_{11}$	20	20	400	436	400	436	2.1
44/4	$C_2^2\times C_{11}$	60	30	1100	1635	800	981	2.1
44					**2071**		**1417**	
45/1	$C_9\times C_5$	24	24	576	912	576	912	2.1
45/2	$C_3^2\times C_5$	192	32	2176	3477	896	1292	2.1
45					**4389**		**2204**	
46/2	$C_2\times C_{23}$	22	22	484	505	484	505	2.1
46					**505**		**505**	
47/1	C_{47}	46	46	2116	2161	2116	2161	3.1
47					**2161**		**2161**	
48/2	$C_{16}\times C_3$	16	16	256	320	256	320	2.1
48/20	$C_4^2\times C_3$	192	28	1600	3120	672	940	2.1
48/23	$C_8\times C_2\times C_3$	32	20	448	560	352	440	2.1
48/44	$C_4\times C_2^2\times C_3$	384	26	2256	4100	584	750	2.1
48/52	$C_2^4\times C_3$	40320	28	82988	198835	640	895	2.1
48					**206935**		**3345**	
49/1	C_{49}	42	42	1764	2044	1764	2044	3.1
49/2	C_7^2	2016	48	13896	16055	2088	2344	
49					**18099**		**4388**	
50/2	$C_2\times C_{25}$	20	20	400	490	400	490	2.1
50/5	$C_2\times C_5^2$	480	24	2336	2847	512	594	2.1
50					**3337**		**1084**	
51/1	$C_3\times C_{17}$	32	32	1024	1355	1024	1355	2.1
51					**1355**		**1355**	
52/2	$C_4\times C_{13}$	24	24	576	620	576	620	2.1
52/5	$C_2^2\times C_{13}$	72	36	1584	2325	1152	1395	2.1
52					**2945**		**2015**	
53/1	C_{53}	52	52	2704	2755	2704	2755	3.1
53					**2755**		**2755**	

G	structure	$\lvert A\rvert$	$\lvert X\rvert$	$\lvert O\rvert$	cq	$\lvert O_c\rvert$	mq	ref
54/2	$C_2 \times C_{27}$	18	18	324	441	324	441	2.1
54/9	$C_2 \times C_9 \times C_3$	108	20	864	1356	336	528	2.1
54/15	$C_2 \times C_3^3$	11232	24	23236	34321	484	605	2.1
54					**36118**		**1574**	
55/2	$C_5 \times C_{11}$	40	40	1600	2071	1600	2071	2.1
55					**2071**		**2071**	
56/2	$C_8 \times C_7$	24	24	576	656	576	656	2.1
56/8	$C_4 \times C_2 \times C_7$	48	30	1008	1148	792	902	2.1
56/13	$C_2^3 \times C_7$	1008	36	7092	13981	1152	1435	2.1
56					**15785**		**2993**	
57/2	$C_3 \times C_{19}$	36	36	1296	1705	1296	1705	2.1
57					**1705**		**1705**	
58/2	$C_2 \times C_{29}$	28	28	784	811	784	811	2.1
58					**811**		**811**	
59/1	C_{59}	58	58	3364	3421	3364	3421	3.1
59					**3421**		**3421**	
60/4	$C_4 \times C_3 \times C_5$	16	16	256	380	256	380	2.1
60/13	$C_2^2 \times C_3 \times C_5$	48	24	704	1425	512	855	2.1
60					**1805**		**1235**	
61/1	C_{61}	60	60	3600	3659	3600	3659	3.1
61					**3659**		**3659**	
62/2	$C_2 \times C_{31}$	30	30	900	929	900	929	2.1
62					**929**		**929**	
63/2	$C_9 \times C_7$	36	36	1296	1968	1296	1968	2.1
63/4	$C_3^2 \times C_7$	288	48	4896	7503	2016	2788	2.1
63					**9471**		**4756**	
64/1	C_{64}	32	32	1024	1024	1024	1024	3.1
64/2	C_8^2	1536	60	13568	22784	3072	3408	
64/26	$C_{16} \times C_4$	256	52	4864	4864	2368	2368	
64/50	$C_{32} \times C_2$	64	40	1792	1792	1408	1408	
64/55	C_4^3	86016	60	206144	441664	4448	4672	
64/83	$C_8 \times C_4 \times C_2$	2048	104	31168	31168	7240	7240	
64/183	$C_{16} \times C_2^2$	768	52	9024	13120	2336	2400	
64/192	$C_4^2 \times C_2^2$	147456	100	550480	1239472	9108	9656	
64/246	$C_8 \times C_2^3$	43008	60	193648	350320	3216	3336	
64/260	$C_4 \times C_2^4$	10321920	69	?	?	?	?	
