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Determining the Distribution of the Duration of Stationary Games for Zero-Order Markov Processes with Final Sequence of States

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Abstract. A zero-order Markov process with final sequence of states represents a stochastic system with independent transitions that stops its evolution as soon as given final sequence of states is reached. The transition time of the system is unitary and the transition probability depends only on the destination state. We consider the following game. Initially, each player defines his distribution of the states. The initial distribution of the states is established according to the distribution given by the first player. After that, the stochastic system passes consecutively to the next state according to the distribution given by the next player. After the last player, the first player acts on the system evolution and the game continues in this way until the given final sequence of states is achieved. Our goal is to study the duration of this game, knowing the distribution established by each player and the final sequence of states of the stochastic system. It is proved that the distribution of the duration of the game is a homogeneous linear recurrent sequence and it is developed a polynomial algorithm to determine the initial state and the generating vector of this recurrence. Using the generating function, the main probabilistic characteristics are determined.

Mathematics subject classification: 65C40, 60J22, 90C39, 90C40.
Keywords and phrases: Zero-Order Markov Process, Final Sequence of States, Duration, Game, Homogeneous Linear Recurrence, Generating Function.

1 Introduction and Problem Formulation

Let L be a discrete stochastic system with finite set of states V, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for all $v \in V$, where $\sum\limits_{v\in V} p^*(v) = 1$.

Also, the transition from one state u to another state v is performed according to the same probability $p^*(v)$ that depends only on the destination state v, for every $u \in V$ and $v \in V$. Additionally we assume that a sequence of states $X = (x_1, x_2, \dots, x_m) \in V^m$ is given and the stochastic system stops transitions as soon as the states $x_1, x_2, \dots, x_m$ are reached consecutively in given order. The time T when the system stops is called evolution time of the stochastic system L with given final sequence of states X.

The stochastic system L described above represents a zero-order Markov process with final sequence of states X. Several interpretations of these Markov processes were analyzed in 1981 by Leo J. Guibas and Andrew M. Odlyzko in [8] and

G. Zbaganu in 1992 in [7]. Various problems related to such systems have been studied in [1]–[5]. Also, in these papers, polynomial algorithms for determining the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time of the given stochastic system L were proposed.

Next, in this paper, a generalization of this problem is studied. The following game is considered. Initially, each player $\mathcal{P}_\ell$ defines his distribution of the states $(p^{*(\ell)}(v))_{v\in V}$, $\ell = \overline{0, r-1}$. The initial distribution of the states is established according to the distribution $(p^{*(0)}(v))_{v\in V}$ given by the first player $\mathcal{P}_0$. After that, the stochastic system passes consecutively to the next state according to the distribution given by the next player. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the system evolution and the game continues in this way until the given final sequence of states X is achieved. The player $\mathcal{P}_{T mod\ r}$, who acts the last on the system, is considered the winner of the game.

Our goal is to study the duration T of this game, knowing the distribution $p^{*(\ell)} = (p^{*(\ell)}(v))_{v\in V}$ established by each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, and the final sequence of states X of the stochastic system L. We will prove that the distribution of the game duration T is a homogeneous linear recurrent sequence ([1],[6]) and a polynomial algorithm to determine the initial state and the generating vector of this recurrence will be developed. Having the generating vector and the initial state of the recurrence, we can use the related algorithm from [1], which was mentioned above, for determining the main probabilistic characteristics of the game duration.

2 The Main Results

2.1 Determining The Distribution of the Game Duration

In this section we will determine the distribution law of the game duration T. Initially, we consider the notations

$$X_k = \{x_k\},\ \overline{X}_k = V\backslash\{x_k\},\ \pi_k^{(\ell)} = p^{*(\ell)}(x_k), w_k^{(\ell)} = \prod_{j=2}^{k} \pi_j^{(\ell\ \oplus(-1)\oplus j)},$$

for each $k = \overline{1, m}$ and $\ell = \overline{0, r-1}$, where $c \oplus d = (c+d) \mod r$, $\forall c, d \in \mathbb{Z}$.

