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Cubic differential systems with two affine real non-parallel invariant straight lines of maximal multiplicity

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Abstract. In this article we classify all differential real cubic systems possessing two affine real non-parallel invariant straight lines of maximal multiplicity. We show that the maximal multiplicity of each of these lines is at most three. The maximal sequences of multiplicities: $m(3,3;1)$, $m(3,2;2)$, $m(3,1;3)$, $m(2,2;3)$, $m_\infty(2,1;3)$, $m_\infty(1,1;3)$ are determined. The normal forms and the corresponding perturbations of the cubic systems which realize these cases are given.

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1 Introduction and the statement of main results

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \quad \gcd(P,Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

associated to system (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 3$ then system (1) is called cubic.

A curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ is said to be an *invariant algebraic curve* of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $deg(K_f) \le n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y)$ holds. We will be interested in invariant algebraic curves of degree one, that is invariant straight lines $\alpha x + \beta y + \gamma = 0$, $(\alpha,\beta) \neq (0,0)$.

Definition 1 (see [5])**.** An invariant algebraic curve f of degree d for the vector field $\mathbb{X}$ has *algebraic multiplicity* m when m is the greatest positive integer such that the m-th power of f divides $E_d(\mathbb{X})$, where

$$E_d(\mathbb{X}) = det\begin{pmatrix} \upsilon_1 & \upsilon_2 & ... & \upsilon_l \\ \mathbb{X}(\upsilon_1) & \mathbb{X}(\upsilon_2) & ... & \mathbb{X}(\upsilon_l) \\ ... & ... & ... & ... \\ \mathbb{X}^{l-1}(\upsilon_1) & \mathbb{X}^{l-1}(\upsilon_2) & ... & \mathbb{X}^{l-1}(\upsilon_l)\end{pmatrix}, \tag{2}$$

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and $v_1, v_2, ..., v_l$ is a basis of $\mathbb{C}_d[x, y]$.

We note that this definition of multiplicity can be applied to the infinite line $Z = 0$ in the case when this line is not full of singular points.

Definition 2 (see [5]). An invariant algebraic curve $f = 0$ of degree d of the vector field $\mathbb{X}$ has *geometric multiplicity* m if m is the largest integer for which there exists a sequence of vector fields $(\mathbb{X}_i)_{i>0}$ of bounded degree, converging to $h\mathbb{X}$, for some polynomial h, not divisible by f, such that each $\mathbb{X}_r$ has m distinct invariant algebraic curves, $f_{r,1} = 0, ..., f_{r,m} = 0$, of degree at most d, which converge to $f = 0$ as r goes to infinity. If we set $h = 1$ in the definition above, then we say that the curve has *strong geometric multiplicity* m.

In [5] it is proved that the notions of algebraic and geometric multiplicity are equivalent.

By present a great number of works have been dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimating the number of invariant straight lines which a polynomial differential system can have was considered in [1]; the problem of coexistence of the invariant straight lines and limit cycles was examined in $\{[16] : n = 2\}$, $\{[9], n = 3\}$, [8]; the problem of coexistence of the invariant straight lines in cubic systems and singular points of center type was studied in [6], [7], [17].

The classification of all cubic systems with the maximum number of invariant straight lines, including the line at infinity, and taking into account their multiplicities, is given in [10].

In [1] it was proved that the cubic system (1) can have at most eight affine invariant straight lines. The cubic systems with exactly eight and exactly seven distinct affine invariant straight lines have been studied in [10], [11], with invariant straight lines of total geometric (parallel) multiplicity eight (seven) - in [2], [3], [4] ([18]), and with six real invariant straight lines along two (three) directions - in [13], [14]. The cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity six were investigated in [15]. In [19] it was shown that in the class of cubic differential systems the maximal multiplicity of an affine real straight line (of the line at infinity) is seven.

In this paper the cubic systems with two affine real non-parallel invariant straight lines of maximal multiplicity are classified.

Denote by $\mathbb{CSL}_k^*$ $(\mathbb{CSL}_{2(r)}^{\times})$ the class of cubic systems with exactly k distinct (with exactly 2 real non-parallel) affine invariant straight lines.

Definition 3. We say that $(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$, where $\mu_j \in \mathbb{N}^*$, $j = 1, \ldots, k, \infty$, $\mu_j \geq \mu_{j+1}, j = 1, \ldots, k-1$, is a sequence of multiplicities of invariant straight lines in the class $\mathbb{CSL}_k^*$ if in $\mathbb{CSL}_k^*$ there exists a system with invariant affine straight lines $l_1, ..., l_k$ which have respectively the multiplicities $\mu_1, \mu_2, ..., \mu_k$ and the line at infinity has the multiplicity μ_∞.

Definition 4. The sequence of multiplicities $(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$ is called maximal with respect to the component $j, j \in \{1, 2, ..., k, \infty\}$ if $(\mu_1, \mu_2, ..., \mu_j + 1, ..., \mu_k; \mu_\infty)$

is not a sequence of multiplicities of invariant straight lines in the class $\mathbb{CSL}_k^*$. We will denote this sequence by $m_j(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$. The sequence of the type $m_j(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$ is called partially maximal. If the sequence $(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$ is maximal with respect to all components, then it is called maximal (or totally maximal) and is denoted by $m(\mu_1, \mu_2, ..., \mu_k; \mu_\infty)$.

Our main results are the following:

Main Theorem *Any cubic system having two affine non-parallel real invariant straight lines of the (partially) maximal multiplicity $m(\mu_1, \mu_2; \mu_\infty)$ ($m_\infty(\mu_1, \mu_2; \mu_\infty)$) via an affine transformation and time rescaling can be written as one of the following forms:*

$m(3,3;1)$	1)	$\dot{x} = x^3, \quad \dot{y} = y(x^2 + ay + by^2),\ b \neq 0;$
$m(3,2;2)$	2.1)	$\dot{x} = ax^3, \quad \dot{y} = y^2, a \neq 0;$
	2.2)	$\dot{x} = x(ax^2 + y), \quad \dot{y} = y^2, \ a \neq 0;$
$m(3,1;3)$	3.1)	$\dot{x} = x^2(ax + by), \quad \dot{y} = y, \ a \neq 0;$
	3.2)	$\dot{x} = x(ay + b), \quad \dot{y} = y(x^2 + ay + b), \ b \neq 0;$
$m(2,2;3)$	4)	$\dot{x} = x, \quad \dot{y} = y(1 + bxy), \ b \neq 0;$
$m_\infty(2,1;3)$	5.1)	$\dot{x} = x^2(a + bx + cy), \quad \dot{y} = y, \quad c(a^2 + b^2) \neq 0;$
	5.2)	$\dot{x} = x, \quad \dot{y} = y(1 + ax + bx^2 + cxy), \quad a(b^2 + c^2) \neq 0;$
	5.3)	$\dot{x} = x(1 + ax + bx^2 + cxy), \quad \dot{y} = y, \quad c(a^2 + b^2) \neq 0;$
	5.4)	$\dot{x} = x(1 + ay), \quad \dot{y} = y(1 + bx + ay + cx^2), \quad abc \neq 0;$
$m_\infty(1,1;3)$	6.1)	$\dot{x} = x, \quad \dot{y} = y(a + bx + cy + dx^2 + exy + fy^2),$ $(a^2+c^2+f^2)(d^2+e^2+f^2)(a^2+b^2+d^2)((a-1)^2+c^2+f^2)\cdot$ $((a-1)^2+b^2+d^2)((a-1)^2+(c^2d-bce+b^2f)^2) \neq 0;$
	6.2)	$\dot{x} = x(a + by), \quad \dot{y} = y(c + dx + ey + x^2), \quad a(c^2 + e^2)((a-c)^2 + (b-e)^2) \neq 0;$
	6.3)	$\dot{x} = x(a + by + cxy + y^2), \ \dot{y} = -y(d + ex + c^2x^2 + cxy),$ $ad(c^2 + e^2 + (a+d)^2)((a+d)^2 + (bc-e)^2) \neq 0;$
	6.4)	$\dot{x} = x(a + by + cxy + dy^2), \ \dot{y} = \alpha y(1 + bx + cx^2 + dxy),$ $\alpha a(c^2 + d^2)(\alpha - a) \neq 0.$

2 The proof of the Main Theorem

2.1 The maximal algebraic multiplicity of the affine invariant straight lines

The goal of this section is to determine the maximal algebraic multiplicity of the invariant straight lines for the cubic systems with two affine real non-parallel invariant straight lines.

We consider the cubic differential system

$$\begin{cases} \dot{x} = P_0 + P_1(x,y) + P_2(x,y) + P_3(x,y) \equiv P(x,y), \\ \dot{y} = Q_0 + Q_1(x,y) + Q_2(x,y) + Q_3(x,y) \equiv Q(x,y), \end{cases} \tag{3}$$

where $P_k = \sum\limits_{i+j=k} a_{ij}x^iy^j$ and $Q_k = \sum\limits_{i+j=k} b_{ij}x^iy^j$, $a_{ij}, b_{ij} \in \mathbb{R}$, $k = \overline{0,3}$.

Suppose that

$$yP_3(x,y) - xQ_3(x,y) \not\equiv 0, \qquad \gcd(P,Q) = 1, \tag{4}$$

i.e. at infinity the system (3) has at most four distinct singular points and the right-hand sides of (3) do not have common divisors of degree greater than 0.

Let the system (3) have two real non-parallel invariant straight lines l_1, l_2. By an affine transformation we can make them to be described by equations $x = 0$ and $y = 0$, respectively. Then, the system (3) looks as

$$\begin{cases} \dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2). \end{cases} \tag{5}$$

We denote by μ_1 the multiplicity of the line $x = 0$, by μ_2 the multiplicity of the line $y = 0$ and by μ_∞ the multiplicity of the line at infinity.

Applying Definition 1, first we determine the maximal algebraic multiplicity of the line $x = 0$, secondly the maximal algebraic multiplicity of the line $y = 0$ and the third step consists in the determination of the maximal algebraic multiplicity of the line at infinity $Z = 0$.

2.1.1 The maximal algebraic multiplicity of the line $x = 0$

In this subsection, we compute the maximal algebraic multiplicity of the invariant straight line $x = 0$ of the system (5). For this purpose, we calculate the determinant $E_1(\mathbb{X})$ from Definition 1. For (5) the determinant $E_1(\mathbb{X})$ is a polynomial in x and y of degree 8. To determine the maximal algebraic multiplicity of the line $x = 0$, we write it in the form:

$$\begin{aligned} E_1(\mathbb{X}) = x(&A_1(y) + A_2(y)x + A_3(y)x^2 + A_4(y)x^3 + A_5(y)x^4 \\ &+A_6(y)x^5 + A_7(y)x^6 + A_8(y)x^7). \end{aligned} \tag{6}$$

Thus for system (5) we have $A_1(y) = -yA_{11}(y)A_{12}(y)$, where $A_{11}(y) = b_{01} + b_{02}y + b_{03}y^2$ and $A_{12}(y) = a_{10}^2 - a_{10}b_{01} + 2a_{10}a_{11}y - 2a_{10}b_{02}y + a_{11}^2y^2 + 2a_{10}a_{12}y^2 + a_{12}b_{01}y^2 - a_{11}b_{02}y^2 - 3a_{10}b_{03}y^2 + 2a_{11}a_{12}y^3 - 2a_{11}b_{03}y^3 + a_{12}^2y^4 - a_{12}b_{03}y^4$.

