Third Hankel determinant for the inverse of reciprocal of bounded turning functions

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Abstract. In this paper we obtain the best possible upper bound to the third Hankel determinants for the functions belonging to the class of reciprocal of bounded turning functions using Toeplitz determinants.

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1 Introduction

Let $A$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc $E = \{z : |z| < 1\}$. Let $S$ be the subclass of $A$ consisting of univalent functions. For a univalent function in the class $A$, it is well known that the $n^{th}$ coefficient is bounded by $n$. The bounds for the coefficients of univalent functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [12] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by many authors in the literature. For example, Noor [10] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in $S$ with bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [7]. One can easily observe that the Fekete-Szego functional is $H_2(1)$. Fekete-Szego then further generalized the estimate
$|a_3 - \mu a_2^2|$ with $\mu$ real and $f \in S$. R. M. Ali [1] found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szego functional $|\gamma_3 - t\gamma_2^2|$, where $t$ is real, for the inverse function of $f$ defined as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n,$$

when it belongs to the class of strongly starlike functions of order $\alpha$ ($0 < \alpha \leq 1$) denoted by $\widetilde{ST}(\alpha)$. In the recent years several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent analytic functions. In particular for $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, $a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$  (2)

For $f \in A$, $a_1 = 1$, so we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|.$$  (3)

The sharp upper bound to the second Hankel functional $H_2(2)$ for the subclass $RT$ of $S$, consisting of functions whose derivative has a positive real part, studied by Mac Gregor [9] was obtained by Janteng [6]. It was known that if $f \in RT$ then $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, \ldots\}$. Also the sharp upper bound for the functional $|a_3 - a_2^2|$ was $\frac{2}{3}$, stated in [2], for the class $RT$. Further, the best possible sharp upper bound for the functional $|a_2 a_3 - a_4|$ was obtained by Babalola [2] and hence the sharp inequality for $|H_3(1)|$, for the class $RT$. The sharp upper bound to $|H_3(1)|$ for the class of inverse of a function whose derivative has a real part was obtained by D. Vamshee Krishna et al. [14].

Motivated by the above mentioned results obtained by different authors in this direction and the results by Babalola [2], in the present paper, we seek an upper bound to the second Hankel determinant, $|t_2 t_3 - t_4|$ and an upper bound to the third Hankel determinant, for certain subclass of analytic functions defined as follows.
Definition 1. A function \( f(z) \in A \) is said to be function whose reciprocal derivative has a positive real part (also called reciprocal of bounded turning function), denoted by \( f \in \overline{RT} \), if and only if
\[
\text{Re}\left(\frac{1}{f'(z)}\right) > 0, \quad \forall z \in E.
\]
(4)

Some preliminary Lemmas required for proving our results are as follows.

2 Preliminary Results

Let \( P \) denote the class of functions consisting of \( p \), such that
\[
p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{5}
\]
which are regular in the open unit disc \( E \) and satisfy \( \text{Re}\left(p(z)\right) > 0 \) for any \( z \in E \).

Here \( p(z) \) is called the Carathéodory function [3].

Lemma 1 (see [11, 13]). If \( p \in P \), then \(|c_k| \leq 2\), for each \( k \geq 1 \) and the inequality is sharp for the function \( 1 + z - z \).

Lemma 2 (see [5]). The power series for \( p(z) \) given in (5) converges in the open unit disc \( E \) to a function in \( P \) if and only if the Toeplitz determinants
\[
D_n = \begin{vmatrix}
2 & c_1 & c_2 & \cdots & c_n \\
c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\
c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2
\end{vmatrix}, \quad \text{for } n = 1, 2, 3, \ldots,
\]
and \( c_{-k} = \overline{c_k} \), are all non-negative. They are strictly positive except for
\[
p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k}z),
\]
\( \rho_k > 0 \), \( t_k \) real and \( t_k \neq t_j \), for \( k \neq j \), where \( p_0(z) = \frac{1 + z}{1 - z} \); in this case
\( D_n > 0 \) for \( n < (m - 1) \) and \( D_n = 0 \) for \( n \geq m \).

