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Certain classes of p-valent analytic functions with negative coefficients and (λ, p)-starlike with respect to certain points

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Abstract. In this article, we consider classes $S^*_{s,\lambda}(p,\alpha,\beta)$, $S^*_{c,\lambda}(p,\alpha,\beta)$, and $S^*_{sc,\lambda}(p,\alpha,\beta)$ of p-valent analytic functions with negative coefficients in the unit disk. They are, respectively, (λ, p)-starlike with respect to symmetric points, (λ, p)-starlike with respect to conjugate points, and (λ, p)-starlike with respect to symmetric conjugate points. Necessary and sufficient coefficient conditions for functions f belonging to these classes are obtained. Several properties such as the coefficient estimates, growth and distortion theorems, extreme points, radii of starlikeness, convexity, and integral operator are studied.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the punctured unit disk $\mathbb{U} = \{z : |z| < 1\}$. For f which belong to A, Salagean [1] introduced the following operator:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{2}$$

Note that

$$D^n f(z) = D\left(D^{n-1} f(z)\right) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}). \tag{3}$$

Let $\mathcal{T}_p$ (p a fixed integer greater than 0) denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \tag{4}$$

that are holomorphic and p-valent in the unit disk $|z| < 1$.

Also let $\mathcal{T}_p$ denote the subclass of $\mathcal{S}_p$ consisting of functions that can be expressed in the form

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_{k+p}| z^{k+p} \tag{5}$$

where $a_{k+p} \geq 0,\ p \in \mathbb{N} = \{1, 2, 3, ...\}\ ,\ n \in \mathbb{N}$, which are analytic in the unit disc $\mathbb{U}$.

We can write the following equalities for the functions $f(z)$ which belong to the class $\mathcal{T}_p$ (see [2]):

$$\begin{aligned}
D^0 f(z) &= f(z), \\
D^1 f(z) &= Df(z) = zf'(z) \\
&= z\left(pz^p - \sum_{k=1}^{\infty} (k+p) a_{k+p} z^{k+p-1}\right) \\
&= pz^p - \sum_{k=1}^{\infty} (k+p) a_{k+p} z^{k+p}, \\
D^2 f(z) &= D(Df(z)) = p^2 z^p - \sum_{k=1}^{\infty} (k+p)^2 a_{k+p} z^{k+p}, \\
&\cdots \\
D^\lambda f(z) &= D\left(D^{\lambda-1} f(z)\right) = p^\lambda z^p - \sum_{k=1}^{\infty} (k+p)^\lambda a_{k+p} z^{k+p}.
\end{aligned} \tag{6}$$

Let S be the subclass of A consisting of functions that are regular and univalent in $\mathbb{U}$. Let S^* be the subclass of S consisting of functions starlike in $\mathbb{U}$. It is known that

$$f \in S^* \ \text{ if and only if } \ \Re\left\{\tfrac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

In [3], Sakaguchi defined the class of starlike functions with respect to symmetric points as follows:

Let $f \in S$. Then f is said to be starlike with respect to symmetric points in $\mathbb{U}$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad (z \in \mathbb{U}), \tag{7}$$

and we denote this class by S_s^*. Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see Sakaguchi [3], Robertson [5], Stankiewicz [6], Wu [7] and Owa et al. [8]. El-Ashwah and Thomas in [9] introduced two other classes, namely the class S_c^* consisting of functions starlike with respect to conjugate points and S_{sc}^* consisting of functions starlike with respect to symmetric conjugate points. The class S^*_{sc} is also studied by Chen et al. [10] (see also [11]).

In [4], Sudharsan et al. introduced the class $S^*_s(\alpha,\beta)$ of functions $f(z)\in S$ and satisfying the following condition (see also [12]):

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-1\right|<\beta\left|\alpha\ \frac{zf'(z)}{f(z)-f(-z)}+1\right| \tag{8}$$

for some $0\le\alpha\le1,\ 0<\beta\le1,\ z\in\mathbb{U}$.

Recently, Aouf el at.[13] introduced the class $S^*_{s,n}\mathcal{T}(\alpha,\beta)$ of functions $f(z)$ being defined by (5). Then $f(z)$ is said to be n-starlike with respect to symmetric points if it satisfies the following condition:

$$\left|\frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}-1\right|<\beta\left|\alpha\ \frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}+1\right|, \tag{9}$$

where $n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}, 0\le\alpha\le1,\ 0<\beta\le1,\ 0\le\frac{2(1-\beta)}{1+\alpha\beta}$, and $z\in\mathbb{U}$.