The algebraic multiplicity μ_1 of the invariant straight line $x = 0$ is at least two if the identity $A_1(y) \equiv 0$ holds. From conditions (4) the polynomial $A_{11}(y)$ is not identically zero, i.e. $|b_{01}| + |b_{02}| + |b_{03}| \neq 0$, therefore it is necessary that $A_{12}(y)$ be identically zero. The identity $A_{12}(y) \equiv 0$ holds if one of the following six sets of conditions is satisfied:

$$a_{10} = a_{11} = a_{12} = 0; \tag{7}$$

$$a_{11} = a_{12} = b_{02} = b_{03} = 0, b_{01} = a_{10},\ a_{10} \neq 0; \tag{8}$$

$$a_{10} = a_{12} = b_{03} = 0, b_{02} = a_{11},\ a_{11} \neq 0; \tag{9}$$

$$a_{12} = b_{03} = 0, b_{01} = a_{10}, b_{02} = a_{11},\ a_{10}a_{11} \neq 0; \tag{10}$$

$$a_{10} = 0, b_{01} = a_{11}(b_{02} - a_{11})/a_{12}, b_{03} = a_{12},\ a_{12} \neq 0; \tag{11}$$

$$b_{01} = a_{10}, b_{02} = a_{11}, b_{03} = a_{12},\ a_{10}a_{12} \neq 0. \tag{12}$$

Lemma 1. *For cubic differential system* {(5), (4)} *the algebraic multiplicity* μ_1 *of the invariant straight line* $x = 0$ *is at least two if and only if one of the following six sets of conditions* (7), (8), (9), (10), (11), (12) *is satisfied.*

We will examine each of the cases (7), (8), (9), (10), (11) and (12) separately.

1) *Conditions* (7).

The algebraic multiplicity is at least three ($\mu_1 \geq 3$) if the identity $A_2(y) \equiv 0$ holds. Here we have $A_2(y) = yA_{11}(a_{20}b_{01} + 2a_{20}b_{02}y + a_{21}b_{02}y^2 + 3a_{20}b_{03}y^2 + 2a_{21}b_{03}y^3)$. The identity $A_2(y) \equiv 0$ holds if one of the following two series of conditions is satisfied:

$$a_{20} = a_{21} = 0; \tag{13}$$

$$a_{20} = b_{02} = b_{03} = 0, a_{21} \neq 0. \tag{14}$$

Under the conditions {(4), (7), (13)}, the cubic system (5) looks as

$$\begin{array}{l} \dot{x} = a_{30}x^3, \quad \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2), \\ a_{30}(|b_{01}| + |b_{02}| + |b_{03}|) \neq 0. \end{array} \tag{15}$$

For this system $A_3(y) = a_{30}yA_{11}(y)(b_{01} + 2b_{02}y + 3b_{03}y^2) \not\equiv 0$, so in this case the multiplicity of the invariant straight line $x = 0$ is three.

If the conditions {(4), (7), (14)} occur, then the system (5) looks as:

$$\dot{x} = x^2(a_{30}x + a_{21}y), \quad \dot{y} = y(b_{01} + b_{11}x + b_{21}x^2 + b_{12}xy), \; a_{21}a_{30}b_{01} \neq 0. \tag{16}$$

The algebraic multiplicity of the line $x = 0$ can not be greater than three, because $A_3(y) = b_{01}y(a_{30}b_{01} - a_{21}(2a_{21} - b_{12})y^2) \not\equiv 0$.

2) *Conditions* (8):

$A_2(y) = -a_{10}^2 y(2(a_{20} - b_{11}) + 3(a_{21} - b_{12})y) \equiv 0 \Rightarrow$

$$b_{11} = a_{20}, b_{12} = a_{21} \tag{17}$$

$\Rightarrow A_3(y) = -3a_{10}^2(a_{30} - b_{21})y \not\equiv 0$, therefore μ_1 can not be greater than three.

In the conditions {(4), (8), (17)} the system (5) takes the form

$$\begin{array}{l} \dot{x} = x(a_{10} + a_{20}x + a_{30}x^2 + a_{21}xy), \\ \dot{y} = y(a_{10} + a_{20}x + b_{21}x^2 + a_{21}xy), \; a_{10}(b_{21} - a_{30}) \neq 0. \end{array} \tag{18}$$

3) *Conditions* (9).

The identity $A_2(y) = y(a_{20}b_{01}^2 - a_{11}(a_{11}a_{20} + 2a_{21}b_{01} - a_{11}b_{11} - b_{01}b_{12})y^2 - 2a_{11}^2(a_{21} - b_{12})y^3) \equiv 0$ holds if one of the following two series of conditions is satisfied:

$$a_{20} = 0, \; b_{11} = a_{21}b_{01}/a_{11}, \; b_{12} = a_{21}; \tag{19}$$

$$b_{01} = 0, \; b_{11} = a_{20}, \; b_{12} = a_{21}, \; a_{20} \neq 0. \tag{20}$$

Under the conditions {(9), (19)} we write the system (5) as

$$\begin{aligned} &\dot{x} = x(a_{11}y + a_{30}x^2 + a_{21}xy),\\ &\dot{y} = y(a_{11}b_{01} + a_{21}b_{01}x + a_{11}^2y + a_{11}b_{21}x^2 + a_{11}a_{21}xy)/a_{11},\ \ b_{21} - a_{30} \neq 0. \end{aligned} \tag{21}$$

Here $A_3(y) = y(a_{30}b_{01}^2 - a_{11}a_{30}b_{01}y - 2a_{11}^2(a_{30} - b_{21})y^2) \not\equiv 0$, therefore the multiplicity μ_1 can not be greater than three.

If the conditions {(9), (20)} are satisfied then the cubic system (5) obtains the following form

$$\begin{aligned} &\dot{x} = x(a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy),\\ &\dot{y} = y(a_{20}x + a_{11}y + b_{21}x^2 + a_{11}y + a_{21}xy),\ \ a_{11}a_{20}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{22}$$

The algebraic multiplicity of the line $x = 0$, for the system (22), can not be greater than three, because $A_3(y) = 2a_{11}^2(b_{21} - a_{30})y^3 \not\equiv 0$.

4) *Conditions* (10):

$A_2(y) = -y(a_{10} + a_{11}y)(2a_{10}(a_{20} - b_{11}) + (a_{11}a_{20} + 3a_{10}a_{21} - a_{11}b_{11} - 3a_{10}b_{12})y + 2a_{11}(a_{21} - b_{12})y^2) \equiv 0 \Rightarrow \{b_{11} = a_{20}, b_{12} = a_{21}\} \Rightarrow A_3(y) = y(b_{21} - a_{30})(a_{10} + a_{11}y)(3a_{10} + 2a_{11}y) \not\equiv 0$, so $\mu_1 = 3$. The cubic system (5) looks as

$$\begin{aligned} &\dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy),\\ &\dot{y} = y(a_{10} + a_{20}x + a_{11}y + b_{21}x^2 + a_{21}xy),\ a_{10}a_{11}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{23}$$

5) *Conditions* (11).

The identity $A_2(y) = y(a_{11} + a_{12}y)(a_{11}a_{20}(a_{11} - b_{02})^2 + 2a_{12}a_{20}(a_{11} - b_{02})^2y + a_{12}(3a_{11}^2a_{21} - 3a_{11}a_{12}a_{20} + 2a_{12}a_{20}b_{02} - 4a_{11}a_{21}b_{02} + a_{21}b_{02}^2 + 2a_{11}a_{12}b_{11} - a_{12}b_{02}b_{11} - a_{11}^2b_{12} + a_{11}b_{02}b_{12})y^2 - 2a_{11}a_{12}^2(a_{21} - b_{12})y^3 - a_{12}^3(a_{21} - b_{12})y^4)/a_{12}^2 \equiv 0$ holds if one of the following four series of conditions is satisfied:

$$a_{20} = 0, b_{02} = 2a_{11}, b_{12} = a_{21}; \tag{24}$$

$$a_{20} = 0, b_{11} = a_{21}(b_{02} - a_{11})/a_{12}, b_{12} = a_{21}, b_{02} \neq 2a_{11}; \tag{25}$$

$$a_{11} = 0, a_{20} \neq 0, b_{02} = 0, b_{12} = a_{21}; \tag{26}$$

$$a_{11} \neq 0, a_{20} \neq 0, b_{02} = a_{11}, b_{11} = a_{20}, b_{12} = a_{21}. \tag{27}$$

a) The conditions {(11), (24)} lead us to the system

$$\begin{aligned} &\dot{x} = x(a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2),\quad \dot{y} = y(a_{11}^2 + 2a_{11}a_{12}y\\ &+a_{12}b_{21}x^2 + a_{12}a_{21}xy + a_{12}^2y^2)/a_{12},\ b_{21} - a_{30} \neq 0, \end{aligned} \tag{28}$$

for which $A_3(y) = y(a_{11}^4a_{30} + 2a_{11}^3a_{12}a_{30}y + a_{12}(a_{11}^2a_{12}b_{21} - a_{11}^2a_{21}^2 + 2a_{11}a_{12}a_{21}b_{11} - a_{12}^2b_{11}^2)y^2 - 2a_{11}a_{12}^3(a_{30} - b_{21})y^3 - a_{12}^4(a_{30} - b_{21})y^4)/a_{12}^2 \not\equiv 0$, so $\mu_1 = 3$.

b) Under the conditions (25) we have $A_3(y) = y(a_{11} + a_{12}y)(a_{11}a_{30}(a_{11} - b_{02})^2 + a_{12}a_{30}(3a_{11} - 2b_{02})(a_{11} - b_{02})y - a_{12}^2(3a_{11} - b_{02})(a_{30} - b_{21})y^2 - a_{12}^3(a_{30} - b_{21})y^3)/a_{12}^2 \not\equiv 0$, therefore in this case $\mu_1 = 3$. The cubic system (5) has the form

$$\begin{aligned} &\dot{x} = x(a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2),\quad \dot{y} = y(a_{11}(b_{02} - a_{11}) +\\ &a_{21}(b_{02} - a_{11})x + a_{12}b_{02}y + a_{12}b_{21}x^2 + a_{12}a_{21}xy + a_{12}^2y^2)/a_{12},\\ &(b_{21} - a_{30})(b_{02} - 2a_{11}) \neq 0. \end{aligned} \tag{29}$$

c) When conditions $\{(11), (26)\}$ hold we have $A_3(y) = -a_{12}y^3(2a_{20}^2 - 3a_{20}b_{11} + b_{11}^2 + a_{12}a_{30}y^2 - a_{12}b_{21}y^2) \not\equiv 0$ and one obtains the following system

$$\begin{aligned} &\dot{x} = x(a_{20}x + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ &\dot{y} = y(b_{11}x + b_{21}x^2 + a_{21}xy + a_{12}y^2), \;\; a_{20}a_{12}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{30}$$

The multiplicity μ_1 is equal to three.

d) The conditions $\{(11), (27)\}$ lead us to the system

$$\begin{aligned} &\dot{x} = x(a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ &\dot{y} = y(a_{20}x + a_{11}y + b_{21}x^2 + a_{21}xy + a_{12}y^2), \;\; a_{11}a_{12}a_{20}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{31}$$

For system (31) we have $A_3(y) = (b_{21} - a_{30})(a_{11} + a_{12}y)(2a_{11} + a_{12}y)y^3 \not\equiv 0$ and therefore $\mu_1 = 3$.

6) *Conditions* (12):
$A_2(y) = -y(a_{10}+a_{11}y+a_{12}y^2)(2a_{10}a_{20}-2a_{10}b_{11}+a_{11}a_{20}y+3a_{10}a_{21}y-a_{11}b_{11}y-3a_{10}b_{12}y+2a_{11}a_{21}y^2-2a_{11}b_{12}y^2+a_{12}a_{21}y^3-a_{12}b_{12}y^3) \equiv 0 \Rightarrow \{b_{11} = a_{20}, b_{12} = a_{21}\}$ $\Rightarrow A_3(y) = y(b_{21} - a_{30})(a_{10} + a_{11}y + a_{12}y^2)(3a_{10} + 2a_{11}y + a_{12}y^2) \not\equiv 0$. Therefore $\mu_1 = 3$. In this case the cubic system (5) looks as

$$\begin{aligned} &\dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ &\dot{y} = y(a_{10} + a_{20}x + a_{11}y + b_{21}x^2 + a_{21}xy + a_{12}y^2), \; a_{10}a_{12}(b_{21} - a_{30}) \neq 0. \end{aligned} \tag{32}$$

In this way we have proved the following two lemmas.