64/267	C_2^6	20158709760	60	?	?	?	?	
64					**?**		**?**	
65/1	$C_5 \times C_{13}$	48	48	2304	2945	2304	2945	2.1
65					**2945**		**2945**	
66/4	$C_2 \times C_3 \times C_{11}$	20	20	400	545	400	545	2.1
66					**545**		**545**	
67/1	C_{67}	66	66	4356	4421	4356	4421	3.1
67					**4421**		**4421**	

G	structure	$\|A\|$	$\|X\|$	$\|O\|$	cq	$\|O_c\|$	mq	ref
68/2	$C_4 \times C_{17}$	32	32	1024	1084	1024	1084	2.1
68/5	$C_2^2 \times C_{17}$	96	48	2816	4065	2048	2439	2.1
68					**5149**		**3523**	
69/1	$C_3 \times C_{23}$	44	44	1936	2525	1936	2525	2.1
69					**2525**		**2525**	
70/4	$C_2 \times C_5 \times C_7$	24	24	576	779	576	779	2.1
70					**779**		**779**	
71/1	C_{71}	70	70	4900	4969	4900	4969	3.1
71					**4969**		**4969**	
72/2	$C_8 \times C_9$	24	24	576	768	576	768	2.1
72/9	$C_4 \times C_2 \times C_9$	48	30	1008	1344	792	1056	2.1
72/14	$C_8 \times C_3^2$	192	32	2176	2928	896	1088	2.1
72/18	$C_2^3 \times C_9$	1008	36	7092	16368	1152	1680	2.1
72/36	$C_4 \times C_2 \times C_3^2$	384	40	3808	5124	1232	1496	2.1
72/50	$C_2^3 \times C_3^2$	8064	48	26792	62403	1792	2380	2.1
72					**88935**		**8468**	
73/1	C_{73}	72	72	5184	5255	5184	5255	3.1
73					**5255**		**5255**	
74/2	$C_2 \times C_{37}$	36	36	1296	1331	1296	1331	2.1
74					**1331**		**1331**	
75/1	$C_3 \times C_{25}$	40	40	1600	2450	1600	2450	2.1
75/3	$C_3 \times C_5^2$	960	48	9344	14235	2048	2970	2.1
75					**16685**		**5420**	
76/2	$C_4 \times C_{19}$	36	36	1296	1364	1296	1364	2.1
76/4	$C_2^2 \times C_{19}$	108	54	3564	5115	2592	3069	2.1
76					**6479**		**4433**	
77/1	$C_7 \times C_{11}$	60	60	3600	4469	3600	4469	2.1
77					**4469**		**4469**	
78/6	$C_2 \times C_3 \times C_{13}$	24	24	576	775	576	775	2.1
78					**775**		**775**	
79/1	C_{79}	78	78	6084	6161	6084	6161	3.1
79					**6161**		**6161**	
80/2	$C_{16} \times C_5$	32	32	1024	1216	1024	1216	2.1
80/20	$C_4^2 \times C_5$	384	56	6400	11856	2688	3572	2.1
80/23	$C_8 \times C_2 \times C_5$	64	40	1792	2128	1408	1672	2.1
80/45	$C_4 \times C_2^2 \times C_5$	768	52	9024	15580	2336	2850	2.1
80/52	$C_2^4 \times C_5$	80640	56	331952	755573	2560	3401	2.1
80					**786353**		**12711**	
81/1	C_{81}	54	54	2916	3996	2916	3996	3.1
81/2	C_9^2	3888	78	35316	54405	5616	8055	
81/5	$C_{27} \times C_3$	324	60	7776	12897	3024	5157	
81/11	$C_9 \times C_3^2$	23328	74	152892	270441	4176	7167	
81/15	C_3^4	24261120	78	?	?	?	?	