This necessary and sufficient condition found in [5] is due to Carathéodory and Toeplitz. We may assume without restriction that \( c_1 > 0 \). On using Lemma 2, for \( n = 2 \), we have
\[
D_2 = \begin{vmatrix}
2 & c_1 & c_2 \\
\overline{c_1} & 2 & c_1 \\
\overline{c_2} & c_1 & 2
\end{vmatrix} = [8 + 2\text{Re}\{c_1^2c_2\} - 2|c_2|^2 - 4|c_1|^2] \geq 0,
\]
which is equivalent to
\[ 2c_2 = c_1^2 + x(4 - c_1^2), \text{ for some } x, |x| \leq 1. \] (6)

For \( n = 3 \),
\[
D_3 = \begin{vmatrix}
2 & c_1 & c_2 & c_3 \\
c_1 & 2 & c_1 & c_2 \\
c_2 & c_1 & 2 & c_1 \\
c_3 & c_2 & c_1 & 2
\end{vmatrix} \geq 0
\]

and is equivalent to
\[
|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2(2c_2 - c_1^2)|^2. \] (7)

Simplifying the relations (6) and (7), we get
\[
4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z
\]
for some \( z \), with \( |z| \leq 1 \). (8)

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [8] and used by several authors in the literature.

3 Main Result

**Theorem 1.** If \( f(z) \in \tilde{RT} \) and \( f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n \) near \( w=0 \) is the inverse function of \( f \) then
\[
|t_2t_4 - t_3^2| \leq \frac{137}{288}.
\]

**Proof.** For \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \tilde{RT} \), there exists an analytic function \( p \in \mathcal{P} \) in the open unit disc \( E \) with \( p(0) = 1 \) and \( \text{Re}(p(z)) > 0 \) such that
\[
\frac{1}{f'(z)} = p(z) \iff 1 = f'(z)p(z).
\] (9)

Replacing \( f'(z) \) and \( p(z) \) with their equivalent series expressions in (9), we have
\[
1 = \left( 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right) \left( 1 + \sum_{n=1}^{\infty} c_n z^n \right).
\]

Upon simplification, we obtain
\[
1 = 1 + (c_1 + 2a_2)z + (c_2 + 2a_2c_1 + 3a_3)z^2 \\
+ (c_3 + 2a_2c_2 + 3a_3c_1 + 4a_4)z^3 \\
+ (c_4 + 2a_2c_3 + 3a_3c_2 + 4a_4c_1 + 5a_5)z^4 \cdots .
\] (10)
Equating the coefficients of like powers of \( z, z^2, z^3 \) and \( z^4 \) respectively on both sides of (10), after simplifying, we get

\[
\begin{align*}
  a_2 &= -\frac{c_1}{2}; \quad a_3 = \frac{1}{3}(c_1^2 - c_2); \quad a_4 = -\frac{1}{4}(c_3 - 2c_1c_2 + c_1^3); \\
  a_5 &= -\frac{1}{5}(c_4 - 2c_1c_3 + 3c_1^2c_2 - c_2^2 - c_1^4). \\
\end{align*}
\]

(11)

Since \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widehat{RT} \), from the definition of inverse function of \( f \), we have

\[
\begin{align*}
  w &= f(f^{-1}(w)) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n (f^{-1}(w))^n = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n (w + \sum_{n=2}^{\infty} t_n w^n)^n.
\end{align*}
\]

After simplifying, we get

\[
(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 \\
+ (t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \cdots = 0.
\]

(12)

Equating the coefficients of like powers of \( w^2, w^3, w^4 \) and \( w^5 \) on both sides of (12), respectively, further simplification gives

\[
\begin{align*}
  t_2 &= -a_2; \quad t_3 = -a_3 + 2a_2^2; \quad t_4 = -a_4 + 5a_2a_3 - 5a_2^3; \\
  t_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_4^2.
\end{align*}
\]

(13)

Using the values of \( a_2, a_3, a_4 \) and \( a_5 \) in (11) along with (13), upon simplification, we obtain

\[
\begin{align*}
  t_2 &= \frac{c_1}{2}; \quad t_3 = \frac{1}{6}[2c_2 + c_1^2]; \quad t_4 = \frac{1}{24}[6c_3 + 8c_1c_2 + c_1^3]; \\
  t_5 &= \frac{1}{120}[24c_4 + 42c_1c_3 + 22c_1^2c_2 + 16c_2^2 + c_1^4].
\end{align*}
\]

(14)

Substituting the values of \( t_2, t_3 \) and \( t_4 \) from (14) in the functional \( |t_2t_4 - t_3^2| \) for the function \( f \in RT \) upon simplification, we obtain

\[
|t_2t_4 - t_3^2| = \frac{1}{144} \left| 18c_1c_3 + 8c_1^2c_2 - 16c_2^2 + c_1^4 \right|
\]

which is equivalent to

\[
|t_2t_4 - t_3^2| = \frac{1}{144} \left| d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4 \right|
\]

where \( d_1 = 18; \quad d_2 = 8; \quad d_3 = -16; \quad d_4 = -1 \).