However, in this paper we consider the subclass $\mathcal{T}$ defined by (5).

Definition 1. Let a function $f(z)$ be defined by (5). Then $f(z)$ is said to be (λ,p)-starlike with respect to symmetric points if it satisfies the following condition:

$$\left|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-D^\lambda f(-z)}-p\right|<\beta\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-D^\lambda f(-z)}+p\right|, \tag{10}$$

where $\lambda\in\mathbb{N}_0=\{0\}\cup\mathbb{N},\ p\in\mathbb{N},\ 0\le\alpha\le1,\ 0<\beta\le1,\ 0\le\frac{2p(1-\beta)}{1+\alpha\beta}$, and $z\in\mathbb{U}$. We denote the class of functions (λ,p)-starlike with respect to symmetric points by $S^*_{s,\lambda}(p,\alpha,\beta)$.

Definition 2. Let a function $f(z)$ be defined by (5). Then $f(z)$ is said to be (λ,p)-starlike with respect to conjugate points if it satisfies the following condition:

$$\left|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)+\overline{D^\lambda f(\overline{z})}}-p\right|<\beta\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^\lambda f(z)+\overline{D^\lambda f(\overline{z})}}+p\right|, \tag{11}$$

where $\lambda\in\mathbb{N}_0=\{0\}\cup\mathbb{N},\ p\in\mathbb{N},\ 0\le\alpha\le1,\ 0<\beta\le1,\ 0\le\frac{2p(1-\beta)}{1+\alpha\beta}$, and $z\in\mathbb{U}$. We denote the class of functions (λ,p)-starlike with respect to conjugate points by $S^*_{c,\lambda}(p,\alpha,\beta)$.

Definition 3. Let a function $f(z)$ be defined by (5). Then $f(z)$ is said to be (λ,p)-starlike with respect to symmetric conjugate points if it satisfies the following condition:

$$\left|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-\overline{D^\lambda f(\overline{-z})}}-p\right|<\beta\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^\lambda f(z)-\overline{D^\lambda f(\overline{-z})}}+p\right|, \tag{12}$$

where $\lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ p \in \mathbb{N},\ 0 \leq \alpha \leq 1,\ 0 < \beta \leq 1,\ 0 \leq \frac{2p(1-\beta)}{1+\alpha\beta}$, and $z \in \mathbb{U}$. We denote the class of functions (λ, p)-starlike with respect to symmetric conjugate points by $S^*_{sc,\lambda}(p, \alpha, \beta)$.

Notice that the above conditions imposed on $\alpha,\ \beta$ and p in the introduction are necessary to ensure that these classes form a subclass of S. For more classes (see in details Halim et al. [14, 15].

2 Coefficient estimates

To prove the following theorems, we will adopt the technique used by Dziok [16], and assume that $\lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ p \in \mathbb{N},\ 0 \leq \alpha \leq 1,\ 0 < \beta \leq 1,\ 0 \leq \frac{2p(1-\beta)}{1+\alpha\beta}$, and $z \in \mathbb{U}$.

Theorem 1. *Let the function $f(z)$ be defined by (5) and $D^\lambda f(z) - D^\lambda f(-z) \neq 0$ for $z \neq 0$. Then $f(z) \in S^*_{s,\lambda}(p, \alpha, \beta)$ if and only if*

$$\sum_{k=1}^{\infty} \left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha + 1 - (-1)^{p+k})\right)\right](k+p)^{\lambda+1}|a_{k+p}|$$
$$\leq p^{\lambda+1}\left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right]. \tag{13}$$