Lemma 2. *Let the cubic system* $\{(3), (4)\}$ *have two affine real non-parallel invariant straight lines. Then the maximal algebraic multiplicity of one of these lines is at most three.*

Lemma 3. *For cubic differential system* $\{(5), (4)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is three if and only if it has one of the following forms:* (15), (16), (18), (21), (22), (23), (28), (29), (30), (31), (32).

2.1.2 The maximal algebraic multiplicity of the line $y = 0$

In this subsection for the systems, enumerated in Lemma 3, we determine the maximal algebraic multiplicity of the line $y = 0$. For this purpose, we write the determinant $E_1(\mathbb{X})$ from Definition 1 in the form:

$$\begin{aligned} E_1(\mathbb{X}) = y(&B_1(x) + B_2(x)y + B_3(x)y^2 + B_4(x)y^3 + B_5(x)y^4 \\ &+B_6(x)y^5 + B_7(x)y^6 + B_8(x)y^7). \end{aligned} \tag{33}$$

The algebraic multiplicity μ_2 of the invariant straight line $y = 0$ is at least two if the identity $B_1(x) \equiv 0$ holds.

Taking into account the condition (4), for each of the systems (16), (18), (22), (23), (31), (32), the polynomial $B_1(x)$ is not identically zero, therefore $\mu_2 = 1$.

In the case of system (15) the identity $B_1(x) \equiv 0$, where $B_1(x) = a_{30}x^3(b_{01}^2 + 2b_{01}b_{11}x - 3a_{30}b_{01}x^2 + b_{11}^2x^2 + 2b_{01}b_{21}x^2 - 2a_{30}b_{11}x^3 + 2b_{11}b_{21}x^3 - a_{30}b_{21}x^4 + b_{21}^2x^4)$ holds if one of the following two series of conditions is satisfied

$$b_{01} = b_{11} = b_{21} = 0; \tag{34}$$

$$b_{01} = b_{11} = 0, b_{21} = a_{30}. \tag{35}$$

The conditions (34) imply $B_2(x) = -a_{30}^2x^5(3b_{02} + 2b_{12}x) \equiv 0 \Rightarrow$

$$b_{02} = b_{12} = 0. \tag{36}$$

Under the conditions (34) and (36) the multiplicity is $\mu_2 = 3$. The cubic system (15) has the form $\dot{x} = a_{30}x^3$, $\dot{y} = b_{03}y^3$, $a_{30}b_{03} \neq 0$. This system is an element of the class $\mathbb{CSL}_4^*$ and for it we have $m(3, 3, 1, 1; 1)$ (see [10]).

The conditions (35) $\Rightarrow B_2(x) = a_{30}^2b_{12}x^6 \equiv 0 \Rightarrow b_{12} = 0 \Rightarrow B_3(x) = a_{30}x^3(2b_{02}^2 + a_{30}b_{03}x^2) \not\equiv 0$, therefore the multiplicity is $\mu_2 = 3$ and the system (15) takes the form

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y(b_{02}y + a_{30}x^2 + b_{03}y^2), \quad a_{30}b_{03} \neq 0. \tag{37}$$

For the system (21) we have $B_1(x) = a_{30}x^3(a_{11}^2b_{01}^2 + 2a_{11}a_{21}b_{01}^2x - b_{01}(3a_{11}^2a_{30} - a_{21}^2b_{01} - 2a_{11}^2b_{21})x^2 - 2a_{11}a_{21}b_{01}(a_{30} - b_{21})x^3 - a_{11}^2b_{21}(a_{30} - b_{21}))/a_{11}^2$ and $\{B_1(x) \equiv 0, (4)\} \Rightarrow$

$$b_{01} = b_{21} = 0, \ a_{11}a_{21}a_{30} \neq 0 \tag{38}$$

$\Rightarrow B_2(x) = -a_{30}^2x^5(3a_{11} + 2a_{21}x) \not\equiv 0$, $\mu_2 = 2$.

In the case of system (28) we have $B_1(x) = a_{30}x^3(a_{11}^4 + 2a_{11}^2a_{12}b_{11}x - a_{12}(3a_{11}^2a_{30} - a_{12}b_{11}^2 - 2a_{11}^2b_{21})x^2 - 2a_{12}^2b_{11}(a_{30} - b_{21})x^3 - a_{12}^2b_{21}(a_{30} - b_{21})x^4)/a_{12}^2 \equiv 0 \Rightarrow$

$$a_{11} = b_{11} = b_{21} = 0, \tag{39}$$

$\Rightarrow B_2(x) = -2a_{21}a_{30}^2x^6 \equiv 0 \Rightarrow a_{21} = 0 \Rightarrow B_3(x) = -3a_{12}a_{30}^2x^5 \not\equiv 0, \quad \mu_2 = 3$. The system (28) looks as:

$$\dot{x} = x(a_{30}x^2 + a_{12}y^2), \quad \dot{y} = a_{12}y^3, \quad a_{30}a_{12} \neq 0. \tag{40}$$

For the system (29) we get $B_1(x) = a_{30}x^3(a_{11}^2(a_{11} - b_{02})^2 + 2a_{11}a_{21}(a_{11} - b_{02})^2x + (a_{11} - b_{02})(a_{11}a_{21}^2 + 3a_{11}a_{12}a_{30} - a_{21}^2b_{02} - 2a_{11}a_{12}b_{21})x^2 + 2a_{12}a_{21}(a_{11} - b_{02})(a_{30} - b_{21})x^3 - a_{12}^2b_{21}(a_{30} - b_{21})x^4)/a_{12}^2$. The identity $B_1(x) \equiv 0$ holds if at least one of the following two sets of conditions is satisfied:

$$a_{11} = a_{21} = b_{21} = 0, \tag{41}$$

$$b_{02} = a_{11}, b_{21} = 0, a_{11} \neq 0. \tag{42}$$

When conditions (41) ((42)) hold the polynomial $B_2(x) = -3a_{30}^2b_{02}x^5$ $(B_2(x) = -a_{30}^2x^5(3a_{11} + 2a_{21}x))$ is not identically zero, therefore $\mu_2 = 2$.

Consider now the system (30). We have: $B_1(x) = -x^4(a_{20}+a_{30}x)(a_{20}b_{11} - b_{11}^2 + 2a_{30}b_{11}x - 2b_{11}b_{21}x + a_{30}b_{21}x^2 - b_{21}^2x^2) \equiv 0 \Rightarrow$

$$b_{11} = 0,\ b_{21} = 0 \tag{43}$$

$\Rightarrow B_2(x) = -a_{21}x^4(a_{20}+a_{30}x)(a_{20}+2a_{30}x) \equiv 0 \Rightarrow a_{21} = 0 \Rightarrow B_3(x) = -a_{12}x^3(a_{20}+a_{30}x)(2a_{20}+3a_{30}x) \not\equiv 0,\ \mu_2 = 3$. The cubic system (30) looks as:

$$\dot{x} = x(a_{20}x + a_{30}x^2 + a_{12}y^2), \quad \dot{y} = a_{12}y^3, \quad a_{12}a_{20}a_{30} \neq 0. \tag{44}$$

The transformation $X = y, Y = x$ reduces (40) and (44) to a system of the form (37).

Lemma 4. *For cubic differential system* $\{(5),\ (4)\}$ *the algebraic multiplicity of the invariant straight lines* $x = 0$ *and* $y = 0$ *are respectively* $\mu_1 = 3$ *and* $\mu_2 \geq 2$ *if and only if it has one of the forms: 1)* $\{(15),\ (34)\}$*, 2)* $\{(15),\ (35)\}$*, 3)* $\{(21),\ (38)\}$*, 4)* $\{(28),\ (39)\}$*, 5)* $\{(29),\ (41)\}$*, 6)* $\{(29),\ (42)\}$*, 7)* $\{(30),\ (43)\}$*.*

Lemma 5. *In the class of cubic systems* $\{(5),\ (4)\} \in \mathbb{CSL}^{\times}_{2(r)}$ *the algebraic multiplicity of the invariant straight lines* $x = 0$ *and* $y = 0$ *is three if and only if it has the form* (37)*.*

2.2 Classification of cubic differential systems with two affine real non-parallel invariant straight lines and the line at infinity of maximal algebraic multiplicity

In this section for cubic system $\{(5),\ (4)\} \in \mathbb{CSL}^{\times}_{2(r)}$ we establish the partially maximal sequences of multiplicities of the type $m_\infty(\mu_1, \mu_2; \mu_\infty)$.

We fix $\mu_1 \in \{1, 2, 3\}$ and $\mu_2 \in \{1, 2, 3\}$, $\mu_1 \geq \mu_2$ and we will determine the maximal multiplicity of the line at infinity such that the sequence $(\mu_1, \mu_2; \mu_\infty)$ should be maximal in the third component. We will investigate the cases:

1. $m(3, 3; \mu_\infty)$, **2.** $m_\infty(3, 2; \mu_\infty)$, **3.** $m_\infty(3, 1; \mu_\infty)$, **4.** $m_\infty(2, 2; \mu_\infty)$, **5.** $m_\infty(2, 1; \mu_\infty)$, **6.** $m_\infty(1, 1; \mu_\infty)$.

We consider the cubic system $\{(5),\ (4)\} \in \mathbb{CSL}^{\times}_{2(r)}$ and its associated homogeneous system

$$\begin{cases} \dot{x} = x(a_{10}Z^2 + a_{20}xZ + a_{11}yZ + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01}Z^2 + b_{11}xZ + b_{02}yZ + b_{21}x^2 + b_{12}xy + b_{03}y^2). \end{cases} \tag{45}$$

For (45) we write $E_1(\mathbb{X})$ in the form

$$\begin{aligned} E_1(\mathbb{X}) = C_0(x, y) + C_1(x, y)Z + C_2(x, y)Z^2 + C_3(x, y)Z^3 + C_4(x, y)Z^4 \\ +C_5(x, y)Z^5 + C_6(x, y)Z^6 + C_7(x, y)Z^7 + C_8(x, y)Z^8, \end{aligned} \tag{46}$$

where $C_j(x, y), j = \overline{0, 8}$ are polynomials in x and y.

The algebraic multiplicity of the line at infinity is $\mu_\infty \in \mathbb{N}^*$ if μ_∞ is the maximal number such that $Z^{(\mu_\infty - 1)}$ divides $E_1(\mathbb{X})$.

2.2.1 Case $m(3,3;\mu_\infty)$

To investigate the maximal algebraic multiplicity of the line at infinity for the system (37) (see Lemma 5), we consider the homogenized system

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y(b_{02}yZ + a_{30}x^2 + b_{03}y^2), \quad a_{30}b_{03} \neq 0. \tag{47}$$

For (47) we have $C_0(x,y) = a_{30}b_{03}x^3y^3(a_{30}x^2 + 3b_{03}y^2) \not\equiv 0$, therefore the algebraic multiplicity of the line at infinity is one and in the class $\mathbb{CSL}^{\times}_{2(r)}$ we have the maximal sequence $m(3,3;1)$.

Lemma 6. *Via an affine transformation and time rescaling any cubic system having two non-parallel real invariant straight lines of the maximal multiplicity* $m(3,3;1)$*, can be brought to the form*

$$\dot{x} = x^3, \quad \dot{y} = y(ay + x^2 + by^2),\ b \neq 0. \tag{48}$$

2.2.2 Case $m_\infty(3,2;\mu_\infty)$

According to Lemma 4, the cubic system $\{(5),(4)\}$ admits the invariant straight lines $x=0$ and $y=0$ of the multiplicities three and two respectively if the cubic system has one of the following seven forms:

1) $\{(15),(34)\}$, 2) $\{(15),(35)\}$, 3) $\{(21),(38)\}$, 4) $\{(28),(39)\}$,
5) $\{(29),(41)\}$, 6) $\{(29),(42)\}$, 7) $\{(30),(43)\}$.