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the game duration T, i.e. $a_n = \mathbb{P}(T = n)$, $n = \overline{0, \infty}$. Since $T \geq m-1$, we have $a_n = 0$, $n = \overline{0, m-2}$. If $T = m-1$, then $v(j) = x_{j+1}$, $j = \overline{0, m-1}$, that implies

$$a_{m-1} = \mathbb{P}(T = m-1) = \pi_1^{(0)}\pi_2^{(1)} \ldots \pi_m^{(m\oplus(-1))} = \pi_1^{(0)} w_m^{(0)}.$$

We consider $\forall n \in \mathbb{Z}$. Let be $S(V) = \{A \mid A \subseteq V\}$. Denote by $P_\Phi^{(\ell)}(n)$ the probability that $T = n$, $v(j) \in \Phi_j$, $j = \overline{0, t-1}$ and the player $\mathcal{P}_\ell$ acts first, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{0, r-1}$. We introduce the following functions on $\mathbb{Z}$, $k=\overline{0,m}$, $\ell=\overline{0,r-1}$:

$$\begin{aligned}\alpha_k^{(\ell)}(n)&=P^{(\ell)}_{(X_1,X_2,\ldots,X_{k-1},\overline{X}_k)}(n),\\ \beta_k^{(\ell)}(n)&=P^{(\ell)}_{(X_1,X_2,\ldots,X_k)}(n),\\ \gamma_k^{(\ell)}(n)&=P^{(\ell)}_{(X_2,X_3,\ldots,X_k)}(n).\end{aligned} \tag{1}$$

We have

$$\beta_k^{(\ell)}(n)=P^{(\ell)}_{(X_1,X_2,\ldots,X_k)}(n)=a_n^{(\ell)}-\sum_{j=1}^{k}\alpha_j^{(\ell)}(n),\ k=\overline{0,m},\ell=\overline{0,r-1}, \tag{2}$$

where $a_n^{(\ell)}=P^{(\ell)}_{(\,)}(n)$, $\ell=\overline{0,r-1}$.

We consider the sets

$$T_s=\{s+1\}\cup\{t\in\{2,3,\ldots,s\}\mid x_{t-1+j}=x_j,\ j=\overline{1,s+1-t}\},\ s=\overline{1,m}.$$

The minimal elements from these sets are

$$t_s=\min_{k\in T_s}k,\ s=\overline{1,m}. \tag{3}$$

The value t_s represents the auto superposition level of the sequence $(x_1,x_2,\ldots,x_s)$, i.e. t_s is the position in the sequence $(x_1,x_2,\ldots,x_s)$ starting with which, if we overlap the same sequence, the superposed elements are equal.

Using the formula (2) for $s=\overline{1,m}$ and $\ell=\overline{0,r-1}$, we obtain

$$\begin{gathered}\gamma_s^{(\ell)}(n)=P^{(\ell)}_{(X_2,X_3,\ldots,X_s)}(n)=\\ =\pi_2^{(\ell)}\pi_3^{(\ell\,\oplus 1)}\ldots\pi_{t_s-1}^{(\ell\,\oplus(t_s-3))}P^{(\ell\,\oplus(t_s-2))}_{(X_{t_s},X_{t_s+1},\ldots,X_s)}(n-t_s+2)=\\ =w_{t_s-1}^{(\ell\,\oplus(-1))}\beta_{s+1-t_s}^{(\ell\,\oplus(t_s-2))}(n-t_s+2)=\\ =w_{t_s-1}^{(\ell\,\oplus(-1))}\left(a_{n-t_s+2}^{(\ell\,\oplus(t_s-2))}-\sum_{j=1}^{s+1-t_s}\alpha_j^{(\ell\,\oplus(t_s-2))}(n-t_s+2)\right).\end{gathered} \tag{4}$$

Particularly, we obtain the relation

$$\gamma_1^{(\ell)}(n)=a_n^{(\ell)},\ \ell=\overline{0,r-1},\ n=\overline{0,\infty}. \tag{5}$$

