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Third Hankel determinant for the inverse of reciprocal of bounded turning functions

B. Venkateswarlu, D. Vamshee Krishna, N. Rani

Abstract. In this paper we obtain the best possible upper bound to the third Hankel determinants for the functions belonging to the class of reciprocal of bounded turning functions using Toeplitz determinants.

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1 Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For a univalent function in the class A, it is well known that the n^{th} coefficient is bounded by n. The bounds for the coefficients of univalent functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [12] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by many authors in the literature. For example, Noor [10] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [7]. One can easily observe that the Fekete-Szego functional is $H_2(1)$. Fekete-Szego then further generalized the estimate

$|a_3 - \mu a_2^2|$ with μ real and $f \in S$. R. M. Ali [1] found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szego functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n,$$

when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. In the recent years several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent analytic functions. In particular for $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, $a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{2}$$

For $f \in A$, $a_1 = 1$, so we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{3}$$

The sharp upper bound to the second Hankel functional $H_2(2)$ for the subclass RT of S, consisting of functions whose derivative has a positive real part, studied by Mac Gregor [9] was obtained by Janteng [6]. It was known that if $f \in RT$ then $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, ...\}$. Also the sharp upper bound for the functional $|a_3 - a_2^2|$ was $\frac{2}{3}$, stated in [2], for the class RT. Further, the best possible sharp upper bound for the functional $|a_2 a_3 - a_4|$ was obtained by Babalola [2] and hence the sharp inequality for $|H_3(1)|$, for the class RT. The sharp upper bound to $|H_3(1)|$ for the class of inverse of a function whose derivative has a real part was obtained by D. Vamshee Krishna et al. [14].

Motivated by the above mentioned results obtained by different authors in this direction and the results by Babalola [2], in the present paper, we seek an upper bound to the second Hankel determinant, $|t_2 t_3 - t_4|$ and an upper bound to the third Hankel determinant, for certain subclass of analytic functions defined as follows.

Definition 1. A function $f(z) \in A$ is said to be function whose reciprocal derivative has a positive real part (also called reciprocal of bounded turning function), denoted by $f \in \widetilde{RT}$, if and only if

$$Re\left(\frac{1}{f'(z)}\right) > 0, \ \forall z \in E. \tag{4}$$

Some preliminary Lemmas required for proving our results are as follows.

2 Preliminary Results

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{5}$$

which are regular in the open unit disc E and satisfy $Re\Big(p(z)\Big) > 0$ for any $z \in E$. Here $p(z)$ is called the Carathe̒dory function [3].

Lemma 1 (see [11, 13])**.** *If $p \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2 (see [5])**.** *The power series for $p(z)$ given in (5) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad for \ \ n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z),$$

$\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [5] is due to Carathe̒dory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 \;=\; c_1^2 + x(4-c_1^2), \text{ for some } x, |x| \leq 1. \tag{6}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq \; 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4-c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4-c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{7}$$

Simplifying the relations (6) and (7), we get

$$4c_3 = c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z$$
$$\text{for some } z, \text{ with } |z| \leq 1. \tag{8}$$

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [8] and used by several authors in the literature.

3 Main Result

Theorem 1. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* w=0 *is the inverse function of* f *then* $|t_2t_4 - t_3^2| \leq \frac{137}{288}$.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widetilde{RT}$, there exists an analytic function $p \in \mathcal{P}$ in the open unit disc E with $p(0) = 1$ and $Re\Big(p(z)\Big) > 0$ such that

$$\frac{1}{f'(z)} = p(z) \Leftrightarrow 1 = f'(z)p(z). \tag{9}$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (9), we have

$$1 = \Big(1 + \sum_{n=2}^{\infty} na_n z^{n-1}\Big)\Big(1 + \sum_{n=1}^{\infty} c_n z^n\Big).$$

Upon simplification, we obtain

$$\begin{aligned} 1 = 1 &+ (c_1 + 2a_2)z + (c_2 + 2a_2c_1 + 3a_3)z^2 \\ &+ (c_3 + 2a_2c_2 + 3a_3c_1 + 4a_4)z^3 \\ &+ (c_4 + 2a_2c_3 + 3a_3c_2 + 4a_4c_1 + 5a_5)z^4 \cdots . \end{aligned} \tag{10}$$

Equating the coefficients of like powers of z, z^2, z^3 and z^4 respectively on both sides of (10), after simplifying, we get

$$a_2 = -\frac{c_1}{2}; \quad a_3 = \frac{1}{3}(c_1^2 - c_2); \quad a_4 = -\frac{1}{4}(c_3 - 2c_1c_2 + c_1^3);$$
$$a_5 = -\frac{1}{5}(c_4 - 2c_1c_3 + 3c_1^2c_2 - c_2^2 - c_1^4). \tag{11}$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widetilde{RT}$, from the definition of inverse function of f, we have

$$w = f(f^{-1}(w)) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n \big(f^{-1}(w)\big)^n \Leftrightarrow w$$
$$= w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n \left(w + \sum_{n=2}^{\infty} t_n w^n\right)^n.$$