Case 1) $\{(15),(34)\}$. Under the condition (34) the cubic system (15) looks as

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y^2(b_{02} + b_{12}x + b_{03}y),\ a_{30}(|b_{02}| + |b_{03}|) \neq 0. \tag{49}$$

For homogeneous system associated to the system (49) we have $C_0(x,y) = -a_{30}x^3y^2(2b_{12}x + 3b_{03}y)(a_{30}x^2 - b_{12}xy - b_{03}y^2) \equiv 0 \Rightarrow b_{03} = b_{12} = 0 \Rightarrow C_1(x,y) = -3a_{30}^2b_{02}x^5y^2 \not\equiv 0$, therefore the multiplicity of the line at infinity is two. The system (49) takes the form $\dot{x} = a_{30}x^3$, $\dot{y} = b_{02}y^2$, $b_{02}a_{30} \neq 0$, and after the time rescaling we can write it as

$$\dot{x} = ax^3, \quad \dot{y} = y^2,\ a \neq 0 \tag{50}$$

(see system 2.1) of the Main Theorem).

From the above it follows for system (50) that $m_\infty(3,2;2) = m(3,2;2)$.

In the Cases 2), 4), 5), 6), 7) we have respectively

$$\dot{x} = a_{30}x^3, \quad \dot{y} = y(a_{30}x^2 + b_{02}y + b_{12}xy + b_{03}y^2),\ a_{30}(b_{02}^2 + b_{03}^2 + b_{12}^2) \neq 0,$$
$$C_0(x,y) = a_{30}x^3y^2(b_{12}x + b_{03}y)(a_{30}x^2 + 2b_{12}xy + 3b_{03}y^2) \not\equiv 0,\ \mu_\infty = 1;$$

$$\dot{x} = x(a_{30}x^2 + a_{21}xy + a_{12}y^2), \quad \dot{y} = y^2(a_{21}x + a_{12}y),\ a_{12}a_{30} \neq 0,$$
$$C_0(x,y) = -a_{30}x^3y^2(2a_{21}a_{30}x^3 + a_{21}^2x^2y + 3a_{12}a_{30}x^2y + 2a_{12}a_{21}xy^2 + a_{12}^2y^3) \not\equiv 0,\ \mu_\infty = 1;$$

$$\dot{x} = x(a_{30}x^2 + a_{12}y^2), \quad \dot{y} = y^2(a_{12}y + b_{02}),\ a_{12}a_{30} \neq 0,$$
$$C_0(x,y) = -a_{12}a_{30}x^3y^3(3a_{30}x^2 + a_{12}y^2) \not\equiv 0,\ \mu_\infty = 1;$$

$$\dot{x} = x(a_{30}x^2 + a_{11}y + a_{21}xy + a_{12}y^2), \quad \dot{y} = y^2(a_{11} + a_{21}x + a_{12}y),$$
$$a_{12}a_{30} \neq 0, \ \ C_0(x,y) = -a_{30}x^3y^2(2a_{21}a_{30}x^3 + a_{21}^2x^2y + 3a_{12}a_{30}x^2y$$
$$+2a_{12}a_{21}xy^2 + a_{12}^2y^3) \not\equiv 0, \ \ \mu_\infty = 1;$$

$$\dot{x} = x(a_{20}x + a_{30}x^2 + a_{21}xy + a_{12}y^2), \quad \dot{y} = y^2(a_{21}x + a_{12}y),$$
$$a_{12}a_{20}a_{30} \neq 0, \quad C_0(x,y) = -a_{30}x^3y^2(2a_{21}a_{30}x^3 + a_{21}^2x^2y+$$
$$3a_{12}a_{30}x^2y + 2a_{12}a_{21}xy^2 + a_{12}^2y^3) \not\equiv 0, \ \ \mu_\infty = 1.$$

Case 3) {(21), (38)}. In this case $C_0(x,y) = -a_{21}a_{30}x^5y^2(2a_{30}x + a_{21}y) \equiv 0 \Rightarrow a_{21} = 0 \Rightarrow C_1(x,y) = -3a_{11}a_{30}^2x^5y^2 \not\equiv 0, \ \ \mu_\infty = 2$. The system {(21), (38)} obtains the form $\dot{x} = x(a_{11}y + a_{30}x^2), \ \ \dot{y} = a_{11}y^2, \ \ a_{11}a_{30} \neq 0$, and after time rescaling we can write it as

$$\dot{x} = x(y + ax^2), \quad \dot{y} = y^2, \ \ a \neq 0 \tag{51}$$

(see system 2.2) of the Main Theorem).

For system (51) we have $m_\infty(3,2;2) = m(3,2;2)$.

Lemma 7. *Any cubic system of the class* $\mathbb{CSL}_{2(r)}^{\times}$ *with invariant straight lines of the maximal multiplicity* $m(3,2;2)$ *via an affine transformation and time rescaling can be written in form* (50) *or* (51).

2.2.3 Case $m_\infty(3,1;\mu_\infty)$

The following cubic systems: (15), (16), (18), (21), (22), (23), (28), (29), (30), (31), (32) possess the invariant straight lines $x = 0$ and $y = 0$ of the multiplicity $\mu_1 = 3$ and $\mu_2 = 1$, respectively (see Lemma 3). Proceeding as in the previous case and taking into account the condition (4), we will examine each system separately.

System (15). For this system we have $C_0(x,y) = -a_{30}x^3yC_{01}(x,y)C_{02}(x,y)$, where $C_{01}(x,y) = a_{30}x^2 - b_{21}x^2 - b_{12}xy - b_{03}y^2$, $C_{02} = (b_{21}x^2 + 2b_{12}xy + 3b_{03}y^2)$. If $C_{01}(x,y) \equiv 0$, then the infinity is degenerate for (15). Let $C_{01}(x,y) \not\equiv 0$, i.e. $|a_{30} - b_{21}| + |b_{12}| + |b_{03}| \neq 0$, and $C_{02}(x,y) \equiv 0$. Then, $b_{03} = b_{12} = b_{21} = 0 \Rightarrow C_1(x,y) = -a_{30}^2x^5y(2b_{11}x+3b_{02}y) \equiv 0 \Rightarrow b_{02} = b_{11} = 0 \Rightarrow C_2(x,y) = -3a_{30}^2b_{01}x^5y \not\equiv 0, \ \ \mu_\infty = 3$. Under the above conditions the system (15) takes the form

$$\dot{x} = a_{30}x^3, \qquad \dot{y} = b_{01}y, \quad a_{30}b_{01} \neq 0. \tag{52}$$

System (16). In this case: $\{(4),\ C_0(x,y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2) \equiv 0\} \Rightarrow \{|a_{30} - b_{21}| + |a_{21} - b_{12}| \neq 0,\ b_{21} = b_{12} = 0\} \Rightarrow C_1(x,y) = -b_{11}x^4y(a_{30}x + a_{21}y)(2a_{30}x + a_{21}y) \equiv 0 \Rightarrow b_{11} = 0 \Rightarrow C_2(x,y) = -b_{01}x^3y(a_{30}x + a_{21}y)(3a_{30}x + 2a_{21}y) \not\equiv 0, \ \ \mu_\infty = 3$. The system (16) has the form

$$\dot{x} = x^2(a_{30}x + a_{21}y), \ \ \dot{y} = b_{01}y, \ \ a_{30}b_{01} \neq 0. \tag{53}$$

Note that the system (52) is a particular case of the system (53), and after time rescaling the last system can be written in the form

$$\dot{x} = x^2(ax + by), \quad \dot{y} = y, \ \ a \neq 0 \tag{54}$$

(see system 3.1) of Main Theorem). The system (54) has not the third affine invariant straight line because $E_1(\mathbb{X}) = -x^3y(-a + 3a^2x^2 + 5abxy + 2b^2y^2)$. The conic $f \equiv -a + 3a^2x^2 + 5abxy + 2b^2y^2 = 0$ is reducible in $\mathbb{C}[x, y]$ only if $b = 0$, i.e. $f = a(-1 + 3ax^2)$, but $f = 0$ is not invariant for $\{(54),\ b = 0\}$. For system (54) we get $m_\infty(3, 1; 3) = m(3, 1; 3)$.

Remark 1. For the homogeneous systems associated to (18) (respectively, (21), (22), (23)) the polynomial $C_0(x, y)$ has the form $C_0(x, y) = (b_{21} - a_{30})x^5y(a_{30}b_{21}x^2 + 2a_{21}a_{30}xy + a_{21}^2y^2)$ and for these systems identity $C_0(x, y) \equiv 0$ holds if one of the following two series of conditions is satisfied:

$$A)\ a_{21} = b_{21} = 0 \qquad \text{and} \qquad B)\ a_{21} = a_{30} = 0.$$

System (18). In conditions A) (B)) we have $C_1(x, y) = -2a_{20}a_{30}^2x^6y \equiv 0$ $(C_1(x, y) = a_{20}b_{21}^2x^6y \equiv 0) \Rightarrow a_{20} = 0 \Rightarrow C_2(x, y) = -3a_{10}a_{30}^2x^5y \not\equiv 0$ $(C_2(x, y) = a_{10}b_{21}^2x^5y \not\equiv 0)$, $\mu_\infty = 3$. We obtain the following two systems:

$$\dot{x} = x(a_{30}x^2 + a_{10}), \quad \dot{y} = a_{10}y, \ \ a_{10}a_{30} \neq 0; \tag{55}$$

$$\dot{x} = a_{10}x, \quad \dot{y} = y(b_{21}x^2 + a_{10}), \ \ a_{10}b_{21} \neq 0. \tag{56}$$

The system (55) has four affine invariant straight lines: $l_1 = x$, $l_2 = y$, $l_{3,4} = x \pm \sqrt{-a_{10}/a_{30}}$ which, together with the line at infinity, form a sequence of multiplicities $(3, 1, 1, 1; 3)$.

System (21). Assume the conditions B) hold, then the system (21) is degenerate, i.e. $\deg(\gcd(P,\, Q)) > 0$ (see (4)). Let $a_{30} \neq 0$. Then, A) $\Rightarrow C_1(x, y) = -3a_{11}a_{30}^2x^5y^2 \not\equiv 0$, $\mu_\infty = 2$.

System (22). When the set of conditions A) (B)) is satisfied, then $C_1(x, y) = a_{20}b_{21}^2x^6y \not\equiv 0$ $(C_1(x, y) = -a_{30}^2x^5y(2a_{20}x + 3a_{11}y) \not\equiv 0)$, $\mu_\infty = 2$.

System (23). Under the conditions A) we have $C_1(x, y) = -a_{30}^2x^5y(2a_{20}x + 3a_{11}y) \not\equiv 0$, so $\mu_\infty = 2$. In the case of conditions B): $C_1(x, y) = a_{20}b_{21}^2x^6y \equiv 0 \Rightarrow a_{20} = 0$, $C_2(x, y) = b_{21}x^3y(a_{10}b_{21}x^2 + 2a_{11}^2y^2) \not\equiv 0$, $\mu_\infty = 3$. The system (23) takes the form

$$\dot{x} = x(a_{11}y + a_{10}), \quad \dot{y} = y(b_{21}x^2 + a_{11}y + a_{10}), \quad a_{10}a_{11}b_{21} \neq 0. \tag{57}$$

It is easy to show that for the systems (28), (29), (30), (31), (32) the algebraic multiplicity of the line at infinity is one.