(15)

(16)

Substituting the values of \( c_2 \) and \( c_3 \) given in (6) and (8) respectively from Lemma 2 on the right-hand side of (15) and using the fact \(|z| < 1\), we have
\[ 4\left| d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_4^4 \right| \leq \left| (d_1 + 2d_2 + d_3 + 4d_4)c_1^4 \right| \]
\[ + \{2d_1c_1 + 2(d_1 + d_2 + d_3)c_1^2|x| - [(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3]|x|^2\}(4 - c_1^2). \] (17)

From (16) and (17), we can now write
\[
(d_1 + 2d_2 + d_3 + 4d_4) = 14; \quad 2d_1 = 36; \quad 2(d_1 + d_2 + d_3) = 20; \]
\[
(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3 = 2(c_1^2 + 18c_1 + 32). \] (18)

Since \( c_1 \in [0, 2] \), using the result \( (c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b) \), where \( a, b \geq 0 \) in (18), we can have
\[
-(c_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3 \leq -2(c_1^2 - 18c_1 + 32). \] (19)

Substituting the calculated values from (18) and (19) on the right-hand side of (17), we have
\[
4\left| d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_4^4 \right| \leq 14c_1^4 + \{36c_1 + 20c_1^3|x| \]
\[ - 2(c_1^2 - 18c_1 + 32)|x|^2\}(4 - c_1^2). \]

Choosing \( c_1 = c \in [0, 2] \), applying triangle inequality and replacing \( |x| \) by \( \mu \) on the right-hand side of the above inequality
\[
2\left| d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_4^4 \right| \leq 7c_1^4 + \{18c + 10c^3\mu + (c^2 - 18c + 32)\mu^2\}(4 - c^2) \]
\[ = F(c, \mu), \quad 0 \leq \mu = |x| \leq 1 \quad \text{and} \quad 0 \leq c \leq 2. \] (20)

We next maximize the function \( F(c, \mu) \) on the closed region \([0, 2] \times [0, 1]\). Differentiating \( F(c, \mu) \) given in (20) partially with respect to \( \mu \), we obtain
\[
\frac{\partial F}{\partial \mu} = [20c + 2(c - 2)(c - 16)\mu](4 - c^2) > 0. \] (21)

For \( 0 < \mu < 1 \) and for fixed \( c \) with \( 0 < c < 2 \), from (21), we observe that \( \frac{\partial F}{\partial \mu} > 0 \). Therefore, \( F(c, \mu) \) becomes an increasing function of \( \mu \) and hence it cannot have a maximum value at any point in the interior of the closed region \([0, 2] \times [0, 1]\). Moreover, for a fixed \( c \in [0, 2] \), we have
\[
\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c). \]

Therefore, replacing \( \mu \) by 1 in \( F(c, \mu) \), upon simplification, we obtain
\[
G(c) = -4c^4 + 12c^2 + 128. \] (22)
\[ G'(c) = -16c^3 + 24c \quad \text{(23)} \]
\[ G''(c) = -48c^2 + 24. \quad \text{(24)} \]

For optimum value of \( G(c) \), consider \( G'(c) = 0 \). From (23), we get
\[ c^2 = \frac{3}{2}. \]

Using the obtained value of \( c^2 \) in (24), which simplifies to give
\[ G''(c) = -48 < 0. \]

Therefore, by the second derivative test, \( G(c) \) has maximum value at \( c = \sqrt{\frac{3}{2}} \in [0, 2] \). Substituting the value of \( c \) in the expression (22), upon simplification, we obtain the maximum value of \( G(c) \) at \( c \) as
\[ G_{\text{max}} = 137. \quad \text{(25)} \]

Simplifying the expressions (20) and (25)
\[ | d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^3 + d_4c_4^1 | \leq \frac{137}{2}. \quad \text{(26)} \]

From the relations (15) and (26), we obtain
\[ | t_2t_4 - t_3^2 | \leq \frac{137}{288}. \quad \text{(27)} \]

This completes the proof of our Theorem. \( \square \)

Remark 1. It is observed that the upper bound to the second Hankel determinant of inverse of a function whose derivative has a positive real part [14] and the inverse of a function whose reciprocal derivative has a positive real part is the same.

Theorem 2. If \( f(z) \in \tilde{RT} \) and \( f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n \) near \( w = 0 \) is the inverse function of \( f \) then \( | t_2t_3 - t_4^2 | = \left( \frac{13}{6} \right)^2 \).