After simplifying, we get

$$(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4$$
$$+(t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \cdots = 0. \tag{12}$$

Equating the coefficients of like powers of w^2, w^3, w^4 and w^5 on both sides of (12), respectively, further simplification gives

$$t_2 = -a_2; \quad t_3 = -a_3 + 2a_2^2; \quad t_4 = -a_4 + 5a_2a_3 - 5a_2^3;$$
$$t_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \tag{13}$$

Using the values of a_2, a_3, a_4 and a_5 in (11) along with (13), upon simplification, we obtain

$$t_2 = \frac{c_1}{2}; \; t_3 = \frac{1}{6}[2c_2 + c_1^2]; \; t_4 = \frac{1}{24}[6c_3 + 8c_1c_2 + c_1^3];$$
$$t_5 = \frac{1}{120}[24c_4 + 42c_1c_3 + 22c_1^2c_2 + 16c_2^2 + c_1^4]. \tag{14}$$

Substituting the values of t_2, t_3 and t_4 from (14) in the functional $|\, t_2t_4 - t_3^2|$ for the function $f \in \widetilde{RT}$ upon simplification, we obtain

$$|\, t_2t_4 - t_3^2| \;=\; \frac{1}{144}\Big|18c_1c_3 + 8c_1^2c_2 - 16c_2^2 - c_1^4\Big|$$

which is equivalent to

$$|\, t_2t_4 - t_3^2| = \frac{1}{144}\Big|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\Big| \tag{15}$$

$$\text{where} \quad d_1 = 18; \; d_2 = 8; \; d_3 = -16; \; d_4 = -1. \tag{16}$$

Substituting the values of c_2 and c_3 given in (6) and (8) respectively from Lemma 2 on the right-hand side of (15) and using the fact $|z| < 1$, we have

$$4|\ d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big|(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + \{2d_1c_1 + 2(d_1 + d_2 + d_3)c_1^2|x| - [(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3]|x|^2\}(4 - c_1^2)\Big|. \quad (17)$$

From (16) and (17), we can now write

$$(d_1 + 2d_2 + d_3 + 4d_4) = 14; \quad 2d_1 = 36; \quad 2(d_1 + d_2 + d_3) = 20;$$
$$(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3 = 2(c_1^2 + 18c_1 + 32). \quad (18)$$

Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in (18), we can have

$$-\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\} \leq -2(c_1^2 - 18c_1 + 32). \quad (19)$$

Substituting the calculated values from (18) and (19) on the right-hand side of (17), we have

$$4|\ d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big|14c_1^4 + \{36c_1 + 20c_1^2|x| - 2(c_1^2 - 18c_1 + 32)|x|^2\}(4 - c_1^2)\Big|.$$

Choosing $c_1 = c \in [0, 2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality

$$2|\ d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big|7c^4 + \{18c + 10c^2\mu + (c^2 - 18c + 32)\mu^2\}(4 - c^2)\Big| = F(c, \mu)\ ,\ 0 \leq \mu = |x| \leq 1 \ \text{ and } \ 0 \leq c \leq 2. \quad (20)$$

We next maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ given in (20) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = [20c + 2(c - 2)(c - 16)\mu](4 - c^2) > 0. \quad (21)$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (21), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c, \mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Moreover, for a fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c).$$

Therefore, replacing μ by 1 in $F(c, \mu)$, upon simplification, we obtain

$$G(c) = -4c^4 + 12c^2 + 128 \quad (22)$$

$$G'(c) = -16c^3 + 24c \tag{23}$$
$$G''(c) = -48c^2 + 24. \tag{24}$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (23), we get

$$c^2 = \frac{3}{2}.$$

Using the obtained value of c^2 in (24), which simplifies to give

$$G''(c) = -48 < 0.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at $c = \sqrt{\frac{3}{2}} \in [0, 2]$. Substituting the value of c in the expression (22), upon simplification, we obtain the maximum value of $G(c)$ at c as

$$G_{max} = 137. \tag{25}$$

Simplifying the expressions (20) and (25)

$$|\ d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \frac{137}{2}. \tag{26}$$

From the relations (15) and (26), we obtain

$$|\ t_2t_4 - t_3^2| \leq \frac{137}{288}. \tag{27}$$

This completes the proof of our Theorem. □

Remark 1. It is observed that the upper bound to the second Hankel determinant of inverse of a function whose derivative has a positive real part [14] and the inverse of a function whose reciprocal derivative has a positive real part is the same.