The values $\alpha_k^{(\ell)}(n)$, $k=\overline{1,m}$, $\ell=\overline{0,r-1}$ are determined in the following way:

$$\alpha_1^{\ell}(n)=P^{(\ell)}_{(\overline{X}_1)}(n)=(1-\pi_1^{(\ell)})a_{n-1}^{(\ell\,\oplus 1)}, \tag{6}$$

$$\alpha_k^{(\ell)}(n)=P^{(\ell)}_{(X_1,X_2,\ldots,X_{k-1},\overline{X}_k)}(n)=\pi_1^{(\ell)}P^{(\ell\,\oplus 1)}_{(X_2,X_3,\ldots,X_{k-1},\overline{X}_k)}(n-1)=$$

$$= \pi_1^{(\ell)} \left(P^{(\ell \oplus 1)}_{(X_2,X_3,\ldots,X_{k-1})}(n-1) - P^{(\ell \oplus 1)}_{(X_2,X_3,\ldots,X_k)}(n-1) \right) =$$
$$= \pi_1^{(\ell)} \left(\gamma^{(\ell \oplus 1)}_{k-1}(n-1) - \gamma^{(\ell \oplus 1)}_{k}(n-1) \right), \ k = \overline{2,m}. \tag{7}$$

Next, we obtain the recurrent formula:

$$a_n^{(\ell)} = \sum_{j=1}^{m} \alpha_j^{(\ell)}(n) = (1-\pi_1^{(\ell)}) a_{n-1}^{(\ell \oplus 1)} + \sum_{j=2}^{m} \pi_1^{(\ell)} \left(\gamma^{(\ell \oplus 1)}_{j-1}(n-1) - \gamma^{(\ell \oplus 1)}_{j}(n-1) \right) =$$
$$= (1-\pi_1^{(\ell)}) a_{n-1}^{(\ell \oplus 1)} + \pi_1^{(\ell)} \left(a_{n-1}^{(\ell \oplus 1)} - \gamma^{(\ell \oplus 1)}_{m}(n-1) \right) =$$
$$= a_{n-1}^{(\ell \oplus 1)} - \pi_1^{(\ell)} \gamma_m^{(\ell \oplus 1)}(n-1), \ \forall n \geq m, \ \ell = \overline{0, r-1}. \tag{8}$$

According to the relations (4) – (7), using the mathematical induction, we can prove that there exist the real coefficients $u_{jk\ell}^{(i)}$ and $v_{jk\ell}^{(i)}$, $j = \overline{1,m}$, $k = \overline{0, j-1}$, $\ell = \overline{0, r-1}$, $i = \overline{0, r-1}$ such that

$$\alpha_j^{(\ell)}(n) = \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} u_{jk\ell}^{(i)} \, a_{n-1-k}^{(i)}, \ \gamma_j^{(\ell)}(n-1) = \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} v_{jk\ell}^{(i)} \, a_{n-1-k}^{(i)}, \ \forall n \in \mathbb{Z}. \tag{9}$$

From the relations (5) and (6), for $i = \overline{0, r-1}$ and $\ell = \overline{0, r-1}$, we obtain

$$u_{1,0,\ell}^{(i)} = \begin{cases} 1 - \pi_1^{(\ell)}, & \text{if } \ i = \ell \oplus 1 \\ 0, & \text{if } \ i \neq \ell \oplus 1 \end{cases} \tag{10}$$

and

$$v_{1,0,\ell}^{(i)} = \begin{cases} 1, & \text{if } \ i = \ell \\ 0, & \text{if } \ i \neq \ell \end{cases} . \tag{11}$$

Using the representation (9), the formula (4) obtains the form

$$\gamma_s^{(\ell)}(n-1) = w_{t_s-1}^{(\ell \oplus (-1))} \left(a^{(\ell \oplus (t_s-2))}_{(n-1)-t_s+2} - \sum_{j=1}^{s+1-t_s} \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} u^{(i)}_{j,k,\ell \oplus (t_s-2)} a^{(i)}_{n-t_s-k} \right) =$$
$$= w_{t_s-1}^{(\ell \oplus (-1))} \left(a^{(\ell \oplus (t_s-2))}_{(n-1)-(t_s-2)} - \sum_{i=0}^{r-1} \sum_{k=t_s-1}^{s-1} a^{(i)}_{n-1-k} \sum_{j=k-t_s+2}^{s+1-t_s} u^{(i)}_{j, \ k-t_s+1, \ \ell \oplus (t_s-2)} \right) =$$
$$= \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} v_{sk\ell}^{(i)} \, a_{n-1-k}^{(i)}, \ s = \overline{1,m}, \ \ell = \overline{0, r-1},$$