Note that systems (56) and (57) may be combined in one system which after an affine transformation and time rescaling can be writing in the form

$$\dot{x} = x(ay + b), \quad \dot{y} = y(x^2 + ay + b), \ \ b \neq 0 \tag{58}$$

(see system 3.2) of the Main Theorem). For system (58) only the lines $x = 0$ and $y = 0$ are affine invariant straight lines as $E_1(\mathbb{X}) = x^3y(3b^2 + 5aby + bx^2 + 2a^2y^2)$ and the algebraic curve $3b^2 + 5aby + bx^2 + 2a^2y^2 = 0$ is not invariant for (58). For system (58) we have $m_\infty(3, 1; 3) = m(3, 1; 3)$.

Lemma 8. *Any cubic system of the class* $\mathbb{CSL}_{2(r)}^{\times}$ *with invariant straight lines of the maximal multiplicity* $m(3, 1; 3)$ *via an affine transformation and time rescaling can be written in the form* (54) *or* (58).

2.2.4 Case $m_\infty(2,2;\mu_\infty)$

In Section 2.2.2 we have obtained the canonical forms of the systems (see Lemma 7) which have the maximal sequence $m(3,2;2)$. For each of these systems the affine invariant straight line $x=0$ $(y=0)$ has the algebraic multiplicity three (two) and the line at infinity l_∞ has multiplicity two. The Poincaré transformation $z=1/x,\ u=y/x$ sends: the line $x=0$ into the line at infinity of the phase plane Ozu, the line at infinity of the phase plane Oxy into the line $z=0$, the line $y=0$ into the line $u=0$, and preserves the multiplicities. This transformation reduces the systems (50) and (51) to the cubic systems, respectively

$$\dot z=-az,\quad \dot u=-u(a-zu);\tag{59}$$

$$\dot z=-z(a+zu),\quad \dot u=-au.\tag{60}$$

Putting in (59) ((60)) $z=x,u=y,t=-\tau/a,a=-1/b$ $(z=y,u=x,t=-\tau/a,a=1/b)$ we obtain the system

$$\dot x=x,\quad \dot y=y(1+bxy),\quad b\neq 0.\tag{61}$$

Lemma 9. *Any cubic system of the class* $\mathbb{CSL}^{\times}_{2(r)}$ *with straight lines of the maximal multiplicity* $m(2,2;3)$ *via an affine transformation and time rescaling can be written in the form* (61).

2.2.5 Case $m_\infty(2,1;\mu_\infty)$

We will examine the sets of conditions (7)–(12) under which the cubic system (5) admits the invariant straight lines $x=0$ and $y=0$ of multiplicities $\mu_1=2$ and $\mu_2=1$, respectively.

1) Conditions (7).

When for cubic system (5) the conditions (7) hold we have $C_0(x,y)=-x^2yC_{01}(x,y)\cdot C_{02}(x,y)$, where $C_{01}(x,y)=((a_{30}-b_{21})x^2+(a_{21}-b_{12})xy-b_{03}y^2)$, $C_{02}(x,y)=(a_{30}b_{21}x^3+2a_{30}b_{12}x^2y+(3a_{30}b_{03}+a_{21}b_{12})xy^2+2a_{21}b_{03}y^3)$.

Taking into account conditions (4) the polynomial $C_{01}(x,y)$ can not be identically zero, so we will require for $C_{02}(x,y)$ to be identically zero. In this case the multiplicity is $\mu_\infty\geq 2$ if one of the following three series of conditions is satisfied

$$a_{30}=a_{21}=0;\tag{62}$$

$$a_{30}=b_{12}=b_{03}=0,\ a_{21}\neq 0;\tag{63}$$

$$b_{21}=b_{12}=b_{03}=0,\ a_{30}\neq 0.\tag{64}$$

The conditions $\{(62),(4)\}$ give us $C_1(x,y)=a_{20}x^2y(b_{21}x^2+b_{12}xy+b_{03}y^2)(b_{21}x^2+2b_{12}xy+3b_{03}y^2)\not\equiv 0,\ \mu_\infty=2$.

For conditions $\{(63),\ (4)\}$ we get $C_1(x,y)=x^3y(a_{20}b_{21}^2x^3+(a_{21}b_{02}b_{21}-a_{21}^2b_{11})xy^2-2a_{21}^2b_{02}y^3)\equiv 0\Rightarrow b_{02}=b_{11}=b_{21}=0\Rightarrow C_2(x,y)=-2a_{21}^2b_{01}x^3y^3\not\equiv 0$, $\mu_\infty=3$. The system $\{(5),\ (4)\}$ has the form

$$\dot x=x^2(a_{20}+a_{21}y),\quad \dot y=b_{01}y,\quad a_{20}a_{21}b_{01}\neq 0.\tag{65}$$

In the case of conditions $\{(64),(4)\}$ we have: $C_1(x,y) = -x^3y(a_{30}x + a_{21}y)\cdot (2a_{30}b_{11}x^2 + 3a_{30}b_{02}xy + a_{21}b_{11}xy + 2a_{21}b_{02}y^2) \equiv 0 \Rightarrow b_{11} = b_{02} = 0 \Rightarrow C_2(x,y) = -b_{01}x^3y(a_{30}x + a_{21}y)(3a_{30}x + 2a_{21}y) \neq 0,\ \mu_\infty = 3$. We obtain the following cubic system

$$\dot{x} = x^2(a_{30}x + a_{21}y + a_{20}), \quad \dot{y} = b_{01}y, \quad a_{30}a_{21}b_{01} \neq 0. \tag{66}$$

After time rescaling $t = \tau/b_{01}$ the systems (65) and (66) can be combined into the system

$$\dot{x} = x^2(a + bx + cy), \quad \dot{y} = y, \ \ c(a^2 + b^2) \neq 0. \tag{67}$$

(see the system 5.1) of Main Theorem).

2) Conditions (8).

Taking into account (4) the polynomial $C_0(x,y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2)$ is identically zero if one of the following three series of conditions is fulfilled: $a_{30} = a_{21} = 0$, i.e. (62), and

$$a_{30} = b_{12} = 0, a_{21} \neq 0; \tag{68}$$

$$b_{21} = b_{12} = 0, a_{30} \neq 0. \tag{69}$$

Under the conditions (62) we have: $\{(4);\ C_1(x,y) = a_{20}x^4y(b_{21}x + b_{12}y)(b_{21}x + 2b_{12}y) \equiv 0\} \Rightarrow \{(4);\ a_{20} = 0\} \Rightarrow C_2(x,y) = a_{10}x^3y(b_{21}x + b_{12}y)(b_{21}x + 2b_{12}y) \not\equiv 0$, $\mu_\infty = 3$. The cubic system looks as

$$\dot{x} = a_{10}x, \quad \dot{y} = y(a_{10} + b_{11}x + b_{21}x^2 + b_{12}xy), \quad a_{10}(b_{21}^2 + b_{12}^2) \neq 0. \tag{70}$$

The conditions (68) give us $C_1(x,y) = x^4y(a_{20}b_{21}^2x^2 - a_{21}^2b_{11}y^2)$. The multiplicity is $\mu_1 = 2$, $\mu_2 = 1$ and $\mu_\infty \geq 3$, if $b_{11} = a_{20} = 0, b_{21} \neq 0$ or $b_{11} = b_{21} = 0, a_{20} \neq 0$. Thus, we have the following two systems, respectively

$$\dot{x} = x(a_{10} + a_{21}xy), \quad \dot{y} = y(a_{10} + b_{21}x^2), \quad a_{10}b_{21}a_{21} \neq 0; \tag{71}$$

$$\dot{x} = x(a_{10} + a_{20}x + a_{21}xy), \quad \dot{y} = a_{10}y, \quad a_{10}a_{20}a_{21} \neq 0. \tag{72}$$

For $\{(71),(4)\}$ $(\{(72),(4)\})$ the polynomial $C_2(x,y) \equiv a_{10}x^3y(b_{21}x - a_{21}y)(b_{21}x + 2a_{21}y)$ $(C_2(x,y) \equiv -2a_{10}a_{21}^2x^3y^3)$ is not identically zero, therefore $\mu_\infty = 3$.

For conditions (69): $C_1(x,y) = -b_{11}x^4y(a_{30}x + a_{21}y)(2a_{30}x + a_{21}y) \equiv 0 \Rightarrow b_{11} = 0$; $\{b_{11} = 0,(4)\} \Rightarrow C_2(x,y) \equiv -a_{10}x^3y(a_{30}x + a_{21}y)(3a_{30}x + 2a_{21}y) \neq 0$, $\mu_\infty = 3 \Rightarrow$

$$\dot{x} = x(a_{10} + a_{20}x + a_{30}x^2 + a_{21}xy), \quad \dot{y} = a_{10}y, \quad a_{10}a_{21}a_{30} \neq 0. \tag{73}$$

The system $\{(70),\ b_{11} = 0, b_{21}b_{12} \neq 0\}$ (respectively, (71) and $\{(73),\ a_{20} = 0,\ a_{30}a_{21} \neq 0\}$) has the affine straight lines $l_1 = x, l_2 = y, l_3 = b_{21}x + b_{12}y$ (respectively, $l_3 = b_{21}x - a_{21}y$ and $l_3 = a_{30}x + a_{21}y$) and it realizes the sequence of multiplicities $(2,1,1;3)$. If for differential system (70): $b_{11} = b_{21} = 0$ $(b_{11} = b_{12} = 0)$, then $\mu_1 = 3 > 2$ $(\mu_2 = 2 > 1)$. Let $a_{10}b_{11}(b_{21}^2 + b_{12}^2) \neq 0$, then, after the time rescaling and change of notation of the coefficients, we can write (70) in the form

$$\dot{x} = x, \quad \dot{y} = y(1 + ax + bx^2 + cxy), \quad a(b^2 + c^2) \neq 0 \tag{74}$$

(see the system 5.2) of Main Theorem).

After time rescaling and change of notation of the coefficients, the systems (72) and (73) can be combined into the system

$$\dot{x} = x(1 + ax + bx^2 + cxy), \quad \dot{y} = y, \quad c(a^2 + b^2) \neq 0. \tag{75}$$

(see the system 5.3) of Main Theorem).

3) Conditions (9).

Taking into account (4) the polynomial $C_0(x, y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2)$ is identically zero if one of the conditions (62), (68), (69) is satisfied.

When the conditions $\{(62), (4)\}$ ($\{(68), (4)\}$ and $\{(69), (4)\}$) hold we obtain $C_1(x, y) = x^3y(b_{21}x + b_{12}y)(a_{20}b_{21}x^2 + 2a_{20}b_{12}xy + a_{11}b_{12}y^2) \not\equiv 0$ (respectively, $C_1(x, y) = x^3y(a_{20}b_{21}^2x^3 - a_{21}^2b_{11}xy^2 + 2a_{11}a_{21}b_{21}xy^2 - 2a_{11}a_{21}^2y^3) \not\equiv 0$ and $C_1(x, y) = -x^3y(a_{30}x + a_{21}y)(2a_{30}b_{11}x^2 + 3a_{11}a_{30}xy + a_{21}b_{11}xy + 2a_{11}a_{21}y^2) \not\equiv 0$), $\mu_\infty = 2$.

4) Conditions (10).

In this case we get $C_0(x, y) = -x^4y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^2 + 2a_{30}b_{12}xy + a_{21}b_{12}y^2)$ and $C_0(x, y)$ is identically zero if at least one of the conditions (62), (68), (69) is satisfied.