Theorem 2. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then* $|\ t_2t_3 - t_4| = \left(\frac{13}{6}\right)^{\frac{3}{2}}$.

Proof. Substituting the values of t_2, t_3 and t_4 from (14) in $|\ t_2t_3 - t_4|$ for the function $f \in \widetilde{RT}$, after simplifying, we get

$$|\ t_2t_3 - t_4| = \frac{1}{24}| - 6c_3 - 4c_1c_2 + c_1^3|. \tag{28}$$

Substituting the values of c_2 and c_3 from (6) and (8) respectively, from Lemma 2 on the right-hand side of (28) and using the fact $|\ z\ |< 1$, after simplifying, we get

$$2| - 6c_3 - 4c_1c_2 + c_1^3| \leq \Big| - 5c_1^3 - 6(4 - c_1^2) - 10c_1(4 - c_1^2)|x|$$

$$+3(c_1+2)(4-c_1^2)|x|^2\Big|. \tag{29}$$

Since $c_1 = c \in [0,2]$, using the result $(c_1+a) \geq (c_1-a)$, where $a \geq 0$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned} 2|-6c_3-4c_1c_2+c_1^3| \leq & \ |5c^3+6(4-c^2)+10c(4-c^2)\mu+3(c-2)(4-c^2)\mu^2| \\ = & \ F(c,\mu) \ , \ 0 \leq \mu = |x| \leq 1 \text{ and } 0 \leq c \leq 2. \end{aligned} \tag{30}$$

Next, we maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = (4-c^2)[10c+6(c-2)\mu] > 0.$$

As described in Theorem 3, further, we obtain

$$G(c) = -8c^3+52c \tag{31}$$

$$G'(c) = -24c^2+52 \tag{32}$$

$$G''(c) = -48c. \tag{33}$$

For optimum value of $G(c)$, consider $G'(c)=0$. From (32), we get

$$c^2 = \frac{13}{6}.$$

Using the obtained value of $c = \sqrt{\frac{13}{6}} \in [0,2]$ in (33), then

$$G''(c) = -8\sqrt{78} < 0.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at $c = \sqrt{\frac{13}{6}}$. Substituting the value of c in the expression (31), upon simplification, we obtain the maximum value of $G(c)$ at c as

$$G_{max} = \frac{104}{3}\sqrt{\frac{13}{6}}. \tag{34}$$

From the expressions (30) and (34), after simplifying, we get

$$|-6c_3-4c_1c_2+c_1^3| \leq \frac{52}{3}\sqrt{\frac{13}{6}}. \tag{35}$$

Simplifying the relations (28) and (35), we obtain

$$|t_2t_3-t_4| \leq \frac{1}{3}\left(\frac{13}{6}\right)^{\frac{3}{2}}.$$

This completes the proof of our Theorem. □

Remark 2. It is observed that the upper bound to the $|t_2t_3 - t_4|$ of a function whose derivative has a positive real part [14] and a function whose reciprocal derivative has a positive real part is the same.

The following theorem is a straight forward verification on applying the same procedure as described in Theorems 3 and 4 and the result is sharp for the values $c_1 = 0,\ c_2 = 2$ and $x = 1$.

Theorem 3. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then* $|\, t_3 - t_2^2| \le \frac{2}{3}$.

Using the fact that $|c_n| \le 2,\quad n \in N = \{1, 2, 3, \cdots\}$, with the help of c_2 and c_3 values given in (6) and (8) respectively together with the values in (14), we at once obtain all the below inequalities.

Theorem 4. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then we have the following inequalities:*
(i) $|t_2| \le 1$ *(ii)* $|t_3| \le \frac{4}{3}$ *(iii)* $|t_4| \le \frac{13}{6}$ *(iv)* $|t_5| \le \frac{59}{15}$.

Using the results of Theorems 3, 4, 5 and 6 in (3), we obtain the following corollary.

Corollary 1. *If* $f(z) \in \widetilde{RT}$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f *then* $|H_3(1)| \le \frac{1}{3}\left[\frac{3157}{360} + \left(\frac{13}{6}\right)^{\frac{5}{2}}\right]$.

Remark 3. It is observed that the upper bound to the third Hankel determinant of a function whose derivative has a positive real part [14] and a function whose reciprocal derivative has a positive real part is the same.

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B. VENKATESWARLU, D. VAMSHEE KRISHNA
Department of Mathematics, GIT, GITAM University
Visakhapatnam-530 045, A. P., India
E-mail: *bvlmaths@gmail.com; vamsheekrishna1972@gmail.com*

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N. RANI
Praveenya Institute of Marine Engineering & Maritime studies, Modavalasa-534 002, Visakhapatnam, A. P., India
E-mail: *raninekkanti1111@gmail.com*