81					**?**		**?**	

G	structure	$\|A\|$	$\|X\|$	$\|O\|$	cq	$\|O_c\|$	mq	ref
82/2	$C_2{\times}C_{41}$	40	40	1600	1639	1600	1639	2.1
82					**1639**		**1639**	
83/1	C_{83}	82	82	6724	6805	6724	6805	3.1
83					**6805**		**6805**	
84/6	$C_4{\times}C_3{\times}C_7$	24	24	576	820	576	820	2.1
84/15	$C_2^2{\times}C_3{\times}C_7$	72	36	1584	3075	1152	1845	2.1
84					**3895**		**2665**	
85/1	$C_5{\times}C_{17}$	64	64	4096	5149	4096	5149	2.1
85					**5149**		**5149**	
86/2	$C_2{\times}C_{43}$	42	42	1764	1805	1764	1805	2.1
86					**1805**		**1805**	
87/1	$C_3{\times}C_{29}$	56	56	3136	4055	3136	4055	2.1
87					**4055**		**4055**	
88/2	$C_8{\times}C_{11}$	40	40	1600	1744	1600	1744	2.1
88/8	$C_4{\times}C_2{\times}C_{11}$	80	50	2800	3052	2200	2398	2.1
88/12	$C_2^3{\times}C_{11}$	1680	60	19700	37169	3200	3815	2.1
88					**41965**		**7957**	
89/1	C_{89}	88	88	7744	7831	7744	7831	3.1
89					**7831**		**7831**	
90/4	$C_2{\times}C_9{\times}C_5$	24	24	576	912	576	912	2.1
90/10	$C_2{\times}C_3^2{\times}C_5$	192	32	2176	3477	896	1292	2.1
90					**4389**		**2204**	
91/1	$C_7{\times}C_{13}$	72	72	5184	6355	5184	6355	2.1
91					**6355**		**6355**	
92/2	$C_4{\times}C_{23}$	44	44	1936	2020	1936	2020	2.1
92/4	$C_2^2{\times}C_{23}$	132	66	5324	7575	3872	4545	2.1
92					**9595**		**6565**	
93/2	$C_3{\times}C_{31}$	60	60	3600	4645	3600	4645	2.1
93					**4645**		**4645**	
94/2	$C_2{\times}C_{47}$	46	46	2116	2161	2116	2161	2.1
94					**2161**		**2161**	
95/1	$C_5{\times}C_{19}$	72	72	5184	6479	5184	6479	2.1
95					**6479**		**6479**	
96/2	$C_{32}{\times}C_3$	32	32	1024	1280	1024	1280	2.1
96/46	$C_8{\times}C_4{\times}C_3$	256	52	4864	6080	2368	2960	2.1
96/59	$C_{16}{\times}C_2{\times}C_3$	64	40	1792	2240	1408	1760	2.1
96/161	$C_4^2{\times}C_2{\times}C_3$	3072	60	24896	49040	3536	4520	2.1
96/176	$C_8{\times}C_2^2{\times}C_3$	768	52	9024	16400	2336	3000	2.1
96/220	$C_4{\times}C_2^3{\times}C_3$	43008	60	193648	437900	3216	4170	2.1
96/231	$C_2^5{\times}C_3$	19998720	54	40096308	98605385	2360	3275	2.1
96					**99118325**		**20965**	
97/1	C_{97}	96	96	9216	9311	9216	9311	3.1
97					**9311**		**9311**	

G	structure	$\lvert A\rvert$	$\lvert X\rvert$	$\lvert O\rvert$	cq	$\lvert O_c\rvert$	mq	ref
98/2	$C_2\times C_{49}$	42	42	1764	2044	1764	2044	2.1
98/5	$C_2\times C_7^2$	2016	48	13896	16055	2088	2344	2.1
98					**18099**		**4388**	
99/1	$C_9\times C_{11}$	60	60	3600	5232	3600	5232	2.1
99/2	$C_3^2\times C_{11}$	480	80	13600	19947	5600	7412	2.1
99					**25179**		**12644**	
100/2	$C_4\times C_{25}$	40	40	1600	1960	1600	1960	2.1
100/5	$C_2^2\times C_{25}$	120	60	4400	7350	3200	4410	2.1
100/8	$C_4\times C_5^2$	960	48	9344	11388	2048	2376	2.1
100/16	$C_2^2\times C_5^2$	2880	72	25696	42705	4096	5346	2.1
100					**63403**		**14092**	
101/1	C_{101}	100	100	10000	10099	10000	10099	3.1
101					**10099**		**10099**	
102/4	$C_2\times C_3\times C_{17}$	32	32	1024	1355	1024	1355	2.1
102					**1355**		**1355**	
103/1	C_{103}	102	102	10404	10505	10404	10505	3.1
103					**10505**		**10505**	
104/2	$C_8\times C_{13}$	48	48	2304	2480	2304	2480	2.1
104/9	$C_4\times C_2\times C_{13}$	96	60	4032	4340	3168	3410	2.1
104/14	$C_2^3\times C_{13}$	2016	72	28368	52855	4608	5425	2.1
104					**59675**		**11315**	