For conditions (62) we find $\{(4),\ C_1(x, y) = x^3y(b_{21}x + b_{12}y)(a_{20}b_{21}x^2 + 2a_{20}b_{12}xy + a_{11}b_{12}y^2) \equiv 0\} \Rightarrow \{(4),\ a_{20} = b_{12} = 0\} \Rightarrow C_2(x, y) = b_{21}x^3y(a_{10}b_{21}x^2 + 2a_{11}^2y^2) \not\equiv 0,\ \mu_\infty = 3$. The cubic system looks as

$$\dot{x} = x(a_{11}y + a_{10}), \quad \dot{y} = y(b_{21}x^2 + b_{11}x + a_{11}y + a_{10}), \quad a_{10}a_{11}b_{21} \neq 0. \tag{76}$$

If $b_{11} = 0$, then the invariant straight line $x = 0$ of (76) has multiplicity $\mu_1 = 3$. Let $b_{11} \neq 0$. Via rescaling the time and change of notation of coefficients, the system (76) can be reduced to the system

$$\dot{x} = x(1 + ay), \quad \dot{y} = y(1 + bx + ay + cx^2), \quad abc \neq 0 \tag{77}$$

(see the system 5.4) of Main Theorem).

In the cases (68) and (69) we have respectively $C_1(x, y) \equiv x^3y(a_{20}b_{21}^2x^3 - a_{21}^2b_{11}xy^2 + 2a_{11}a_{21}b_{21}xy^2 - 2a_{11}a_{21}^2y^3) \neq 0$ and $C_1(x, y) = -x^3y(a_{30}x + a_{21}y) \cdot (2a_{30}b_{11}x^2 + (3a_{11}a_{30} + a_{21}b_{11})xy + 2a_{11}a_{21}y^2) \not\equiv 0$, thus μ_∞ can not be greater than two.

5) Conditions (11) *and Conditions* (12). Taking into account (4), in each of this conditions, we have $C_0(x, y) = -x^2y((a_{30} - b_{21})x + (a_{21} - b_{12})y)(a_{30}b_{21}x^4 + 2a_{30}b_{12}x^3y + (3a_{12}a_{30} + a_{21}b_{12} - a_{12}b_{21})x^2y^2 + 2a_{12}a_{21}xy^3 + a_{12}^2y^4) \not\equiv 0,\ \mu_\infty = 1$.

Lemma 10. *Any cubic system of the class $\mathbb{CSL}_{2(r)}^{\times}$ with straight lines of the partially maximal multiplicity $m_\infty(2, 1; 3)$ via an affine transformation and time rescaling can be written in one of the following four forms* (67), (70), (75) *and* (77).

2.2.6 Case $m_\infty(1,1;\mu_\infty)$

We consider the homogenized system associated to the system (5)

$$\begin{cases} \dot{x} = x(a_{10}Z^2 + a_{20}xZ + a_{11}yZ + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01}Z^2 + b_{11}xZ + b_{02}yZ + b_{21}x^2 + b_{12}xy + b_{03}y^2). \end{cases} \quad (78)$$

For (78) we have $C_0(x,y) = -xyC_{01}(x,y)C_{02}(x,y)$, where $C_{01}(x,y) = (a_{30}-b_{21})x^2 + (a_{21}-b_{12})xy + (a_{12}-b_{03})y^2$ and $C_{02}(x,y) = (a_{30}b_{21}x^4 + 2a_{30}b_{12}x^3y + (3a_{30}b_{03} + a_{21}b_{12} - a_{12}b_{21})x^2y^2 + 2a_{21}b_{03}xy^3 + a_{12}b_{03}y^4)$. If $C_{01} \equiv 0$, then the system (78) has degenerate infinity. Let $C_{01} \not\equiv 0$. The identity $C_{02}(x,y) \equiv 0$ holds if at least one of the following four series of conditions is fulfilled

$$a_{30} = a_{21} = a_{12} = 0; \quad (79)$$

$$a_{30} = a_{21} = b_{21} = b_{03} = 0,\ a_{12} \neq 0; \quad (80)$$

$$a_{30} = b_{03} = 0, b_{12} = a_{12}b_{21}/a_{21}; \quad (81)$$

$$b_{21} = b_{12} = b_{03} = 0, a_{30} \neq 0. \quad (82)$$

1) Conditions $\{(79),\ (4)\}$: $C_1(x,y) = -xyC_{01}(x,y)(a_{20}b_{21}x^3 + 2a_{20}b_{12}x^2y + 3a_{20}b_{03}xy^2 + a_{11}b_{12}xy^2 + 2a_{11}b_{03}y^3) \equiv 0 \Rightarrow$

$$a_{20} = a_{11} = 0 \quad (83)$$

or

$$a_{20} = b_{12} = b_{03} = 0, a_{11} \neq 0. \quad (84)$$

For conditions $\{(83),\ (4)\}$ we have the system

$$\begin{gathered} \dot{x} = a_{10}x, \quad \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2), \\ a_{10}(b_{21}^2 + b_{12}^2 + b_{03}^2)(b_{01}^2 + b_{02}^2 + b_{03}^2) \neq 0 \end{gathered} \quad (85)$$

for which $C_2(x,y) = -a_{10}xyC_{01}(x,y)(b_{21}x^2 + 2b_{12}xy + 3b_{03}y^2) \not\equiv 0, \quad \mu_\infty = 3$, and for conditions $\{(84),\ (4)\}$ the cubic system looks as

$$\dot{x} = x(a_{10}+a_{11}y), \quad \dot{y} = y(b_{01}+b_{11}x+b_{02}y+b_{21}x^2), \quad a_{10}a_{11}b_{21}(b_{01}^2+b_{02}^2) \neq 0. \quad (86)$$

For (86) we find $C_2(x,y) = b_{21}x^3y(a_{10}b_{21}x^2 + a_{11}^2y^2 + a_{11}b_{02}y^2) \not\equiv 0,\ \mu_\infty = 3$. Via rescaling the time an change of notation of coefficients, (85) can be reduced to the system

$$\dot{x} = x,\ \dot{y} = y(a + bx + cy + dx^2 + exy + fy^2),\ (a^2 + c^2 + f^2)(d^2 + e^2 + f^2) \neq 0 \quad (87)$$

(see the system 6.1) of Main Theorem).

In 6.1) the condition $(a^2 + b^2 + d^2)((a-1)^2 + (c^2d - bce + b^2f)^2) \neq 0$ means that the system (87) has only the following two affine invariant straight lines $x = 0,\ y = 0$ and the condition $((a-1)^2 + c^2 + f^2)((a-1)^2 + b^2 + d^2) \neq 0$ means that each of these affine straight lines has the algebraic multiplicity one.

2) Conditions $\{(80), (4)\}$. The polynomial $C_1(x,y) = xy^3(2a_{20}b_{12}^2x^3 - b_{12}(a_{12}a_{20} + a_{12}b_{11} - a_{11}b_{12})x^2y - a_{12}^2b_{02}y^3)$ is identically zero if one of the following two series of conditions is satisfied

$$b_{02} = b_{12} = 0; \tag{88}$$

$$a_{20} = b_{02} = 0, b_{11} = a_{11}b_{12}/a_{12},\ b_{12} \neq 0. \tag{89}$$

The conditions $\{(88), (4)\}$ and $\{(89), (4)\}$ lead us, respectively, to the following two systems

$$\dot{x} = x(a_{12}y^2 + a_{20}x + a_{11}y + a_{10}), \quad \dot{y} = y(b_{11}x + b_{01}),\ a_{12}b_{01}(a_{10}^2 + a_{20}^2) \neq 0, \tag{90}$$

$C_2(x,y) = -a_{12}xy^3(b_{11}(a_{20} + b_{11})x^2 + a_{12}b_{01}y^2) \not\equiv 0,\ \mu_\infty = 3;$

$$\dot{x} = x(a_{12}y^2 + a_{11}y + a_{10}),\ \dot{y} = y(a_{12}b_{12}xy + a_{11}b_{12}x + a_{12}b_{01})/a_{12}, a_{10}b_{12}b_{01} \neq 0, \tag{91}$$

$C_2(x,y) = -xy^3(-2a_{10}b_{12}^2x^2 + a_{12}b_{01}b_{12}xy + a_{12}^2b_{01}y^2) \not\equiv 0,\ \mu_\infty = 3.$

Via an affine transformation of coordinates and time rescaling (90) can be reduced to the system

$$\dot{x} = x(a + by),\ \ \dot{y} = y(c + dx + ey + x^2),\ \ a(c^2 + e^2) \neq 0 \tag{92}$$

(see system 6.2) of the Main Theorem). In 6.2) the inequality $(a-c)^2 + (b-e)^2 \neq 0$ means that $\mu_1 = 1$.

Note that (86) modulo time rescaling is a particular case of the system (92).

3) Conditions $\{(81), (4)\}$. In this case the polynomial $C_1(x,y) = -xy(a_{21}x + a_{12}y)(-a_{20}a_{21}b_{21}^2x^4 - 2a_{12}a_{20}b_{21}^2x^3y + (a_{21}^3b_{11} + a_{12}a_{20}a_{21}b_{21} - a_{11}a_{21}^2b_{21} - a_{21}^2b_{02}b_{21} + a_{12}a_{21}b_{11}b_{21} - a_{11}a_{12}b_{21}^2)x^2y^2 + 2a_{21}^3b_{02}xy^3 + a_{12}a_{21}^2b_{02}y^4)/a_{21}^2$ is identically zero if one of the following three series of conditions is satisfied:

$$b_{11} = b_{02} = b_{21} = 0, a_{20} \neq 0; \tag{93}$$

$$a_{20} = b_{02} = 0, a_{12} = -a_{21}^2/b_{21}; \tag{94}$$

$$a_{20} = b_{02} = 0, b_{11} = a_{11}b_{21}/a_{21}. \tag{95}$$

The conditions (93), (94), (95) give us, respectively, the systems:

$$\dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{21}xy + a_{12}y^2),\ \dot{y} = b_{01}y,\ \ b_{01}(a_{10}^2 + a_{20}^2)(a_{21}^2 + a_{12}^2) \neq 0 \tag{96}$$

with $C_2(x,y) \equiv -b_{01}xy^3(a_{21}x + a_{12}y)(2a_{21}x + a_{12}y) \neq 0;$

$$\begin{aligned} &\dot{x} = x(a_{10}b_{21} + a_{11}b_{21}y + a_{21}b_{21}xy - a_{21}^2y^2)/b_{21}, \\ &\dot{y} = y(b_{01} + b_{11}x + b_{21}x^2 - a_{21}xy),\ a_{10}b_{01} \neq 0 \end{aligned} \tag{97}$$

with $C_2(x,y) \equiv xy(a_{10}b_{21}^4x^4 - 2a_{10}a_{21}b_{21}^3x^3y + a_{21}^2b_{11}^2b_{21}x^2y^2 + a_{10}a_{21}^2b_{21}^2x^2y^2 - a_{21}^2b_{01}b_{21}^2x^2y^2 - 2a_{11}a_{21}b_{11}b_{21}^2x^2y^2 + a_{11}^2b_{21}^3x^2y^2 + 2a_{21}^3b_{01}b_{21}xy^3 - a_{21}^4b_{01}y^4)/b_{21}^2 \neq 0;$

$$\begin{aligned} &\dot{x} = x(a_{10} + a_{11}y + a_{21}xy + a_{12}y^2), \\ &\dot{y} = y(a_{21}b_{01} + a_{11}b_{21}x + a_{21}b_{21}x^2 + a_{12}b_{21}xy)/a_{21} \end{aligned} \tag{98}$$

with $C_2(x,y) \equiv -xy(a_{21}x + a_{12}y)(-a_{10}a_{21}b_{21}^2x^3 - a_{10}a_{21}^2b_{21}x^2y - 2a_{10}a_{12}b_{21}^2x^2y + 2a_{21}^3b_{01}xy^2 + a_{12}a_{21}b_{01}b_{21}xy^2 + a_{12}a_{21}^2b_{01}y^3)/a_{21}^2 \neq 0$. Thus, in the case of conditions $\{(81), (4)\}$ the multiplicity μ_∞ is three.