Proof. Substituting the values of \( t_2, t_3 \) and \( t_4 \) from (14) in \( | t_2t_3 - t_4 | \) for the function \( f \in \tilde{RT} \), after simplifying, we get
\[ | t_2t_3 - t_4 | = \frac{1}{24} | -6c_3 - 4c_1c_2 + c_1^3 |. \quad \text{(28)} \]

Substituting the values of \( c_2 \) and \( c_3 \) from (6) and (8) respectively, from Lemma 2 on the right-hand side of (28) and using the fact \( | z | < 1 \), after simplifying, we get
\[ 2| -6c_3 - 4c_1c_2 + c_1^3 | \leq \left| -5c_1^3 - 6(4 - c_1^2) - 10c_1(4 - c_1^2)|x| \right| \]
\[+ 3(c_1 + 2)(4 - c_1^2)|x|^2].\]  

(29)

Since \(c_1 = c \in [0, 2]\), using the result \((c_1 + a) \geq (c_1 - a)\), where \(a \geq 0\), applying triangle inequality and replacing \(|x|\) by \(\mu\) on the right-hand side of the above inequality, we have

\[2| - 6c_3 - 4c_1c_2 + c_1^3| \leq |5c^3 + 6(4 - c^2) + 10(c - 2)(4 - c^2)\mu| = F(c, \mu), \quad 0 \leq \mu = |x| \leq 1 \text{ and } 0 \leq c \leq 2.\]  

(30)

Next, we maximize the function \(F(c, \mu)\) on the closed square \([0, 2] \times [0, 1]\). Differentiating \(F(c, \mu)\) partially with respect to \(\mu\), we get

\[\frac{\partial F}{\partial \mu} = (4 - c^2)[10c + 6(c - 2)\mu] > 0.\]

As described in Theorem 3, further, we obtain

\[G(c) = -8c^3 + 52c\]

(31)

\[G'(c) = -24c^2 + 52\]

(32)

\[G''(c) = -48c.\]

(33)

For optimum value of \(G(c)\), consider \(G'(c) = 0\). From (32), we get

\[c^2 = \frac{13}{6}.\]

Using the obtained value of \(c = \sqrt{\frac{13}{6}} \in [0, 2]\) in (33), then

\[G''(c) = -8\sqrt{\frac{13}{6}} < 0.\]

Therefore, by the second derivative test, \(G(c)\) has maximum value at \(c = \sqrt{\frac{13}{6}}\).

Substituting the value of \(c\) in the expression (31), upon simplification, we obtain the maximum value of \(G(c)\) at \(c\) as

\[G_{\text{max}} = \frac{104}{3}\sqrt{\frac{13}{6}}.\]  

(34)

From the expressions (30) and (34), after simplifying, we get

\[| - 6c_3 - 4c_1c_2 + c_1^3| \leq \frac{52}{3}\sqrt{\frac{13}{6}}.\]  

(35)

Simplifying the relations (28) and (35), we obtain

\[| t_2t_3 - t_4| \leq \frac{1}{3}\left(\frac{13}{6}\right)^{\frac{3}{4}}.\]

This completes the proof of our Theorem. \(\Box\)
Remark 2. It is observed that the upper bound to the $|t_2t_3 - t_4|$ of a function whose derivative has a positive real part [14] and a function whose reciprocal derivative has a positive real part is the same.

The following theorem is a straightforward verification on applying the same procedure as described in Theorems 3 and 4 and the result is sharp for the values $c_1 = 0$, $c_2 = 2$ and $x = 1$.

**Theorem 3.** If $f(z) \in \tilde{RT}$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of $f$ then $|t_3 - t_2^2| \leq \frac{2}{3}$.

Using the fact that $|c_n| \leq 2$, $n \in N = \{1, 2, 3, \cdots \}$, with the help of $c_2$ and $c_3$ values given in (6) and (8) respectively together with the values in (14), we at once obtain all the below inequalities.

**Theorem 4.** If $f(z) \in \tilde{RT}$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of $f$ then we have the following inequalities:

(i) $|t_2| \leq 1$  
(ii) $|t_3| \leq \frac{1}{3}$  
(iii) $|t_4| \leq \frac{13}{6}$  
(iv) $|t_5| \leq \frac{59}{15}$.

Using the results of Theorems 3, 4, 5 and 6 in (3), we obtain the following corollary.

**Corollary 1.** If $f(z) \in \tilde{RT}$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of $f$ then $|H_3(1)| \leq \frac{1}{3} \left[ \frac{3157}{360} + \left( \frac{13}{6} \right)^{5/2} \right]$.

Remark 3. It is observed that the upper bound to the third Hankel determinant of a function whose derivative has a positive real part [14] and a function whose reciprocal derivative has a positive real part is the same.

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