Proof. Use (5), (6) and (10), that is

$$\left|\frac{D^{\lambda+1}f(z)}{D^\lambda f(z) - D^\lambda f(-z)} - p\right| < \beta\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^\lambda f(z) - D^\lambda f(-z)} + p\right|$$
$$= \left|(-1)^p p^{\lambda+1} z^p - \sum_{k=1}^{\infty}\left[k + p(-1)^{p+k}\right](k+p)^\lambda |a_{k+p}| z^{k+p}\right|$$
$$< \beta\left|(\alpha + 1 - (-1)^p)p^{\lambda+1}z^p - \sum_{k=1}^{\infty}[\alpha k + p(\alpha + 1 - (-1)^{p+k})]|a_{k+p}|z^{k+p}\right|$$

and

$$-p^{\lambda+1}\left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right]|z|^p$$
$$+ \sum_{k=1}^{\infty}\left[k + p(-1)^{p+k} + \beta\{\alpha k + p(\alpha + 1 - (-1)^{p+k})\}\right](k+p)^\lambda |a_{k+p}||z|^{k+p} \leq 0.$$

Letting $|z| = 1$, we have

$$\sum_{k=1}^{\infty}\left[k + p(-1)^{p+k} + \beta\{\alpha k + p(\alpha + 1 - (-1)^{p+k})\}\right](k+p)^\lambda |a_{k+p}| \leq 0.$$

$$-p^{\lambda+1}\left[(-1)^p + \beta(\alpha + 1 - (-1)^p)\right] \leq 0.$$

Therefore, by the maximum modulus theorem, we have $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$.

For the converse, let us suppose that

$$\frac{\left|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)-D^{\lambda}f(-z)}-p\right|}{\left|\alpha\ \frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)-D^{\lambda}f(-z)}+p\right|}<\beta.$$

This implies that

$$\left|\frac{-\left[-(-1)^p p^{\lambda+1}z^p+\sum_{k=1}^{\infty}\left[k+p(-1)^{p+k}\right](k+p)^{\lambda}|a_{k+p}|z^{k+p}\right]}{(\alpha+1-(-1)^p)p^{\lambda+1}z^p-\sum_{k=1}^{\infty}[\alpha k+p(\alpha+1-(-1)^{p+k})]|a_{k+p}|z^{k+p}}\right|<\beta.$$

Since $|\Re(z)|\le|z|$ for all z, we have

$$\Re\left\{\frac{-(-1)^p p^{\lambda+1}z^p+\sum_{k=1}^{\infty}\left[k+p(-1)^{p+k}\right](k+p)^{\lambda}|a_{k+p}||z|^{k+p}}{(\alpha+1-(-1)^p)p^{\lambda+1}z^p-\sum_{k=1}^{\infty}[\alpha k+p(\alpha+1-(-1)^{p+k})]|a_{k+p}||z|^{k+p}}\right\}<\beta. \tag{14}$$

If we choose values of z on the real axis, then $\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)-D^{\lambda}f(-z)}$ is real and $D^{\lambda}f(z)-D^{\lambda}f(-z)\neq 0$ for $z\neq 0$. Upon clearing the denominator in (14) and letting $z\to 1^-$ through real values, we obtain

$$\sum_{k=1}^{\infty}\left[k+p(-1)^{p+k}\right](k+p)^{\lambda}|a_{k+p}|+\sum_{k=1}^{\infty}\beta[\alpha k+p(\alpha+1-(-1)^{p+k})](k+p)^{\lambda}|a_{k+p}|$$
$$\le\beta(\alpha+1-(-1)^p)p^{\lambda+1}+(-1)^p p^{\lambda+1}.$$

This gives the required condition.

Corollary 1. *Let the function $f(z)$ defined by (5) be in the class $S^*_{s,\lambda}(p,\alpha,\beta)$. Then we have*

$$|a_{k+p}|\le\frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}\quad(k\ge 1), \tag{15}$$

where $p\in\mathbb{N},\ \lambda\in\mathbb{N}_0$ and $z\in\mathbb{U}$.

The equality in (15) is attained for the function $f(z)$ given by

$$f_k(z)=z^p-\frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}z^{k+p}\quad(k\ge 1), \tag{16}$$

where $p\in\mathbb{N},\ \lambda\in\mathbb{N}_0$ and $z\in\mathbb{U}$.