Via an affine transformation of coordinates and time rescaling (96) can be reduced to the system (85). If $a_{21} = 0$, then the system (97) is modulo time rescaling a particular case of the system (92). Let $a_{21} \neq 0$. Then, after the time rescaling $t \to -b_{21}t/a_{21}^2$, the system (97) has the form

$$\dot{x} = x(a + by + cxy + y^2),\ \dot{y} = -y(d + ex + c^2x^2 + cxy),\ \ ad \neq 0, \qquad (99)$$

where $a = -a_{10}b_{21}/a_{21}^2, b = -a_{11}b_{21}/a_{21}^2, c = -b_{21}/a_{21}, d = b_{01}b_{21}/a_{21}^2); e = b_{11}b_{21}/a_{21}^2)$ (see the system 6.3) of the Main Theorem). In 6.3) the condition $c^2 + e^2 + (a + d)^2 \neq 0$ $((a + d)^2 + (bc - e)^2 \neq 0)$ means that $\mu_2 = 1$ (only $x = 0$ and $y = 0$ are affine invariant straight lines for 6.3)).

If $b_{21} = 0$, then the system (98) modulo affine transformation and time rescaling is a particular case of the system (87). Let $b_{21} \neq 0$. The time rescaling $t \to b_{21}t/(a_{21}b_{01})$ reduces (98) to the following system

$$\dot{x} = x(a + by + cxy + dy^2),\ \dot{y} = \alpha y(1 + bx + cx^2 + dxy),\ \ \alpha a(c^2 + d^2) \neq 0, \quad (100)$$

where $a = a_{10}b_{21}/(a_{21}b_{01})$, $b = a_{11}b_{21}/(a_{21}b_{01})$, $c = b_{21}/b_{01}$, $d = a_{12}b_{21}/(a_{21}b_{01})$, $\alpha = b_{21}/a_{21}$ (see the system 6.4) of the Main Theorem). In 6.4) the inequality $\alpha - a \neq 0$ means that the differential system has only the affine invariant straight lines $x = 0$ and $y = 0$.

4) Conditions $\{(82), (4)\}$:

$C_1(x,y) = -xy(a_{30}x^2 + a_{21}xy + a_{12}y^2)(2a_{30}b_{11}x^3 + (3a_{30}b_{02} + a_{21}b_{11})x^2y + 2a_{21}b_{02}xy^2 + a_{12}b_{02}y^3) \equiv 0 \Rightarrow b_{11} = b_{02} = 0 \Rightarrow C_2(x,y) = -b_{01}xy(a_{30}x^2 + a_{21}xy + a_{12}y^2)(3a_{30}x^2 + 2a_{21}xy + a_{12}y^2) \not\equiv 0 \Rightarrow \mu_\infty = 3$. The cubic system looks as:

$$\begin{aligned} &\dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2),\ \dot{y} = b_{01}y, \\ &\quad a_{30}b_{01}(a_{10}^2 + a_{20}^2 + a_{30}^2) \neq 0. \end{aligned} \qquad (101)$$

Modulo affine transformation the system (101) is a particular case of the system (85).

Lemma 11. *Any cubic system of the class* $\mathbb{CSL}_{2(r)}^{\times}$ *with straight lines of the partially maximal multiplicity* $m_\infty(1, 1; 3)$ *via an affine transformation and time rescaling can be written in one of the following four forms* (87), (92), (99) *and* (100).

The proof of the Main Theorem follows from Lemmas 8–11.

2.3 Geometric multiplicity

In this section for the normal forms given in Main Theorem we construct the corresponding perturbed cubic systems which show that for invariant straight lines ($x = 0$, $y = 0$ and $Z = 0$) the algebraic and geometric multiplicities coincide.

1) $m(3,3;1)$: $\dot{x} = x^3, \quad \dot{y} = y(x^2 + ay + by^2),\ b \neq 0.$

The perturbed cubic system is

$\dot{x} = x(x - a\epsilon + 2bx\epsilon^2)(x + a\epsilon + 2bx\epsilon^2), \quad \dot{y} = y(x^2 + ay + by^2 + a^2\epsilon^2 + 3bx^2\epsilon^2 + 4aby\epsilon^2 + +4b^2y^2\epsilon^2 + a^2b\epsilon^4 + 4ab^2y\epsilon^4 + 4b^3y^2\epsilon^4 - 4b^3x^2\epsilon^6),\ b \neq 0.$

The invariant straight lines are $l_1 = x,\ l_2 = y,\ l_3 = x - a\epsilon + 2bx\epsilon^2,\ l_4 = x + a\epsilon + 2bx\epsilon^2,\ \ l_5 = y - x\epsilon + a\epsilon^2 + 2by\epsilon^2 - 2bx\epsilon^3,\ l_6 = y + x\epsilon + a\epsilon^2 + 2by\epsilon^2 + 2bx\epsilon^3.$

If $\epsilon \to 0$, then the invariant straight lines $l_1, l_3, l_4 \to l_1$ and $l_2, l_5, l_6 \to l_2$.

2.1) $m(3,2;2)$: $\dot{x} = ax^3, \quad \dot{y} = y^2, a \neq 0.$

The perturbed cubic system is $\dot{x} = ax(x-\epsilon)(x+\epsilon), \quad \dot{y} = y(y-\epsilon)(\epsilon y+1), a \neq 0.$

The invariant straight lines are $l_1 = x,\ l_2 = y,\ l_3 = x - \epsilon,\ l_4 = x + \epsilon,\ l_5 = y - \epsilon,\ l_6 = \epsilon y + 1.$

If $\epsilon \to 0$, then the invariant straight lines $l_1, l_3, l_4 \to l_1$; $l_2, l_5 \to l_2$ and $l_6 \to l_\infty$.

2.2) $m(3,2;2)$: $\dot{x} = x(ax^2 + y), \quad \dot{y} = y^2, \ a \neq 0.$

The perturbed cubic system is $\dot{x} = x(ax^2 + y + \epsilon - a\epsilon^4), \quad \dot{y} = y(y+\epsilon)(1 + ay\epsilon^2 - a\epsilon^3), a \neq 0.$

The invariant straight lines are $l_1 = x,\ l_2 = y,\ l_3 = x - y\epsilon,\ l_4 = x + y\epsilon,\ l_5 = y - \epsilon,\ l_6 = ay\epsilon^2 - a\epsilon^3 + 1.$

If $\epsilon \to 0$, then the invariant straight lines $l_1, l_3, l_4 \to l_1$; $l_2, l_5 \to l_2$ and $l_6 \to l_\infty$.

3.1) $m(3,1;3)$: $\dot{x} = x^2(ax + by), \quad \dot{y} = y, \ a \neq 0.$

The perturbed cubic system is

$\dot{x} = x(ax^2 + bxy - a\epsilon^2 + 4a^2x^2\epsilon^2 + 4abxy\epsilon^2 + 2b^2y^2\epsilon^2 - 4a^2\epsilon^4 + 4a^3x^2\epsilon^4 + 4a^2bxy\epsilon^4 + ab^2y^2\epsilon^4 - 4a^3\epsilon^6), \ \dot{y} = y(-1 + by\epsilon - 2a\epsilon^2)(1 + by\epsilon + 2a\epsilon^2), \ a \neq 0.$

The invariant straight lines are $l_1 = x,\ l_2 = y,\ l_3 = x - \epsilon + 2ax\epsilon^2 + by\epsilon^2 - 2a\epsilon^3,\ l_4 = x + \epsilon + 2ax\epsilon^2 + by\epsilon^2 + 2a\epsilon^3,\ l_5 = by\epsilon - 2a\epsilon^2 - 1,\ l_6 = by\epsilon + 2a\epsilon^2 + 1.$

If $\epsilon \to 0$, then invariant straight lines $l_1, l_3, l_4 \to l_1$ and $l_5, l_6 \to l_\infty$.

3.2) $m(3,1;3)$: $\dot{x} = x(ay + b), \quad \dot{y} = y(x^2 + ay + b), \ b \neq 0.$

The perturbed cubic system is

$\dot{x} = -x(-b - ay - 4b^2\epsilon^2 + bx^2\epsilon^2 - 4aby\epsilon^2 - 2a^2y^2\epsilon^2 - 4b^3\epsilon^4 + 4b^2x^2\epsilon^4 - 4ab^2y\epsilon^4 - a^2by^2\epsilon^4 + 4b^3x^2\epsilon^6), \ \dot{y} = y(b + x^2 + ay + 4b^2\epsilon^2 + 3bx^2\epsilon^2 + 4aby\epsilon^2 + a^2y^2\epsilon^2 + 4b^3\epsilon^4 + 4ab^2y\epsilon^4 + a^2by^2\epsilon^4 - 4b^3x^2\epsilon^6), \ b \neq 0.$

The invariant straight lines are $l_1 = x,\ l_2 = y,\ l_3 = x - ay\epsilon + 2bx\epsilon^2,\ l_4 = x + ay\epsilon + 2bx\epsilon^2,\ l_5 = x\epsilon - 2b\epsilon^2 - ay\epsilon^2 + 2bx\epsilon^3 - 1,\ l_6 = x\epsilon + 2b\epsilon^2 + ay\epsilon^2 + 2bx\epsilon^3 + 1.$

If $\epsilon \to 0$, then invariant straight lines $l_1, l_3, l_4 \to l_1$ and $l_5, l_6 \to l_\infty$.

4) $m(2,2;3)$: $\dot{x} = x, \quad \dot{y} = y(1 + bxy), \ b \neq 0.$

The perturbed cubic system is $\dot{x} = -x(x\epsilon - 1)(x\epsilon + 1), \quad \dot{y} = y(1 + bxy + by^2\epsilon - y^2\epsilon^4), \ b \neq 0.$

The invariant straight lines are $l_1 = x,\ l_2 = y,\ l_3 = x + \epsilon y,\ l_4 = by + x\epsilon^2 - y\epsilon^3,\ l_5 = x\epsilon + 1,\ l_6 = x\epsilon - 1.$

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$; $l_2, l_4 \to l_2$ and $l_5, l_6 \to l_\infty$.

5.1) $m_\infty(2,1;3)$: $\dot{x} = x^2(a+bx+cy)$, $\dot{y} = y$, $c(a^2+b^2) \neq 0$.

The perturbed cubic system is $\dot{x} = x(a+bx+cy)(x+\epsilon)$, $\dot{y} = -y(-1+\epsilon y)(1+\epsilon y)$, $c(a^2+b^2) \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x+\epsilon$, $l_4 = \epsilon y - 1$, $l_5 = \epsilon y + 1$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

5.2) $m_\infty(2,1;3)$: $\dot{x} = x$, $\dot{y} = y(1+ax+bx^2+cxy)$, $a(b^2+c^2) \neq 0$.

The perturbed cubic system is

$\dot{x} = -x(-1+x\epsilon)(1+x\epsilon)$, $\dot{y} = y(1+ax+bx^2+cxy+ay\epsilon+bxy\epsilon+cy^2\epsilon - x^2\epsilon^2)$, $a(b^2+c^2) \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x+\epsilon y$, $l_4 = \epsilon x + 1$, $l_5 = \epsilon x - 1$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

5.3) $m_\infty(2,1;3)$: $\dot{x} = x(1+ax+bx^2+cxy)$, $\dot{y} = y$, $c(a^2+b^2) \neq 0$;

The perturbed cubic system is

$\dot{x} = x(1+ax+bx^2+cxy+ay\epsilon+bxy\epsilon+cy^2\epsilon-y^2\epsilon^2)$, $\dot{y} = -y(-1+y\epsilon)(1+y\epsilon)$, $c(a^2+b^2) \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x+\epsilon y$, $l_4 = \epsilon y + 1$, $l_5 = \epsilon y - 1$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

5.4) $m_\infty(2,1;3)$: $\dot{x} = x(1+ay)$, $\dot{y} = y(1+bx+ay+cx^2)$, $abc \neq 0$.