where

$$v_{s,k,\ell}^{(i)} = \begin{cases} 0, & \text{if } \ k \leq t_s - 3 \\ 0, & \text{if } \ \begin{cases} k = t_s - 2, \\ i \neq \ell \oplus (t_s - 2) \end{cases} \\ w_{t_s-1}^{(\ell \oplus (-1))}, & \text{if } \ \begin{cases} k = t_s - 2, \\ i = \ell \oplus (t_s - 2) \end{cases} \\ -w_{t_s-1}^{(\ell \oplus (-1))} \sum\limits_{j=k-t_s+2}^{s+1-t_s} u^{(i)}_{j, \ k-t_s+1, \ \ell \oplus (t_s-2)}, & \text{if } \ k \geq t_s - 1 \end{cases} \tag{12}$$

$s=\overline{1,m}$, $k=\overline{0,s-1}$, $\ell=\overline{0,r-1}$, $i=\overline{0,r-1}$, and the formula (7) becomes

$$\alpha_s^{(\ell)}(n)=\pi_1^{(\ell)}\left(\gamma_{s-1}^{(\ell\,\oplus 1)}(n-1)-\gamma_s^{(\ell\,\oplus 1)}(n-1)\right)=$$

$$=\pi_1^{(\ell)}\left(\sum_{i=0}^{r-1}\sum_{k=0}^{s-2}v_{s-1,\ k,\ \ell\,\oplus 1}^{(i)}a_{n-1-k}^{(i)}-\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}v_{s,\ k,\ \ell\,\oplus 1}^{(i)}a_{n-1-k}^{(i)}\right)=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}u_{sk\ell}^{(i)}\,a_{n-1-k}^{(i)},\ s=\overline{2,m},\ \ell=\overline{0,r-1},$$

where

$$u_{s,k,\ell}^{(i)}=\begin{cases}\pi_1^{(\ell)}\left(v_{s-1,\ k,\ \ell\,\oplus 1}^{(i)}-v_{s,\ k,\ \ell\,\oplus 1}^{(i)}\right), & \text{if } \ k\le s-2\\ -\pi_1^{(\ell)}v_{s,\ k,\ \ell\,\oplus 1}^{(i)}, & \text{if } \ k=s-1\end{cases}, \tag{13}$$

$s=\overline{2,m}$, $k=\overline{0,s-1}$, $\ell=\overline{0,r-1}$, $i=\overline{0,r-1}$. The formula (8) obtains the form

$$a_n^{(\ell)}=a_{n-1}^{(\ell\,\oplus 1)}-\pi_1^{(\ell)}\gamma_m^{(\ell\,\oplus 1)}(n-1)=a_{n-1}^{(\ell\,\oplus 1)}-\pi_1^{(\ell)}\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}v_{m,\ k,\ \ell\,\oplus 1}^{(i)}a_{n-1-k}^{(i)}=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}q_{ik}^{(\ell)}a_{n-1-k}^{(i)},\ \forall n\ge m,\ \ell=\overline{0,r-1}, \tag{14}$$

where

$$q_{ik}^{(\ell)}=\begin{cases}1-\pi_1^{(\ell)}v_{m,\ 0,\ \ell\,\oplus 1}^{(\ell\,\oplus 1)}, & \text{if } \ i=\ell\oplus 1 \text{ and } \ k=0\\ -\pi_1^{(\ell)}v_{m,\ k,\ \ell\,\oplus 1}^{(i)}, & \text{otherwise}\end{cases}. \tag{15}$$