105/2	$C_3\times C_5\times C_7$	48	48	2304	3895	2304	3895	2.1
105					**3895**		**3895**	
106/2	$C_2\times C_{53}$	52	52	2704	2755	2704	2755	2.1
106					**2755**		**2755**	
107/1	C_{107}	106	106	11236	11341	11236	11341	3.1
107					**11341**		**11341**	
108/2	$C_4\times C_{27}$	36	36	1296	1764	1296	1764	2.1
108/5	$C_2^2\times C_{27}$	108	54	3564	6615	2592	3969	2.1
108/12	$C_4\times C_9\times C_3$	216	40	3456	5424	1344	2112	2.1
108/29	$C_2^2\times C_9\times C_3$	648	60	9504	20340	2688	4752	2.1
108/35	$C_4\times C_3^3$	22464	48	92944	137284	1936	2420	2.1
108/45	$C_2^2\times C_3^3$	67392	72	255596	514815	3872	5445	2.1
108					**686242**		**20462**	
109/1	C_{109}	108	108	11664	11771	11664	11771	3.1
109					**11771**		**11771**	
110/6	$C_2\times C_5\times C_{11}$	40	40	1600	2071	1600	2071	2.1
110					**2071**		**2071**	
111/2	$C_3\times C_{37}$	72	72	5184	6655	5184	6655	2.1
111					**6655**		**6655**	

G	structure	$\lvert A\rvert$	$\lvert X\rvert$	$\lvert O\rvert$	cq	$\lvert O_c\rvert$	mq	ref
112/2	$C_{16}\times C_7$	48	48	2304	2624	2304	2624	2.1
112/19	$C_4^2\times C_7$	576	84	14400	25584	6048	7708	2.1
112/22	$C_8\times C_2\times C_7$	96	60	4032	4592	3168	3608	2.1
112/37	$C_4\times C_2^2\times C_7$	1152	78	20304	33620	5256	6150	2.1
112/43	$C_2^4\times C_7$	120960	84	746892	1630447	5760	7339	2.1
112					**1696867**		**27429**	
113/1	C_{113}	112	112	12544	12655	12544	12655	3.1
113					**12655**		**12655**	
114/6	$C_2\times C_3\times C_{19}$	36	36	1296	1705	1296	1705	2.1
114					**1705**		**1705**	
115/1	$C_5\times C_{23}$	88	88	7744	9595	7744	9595	2.1
115					**9595**		**9595**	
116/2	$C_4\times C_{29}$	56	56	3136	3244	3136	3244	2.1
116/5	$C_2^2\times C_{29}$	168	84	8624	12165	6272	7299	2.1
116					**15409**		**10543**	
117/2	$C_9\times C_{13}$	72	72	5184	7440	5184	7440	2.1
117/4	$C_3^2\times C_{13}$	576	96	19584	28365	8064	10540	2.1
117					**35805**		**17980**	
118/2	$C_2\times C_{59}$	58	58	3364	3421	3364	3421	2.1
118					**3421**		**3421**	
119/1	$C_7\times C_{17}$	96	96	9216	11111	9216	11111	2.1
119					**11111**		**11111**	
120/4	$C_8\times C_3\times C_5$	32	32	1024	1520	1024	1520	2.1
120/31	$C_4\times C_2\times C_3\times C_5$	64	40	1792	2660	1408	2090	2.1
120/47	$C_2^3\times C_3\times C_5$	1344	48	12608	32395	2048	3325	2.1
120					**36575**		**6935**	
121/1	C_{121}	110	110	12100	13288	12100	13288	3.1
121/2	C_{11}^2	13200	120	144200	158199	13400	14508	
121					**171487**		**27796**	
122/2	$C_2\times C_{61}$	60	60	3600	3659	3600	3659	2.1
122					**3659**		**3659**	
123/1	$C_3\times C_{41}$	80	80	6400	8195	6400	8195	2.1
123					**8195**		**8195**	
124/2	$C_4\times C_{31}$	60	60	3600	3716	3600	3716	2.1
124/4	$C_2^2\times C_{31}$	180	90	9900	13935	7200	8361	2.1
124					**17651**		**12077**	
125/1	C_{125}	100	100	10000	12325	10000	12325	3.1
125/2	$C_{25}\times C_5$	2000	104	47200	66580	9280	13270	
125/5	C_5^3	1488000	120	?	?	?	?	
125					**?**		**?**	
126/6	$C_2\times C_9\times C_7$	36	36	1296	1968	1296	1968	2.1
126/16	$C_2\times C_3^2\times C_7$	288	48	4896	7503	2016	2788	2.1
126					**9471**		**4756**	
127/1	C_{127}	126	126	15876	16001	15876	16001	3.1
127					**16001**		**16001**	

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