The perturbed cubic system is

$\dot{x} = x(1+x\epsilon)(1+ay-x\epsilon)$, $\dot{y} = y(c^2+bc^2x+c^3x^2+ac^2y+2bc\epsilon+2b^2cx\epsilon+2bc^2x^2\epsilon+abcy\epsilon-2ac^2xy\epsilon+b^2\epsilon^2+4c\epsilon^2+b^3x\epsilon^2+4bcx\epsilon^2+b^2cx^2\epsilon^2+3c^2x^2\epsilon^2+4acy\epsilon^2-abcxy\epsilon^2+2a^2cy^2\epsilon^2+4b\epsilon^3+4b^2x\epsilon^3+2bcx^2\epsilon^3+2aby\epsilon^3+ab^2xy\epsilon^3-2acxy\epsilon^3+4\epsilon^4+4bx\epsilon^4-b^2x^2\epsilon^4+4ay\epsilon^4+4abxy\epsilon^4-4bx^2\epsilon^5+4axy\epsilon^5-4x^2\epsilon^6)/(c+b\epsilon+2\epsilon^2)^2$, $c \neq 0$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = -cx-bx\epsilon+ay\epsilon-2x\epsilon^2$, $l_4 = x\epsilon+1$, $l_5 = c+b\epsilon-cx\epsilon+2\epsilon^2-bx\epsilon^2+2ay\epsilon^2-2x\epsilon^3$.

If $\epsilon \to 0$, then $l_1, l_3 \to l_1$ and $l_4, l_5 \to l_\infty$.

6.1) $m_\infty(1,1;3)$: $\dot{x} = x$, $\dot{y} = y(a+bx+cy+dx^2+exy+fy^2)$, $(a^2+c^2+f^2)(d^2+e^2+f^2)(a^2+b^2+d^2)((a-1)^2+c^2+f^2)((a-1)^2+b^2+d^2)((a-1)^2+(c^2d-bce+b^2f)^2) \neq 0$;

The perturbed cubic system is $\dot{x} = x(\epsilon x+1)(\epsilon x-1)$, $\dot{y} = y(a+bx+cy+dx^2+exy+fy^2)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = \epsilon x+1$, $l_4 = \epsilon x - 1$.

If $\epsilon \to 0$, then $l_3, l_4 \to l_\infty$.

6.2) $m_\infty(1,1;3)$: $\dot{x} = x(a+by)$, $\dot{y} = y(c+dx+ey+x^2)$, $a(c^2+e^2)((a-c)^2+(b-e)^2) \neq 0$.

The perturbed cubic system is

$\dot{x} = -x(1+x\epsilon)(-a-by+x\epsilon)$, $\dot{y} = y(a^5c+a^5dx+a^5x^2+a^5ey+2a^4cd\epsilon-a^5x\epsilon+a^6x\epsilon+2a^4d^2x\epsilon+2a^4dx^2\epsilon+2a^4dey\epsilon-a^4bxy\epsilon-a^4exy\epsilon+2a^5c\epsilon^2+2a^3c^2\epsilon^2+a^3cd^2\epsilon^2-$

$2a^4dx\epsilon^2 + 4a^5dx\epsilon^2 + 2a^3cdx\epsilon^2 + a^3d^3x\epsilon^2 + a^5x^2\epsilon^2 + 2a^3cx^2\epsilon^2 + a^3d^2x^2\epsilon^2 + 2a^5ey\epsilon^2 + 2a^3cey\epsilon^2 + a^3d^2ey\epsilon^2 - a^3bdxy\epsilon^2 + a^4bdxy\epsilon^2 - 2a^3dexy\epsilon^2 + a^3bey^2\epsilon^2 + a^4bey^2\epsilon^2 + 2a^4cd\epsilon^3 + 2a^2c^2d\epsilon^3 - 2a^5x\epsilon^3 + 2a^6x\epsilon^3 - 2a^3cx\epsilon^3 + 2a^4cx\epsilon^3 - a^3d^2x\epsilon^3 + 3a^4d^2x\epsilon^3 + 2a^2cd^2x\epsilon^3 + 2a^2cdx^2\epsilon^3 + 2a^4dey\epsilon^3 + 2a^2cdey\epsilon^3 - a^4bxy\epsilon^3 + a^5bxy\epsilon^3 + 2a^3bcxy\epsilon^3 + a^3bd^2xy\epsilon^3 - 2a^4exy\epsilon^3 - 2a^2cexy\epsilon^3 - a^2d^2exy\epsilon^3 + a^2bdey^2\epsilon^3 + a^3bdey^2\epsilon^3 + a^5c\epsilon^4 + 2a^3c^2\epsilon^4 + ac^3\epsilon^4 - 2a^4dx\epsilon^4 + 3a^5dx\epsilon^4 - 2a^2cdx\epsilon^4 + 4a^3cdx\epsilon^4 + ac^2dx\epsilon^4 - a^5x^2\epsilon^4 + ac^2x^2\epsilon^4 - a^3d^2x^2\epsilon^4 + a^5ey\epsilon^4 + 2a^3cey\epsilon^4 + ac^2ey\epsilon^4 + 2a^4bdxy\epsilon^4 + abcdxy\epsilon^4 + 3a^2bcdxy\epsilon^4 - 2a^3dexy\epsilon^4 - 2acdexy\epsilon^4 - ab^2cy^2\epsilon^4 - 2a^2b^2cy^2\epsilon^4 - a^3b^2cy^2\epsilon^4 + a^3bey^2\epsilon^4 + a^4bey^2\epsilon^4 + abcey^2\epsilon^4 + a^2bcey^2\epsilon^4 - a^5x\epsilon^5 + a^6x\epsilon^5 - 2a^3cx\epsilon^5 + 2a^4cx\epsilon^5 - ac^2x\epsilon^5 + a^2c^2x\epsilon^5 - 2a^4dx^2\epsilon^5 - 2a^2cdx^2\epsilon^5 + a^5bxy\epsilon^5 + a^2bcxy\epsilon^5 + 3a^3bcxy\epsilon^5 + bc^2xy\epsilon^5 + 2abc^2xy\epsilon^5 - a^4exy\epsilon^5 - 2a^2cexy\epsilon^5 - c^2exy\epsilon^5 - a^5x^2\epsilon^6 - 2a^3cx^2\epsilon^6 - ac^2x^2\epsilon^6)/(a(a^2+ad\epsilon+a^2\epsilon^2+c\epsilon^2)^2)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = x\epsilon + 1$, $l_4 = a^3 + a^2d\epsilon - a^2x\epsilon + a^3\epsilon^2 + ac\epsilon^2 - adx\epsilon^2 + aby\epsilon^2 + a^2by\epsilon^2 - a^2x\epsilon^3 - cx\epsilon^3$.

If $\epsilon \to 0$, then $l_3, l_4 \to l_\infty$.

6.3) $m_\infty(1,1;3)$: $\dot{x} = x(a + by + cxy + y^2)$, $\dot{y} = -y(d + ex + c^2x^2 + cxy)$, $ad(c^2 + e^2 + (a + d)^2)((a + d)^2 + (bc - e)^2) \neq 0$.

The perturbed cubic system is

$\dot{x} = x(a + by + cxy + y^2 - bcxy\epsilon + exy\epsilon - a^2\epsilon^2 - ab^2\epsilon^2 + 2abcx\epsilon^2 - 2aex\epsilon^2 - ac^2x^2\epsilon^2 - aby\epsilon^2 - b^3y\epsilon^2 - acxy\epsilon^2 - 2bexy\epsilon^2 - ay^2\epsilon^2 - b^2y^2\epsilon^2 - a^2b\epsilon^3 - 2a^2cx\epsilon^3 + 2abex\epsilon^3 + 2abc^2x^2\epsilon^3 - 2acex^2\epsilon^3 - ab^2y\epsilon^3 - 2aexy\epsilon^3 + b^2exy\epsilon^3 - aby^2\epsilon^3 - 2a^2bcx\epsilon^4 + 2a^2ex\epsilon^4 - a^2c^2x^2\epsilon^4 + ab^2c^2x^2\epsilon^4 + 2abcex^2\epsilon^4 + 2abexy\epsilon^4 - a^2bc^2x^2\epsilon^5 + 2a^2cex^2\epsilon^5 + a^2exy\epsilon^5)$, $\dot{y} = y(-d - ex - c^2x^2 - cxy - acx\epsilon + b^2cx\epsilon + 2bc^2x^2\epsilon - 2cex^2\epsilon + bcxy\epsilon - exy\epsilon + ad\epsilon^2 + b^2d\epsilon^2 + b^3cx\epsilon^2 - 2bcdx\epsilon^2 + b^2ex\epsilon^2 + 2dex\epsilon^2 - 2ac^2x^2\epsilon^2 + b^2c^2x^2\epsilon^2 + c^2dx^2\epsilon^2 + 2bcex^2\epsilon^2 - acxy\epsilon^2 + 2b^2cxy\epsilon^2 + 2cdxy\epsilon^2 + dy^2\epsilon^2 + abd\epsilon^3 - a^2cx\epsilon^3 + 2acdx\epsilon^3 + 2abex\epsilon^3 - 2bdex\epsilon^3 + abc^2x^2\epsilon^3 - 2bc^2dx^2\epsilon^3 + 2cdex^2\epsilon^3 - 2bcdxy\epsilon^3 + b^2exy\epsilon^3 + 2dexy\epsilon^3 - a^2bcx\epsilon^4 + 2abcdx\epsilon^4 + a^2ex\epsilon^4 - 2adex\epsilon^4 - a^2c^2x^2\epsilon^4 + ab^2c^2x^2\epsilon^4 + ac^2dx^2\epsilon^4 - b^2c^2dx^2\epsilon^4 + 2abcex^2\epsilon^4 - 2bcdex^2\epsilon^4 - 2b^2cdxy\epsilon^4 + 2abexy\epsilon^4 - 2bdexy\epsilon^4 - ady^2\epsilon^4 - b^2dy^2\epsilon^4 - a^2bc^2x^2\epsilon^5 + abc^2dx^2\epsilon^5 + 2a^2cex^2\epsilon^5 - 2acdex^2\epsilon^5 + a^2exy\epsilon^5 - 2adexy\epsilon^5 - abdy^2\epsilon^5)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = 1 + cx\epsilon + y\epsilon$, $l_4 = -1 + cx\epsilon + y\epsilon + a\epsilon^2 + b^2\epsilon^2 - 2bcx\epsilon^2 + 2ex\epsilon^2 + ab\epsilon^3 + acx\epsilon^3 - b^2cx\epsilon^3 - 2bex\epsilon^3 - ay\epsilon^3 - b^2y\epsilon^3 + abcx\epsilon^4 - 2aex\epsilon^4 - aby\epsilon^4$.

If $\epsilon \to 0$, then $l_3, l_4 \to l_\infty$.

6.4) $m_\infty(1,1;3)$: $\dot{x} = x(a + by + cxy + dy^2)$, $\dot{y} = \alpha y(1 + bx + cx^2 + dxy)$, $\alpha a(c^2 + d^2)(\alpha - a) \neq 0$.

The perturbed cubic system is

$\dot{x} = -x(-a - by - cxy - dy^2 - axy\alpha\epsilon^2 + ax^2\alpha^2\epsilon^2 - 2xy\alpha^2\epsilon^2)$, $\dot{y} = -y\alpha(-1 - bx - cx^2 - dxy - axy\epsilon^2 + y^2\epsilon^2 + ax^2\alpha\epsilon^2 - 2xy\alpha\epsilon^2 - x^2\alpha^2\epsilon^2)$.

The invariant straight lines are $l_1 = x$, $l_2 = y$, $l_3 = 1 - y\epsilon + x\alpha\epsilon$, $l_4 = -1 - y\epsilon + x\alpha\epsilon$.

If $\epsilon \to 0$, then the lines $l_3, l_4 \to \infty$.

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