Next, we consider the column vectors $A_n=((a_n^{(\ell)})_{\ell=0}^{r-1})^T$, $n=\overline{0,\infty}$. Also, we define the matrices $Q^{(k)}=(q_{ik}^{(\ell)})_{\ell,\ i=\overline{0,\ r-1}}$, $k=\overline{0,m-1}$ and we consider the sequence $Q=(Q^{(k)})_{k=0}^{m-1}$. From the relation (14), we have the homogeneous linear recurrence $A_n=\sum\limits_{k=0}^{m-1}Q^{(k)}A_{n-1-k}$, $\forall n\ge m$, i.e. $A=(A_n)_{n=0}^{\infty}\in Rol^*[\mathcal{M}_r(\mathbb{R})][m]$ with generating vector $Q\in G^*[\mathcal{M}_r(\mathbb{R})][m](A)$. So, the vectorial sequence A is homogeneous linear recurrent on the matrix field $\mathcal{M}_r(\mathbb{R})$ with generating vector Q. Applying the results obtained in [1], we have $A\in Rol^*[\mathbb{R}][mr]$ with characteristic polynomial $H(z)=|I_r-zG_m^{[Q]}(z)|\in H^*[\mathbb{R}][mr](A)$, which implies that $a^{(\ell)}=(a_n^{(\ell)})_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr]$ and $H(z)=|I_r-zG_m^{[Q]}(z)|\in H^*[\mathbb{R}][mr](a^{(\ell)})$, $\ell=\overline{0,r-1}$. Because the game is started by player $\mathcal{P}^{(0)}$, then the distribution a of the game duration T coincides with $a^{(0)}$, i.e. $a=(a_n)_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr]$ with characteristic polynomial $H(z)=|I_r-zG_m^{[Q]}(z)|\in H^*[\mathbb{R}][mr](a)$.

Next, we will use only the relation $a\in Rol^*[\mathbb{C}][mr]$, the minimal generating vector being determined by use of the minimization method based on the matrix rank, described in [1], that is available also for degenerated homogeneous linear recurrences. So, according to this method, we have that the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{16}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n \in \mathbb{N}^* \tag{17}$$

and R is the rank of the matrix $A_{mr}^{[a]}$.

For this, we need to have only the values a_k, $k = \overline{0, 2mr-1}$. These values are determined the formula

$$a_k = a_k^{(0)},\ k = \overline{0, 2mr-1}, \tag{18}$$

using the relations (4) – (8) and the initial conditions

$$\begin{gathered} a_n = a_n^{(\ell)} = P^{(\ell)}(n) = 0,\ \ell = 0, r-1,\ n = \overline{0, m-2}, \\ \alpha_k^{(\ell)}(n) = 0,\ k = \overline{1,m},\ n = \overline{0, m-1},\ \ell = \overline{0, r-1}, \\ a_{m-1}^{(\ell)} = \pi_1^{(\ell)} w_m^{(\ell)},\ \ell = \overline{0, r-1}. \end{gathered} \tag{19}$$

2.2 Describing the developed algorithm

In the previous subsection we theoretically grounded the following algorithm for determining the main probabilistic characteristics (the distribution $(\mathbb{P}(T=n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = 1, 2, \ldots$) of the game duration T.

Algorithm 1.

Input: $X = (x_1, x_2, \ldots, x_m) \in V^m$, $\pi_k^{(\ell)}$, $k = \overline{1,m}$, $\ell = \overline{0, r-1}$;
Output: $\mathbb{E}(T),\ \mathbb{V}(T),\ \sigma(T),\ \nu_k(T),\ k = \overline{1,t},\ t \geq 2$.

1. Determine the values a_k, $k = \overline{0, 2mr-1}$, using the formula (18), the relations (4) – (8) and the initial conditions (19);

2. Find the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$ by solving the system (16), taking into account the relation (17);

3. Consider the distribution $a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T=n))_{n=0}^{\infty}$ of the game duration T as a homogeneous linear recurrence with the initial state $I_R^{[a]} = (a_n)_{n=0}^{R-1}$ and the minimal generating vector $q = (q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = \overline{1,t}$, of the game duration T by using the corresponding algorithm from [1].

3 Conclusions

In this paper the stationary games defined on zero-order Markov processes with final sequence of states were studied and the duration of these games was analyzed. It was proved that the game duration is a discrete random variable with homogeneous linear recurrent distribution. Based on this fact, the generating function is applied for determining the main probabilistic characteristics of the game duration. The developed algorithm has polynomial time complexity. Also, the algorithm is applicable for the case when the set of the states is infinite.

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