Theorem 2. *Let the function $f(z)$ be defined by (5) and $D^{\lambda}f(z)-D^{\lambda}f(-z)\neq 0$ for $z\neq 0$. Then $f(z)\in S^*_{c,\lambda}(p,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left(\beta(\alpha+2)-1\right)\right](k+p)^{\lambda+1}|a_{k+p}|$$
$$\le p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]. \tag{17}$$

Corollary 2. *Let the function $f(z)$ defined by (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$. Then we have*

$$|a_{k+p}| \leq \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[k(1+\alpha\beta)+(\beta(\alpha+2)-1)\right](k+p)^{\lambda+1}} \quad (k \geq 1), \tag{18}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

The equality in (18) is attained for the function $f(z)$ given by

$$f_k(z) = z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[k(1+\alpha\beta)+(\beta(\alpha+2)-1)\right](k+p)^{\lambda+1}} z^{k+p} \quad (k \geq 1), \tag{19}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Theorem 3. *Let the function $f(z)$ be defined by (5) and $D^\lambda f(z) - D^\lambda f(-z) \neq 0$ for $z \neq 0$. Then $f(z) \in S^*_{sc,\lambda}(p,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}|a_{k+p}|$$
$$\leq p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]. \tag{20}$$

Corollary 3. *Let the function $f(z)$ defined by (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$. Then we have*

$$|a_{k+p}| \leq \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}} \quad (k \geq 1), \tag{21}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

The equality in (21) is attained for the function $f(z)$ given by

$$f_k(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}} z^{k+p} \quad (k \geq 1), \tag{22}$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

3 Growth and Distortion theorems

Theorem 4. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Then we have*

$$p^i|z|^p - \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta)+p\left((-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}}|z|^{p+1}$$
$$\leq \left|D^i f(z)\right|$$
$$\leq p^i|z|^p + \frac{p^{\lambda+1}\left[(-1)^p+\beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta)+p\left((-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}}|z|^{p+1} \tag{23}$$

for $z \in \mathbb{U}$, where $0 \leq i \leq n$. The result is sharp.

Proof. Note that $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ if and only if $D^i f(z) \in S^*_{s,\lambda-i}(p,\alpha,\beta)$, and that

$$D^i f(z) = p^i z^p - \sum_{k=1}^{\infty} (k+p)^i |a_{k+p}| z^{k+p}. \tag{24}$$

Using Theorem 1, we know that

$$\sum_{k=1}^{\infty}(k+p)^i|a_{k+p}| \leq \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right](p+1)^i}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda}}. \tag{25}$$

That is

$$\sum_{k=1}^{\infty}(k+p)^i|a_{k+p}| \leq \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}}. \tag{26}$$

It follows from (24) and (26) that

$$\begin{aligned} \left|D^i f(z)\right| &\geq p^i|z|^p - |z|^{p+1}\sum_{k=1}^{\infty}(k+p)^i|a_{k+p}| \\ &\geq p^i|z|^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+2})\right)\right](p+1)^{\lambda-i}}|z|^{p+1}. \end{aligned} \tag{27}$$

Also

$$\begin{aligned} \left|D^i f(z)\right| &\leq p^i|z|^p + |z|^{p+1}\sum_{k=1}^{\infty}(k+p)^i|a_{k+p}|z^{k+p} \\ &\leq p^i|z|^p + \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}}|z|^{p+1}. \end{aligned} \tag{28}$$

Finally, we note that the equality in (23) is attained by the function

$$D^i f(z) = p^i z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda-i}} z^{p+1} \tag{29}$$

or by

$$f(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^{\lambda}} z^{p+1} \tag{30}$$

and this completes the proof of Theorem 4.

Corollary 4. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Then we have*

$$|z|^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^\lambda}|z|^{p+1}$$
$$\leq |f(z)|$$
$$\leq |z|^p + \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left((-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})\right)\right](p+1)^\lambda}|z|^{p+1} \tag{31}$$

for $z \in \mathbb{U}$. The result is sharp for the function $f(z)$ given by (30).

Proof. For $i = 0$ in Theorem 4, we can easily show (30).

Similarly, we can prove the following result.

Theorem 5. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{c,\lambda}(p,\alpha,\beta)$. Then we have*

$$p^i|z|^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda-i}}|z|^{p+1}$$
$$\leq \left|D^i f(z)\right|$$
$$\leq p^i|z|^p + \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+1)-1\right)\right](p+1)^{\lambda-i}}|z|^{p+1} \tag{32}$$

for $z \in \mathbb{U}$, where $0 \leq i \leq n$. The result is sharp, for the function $f(z)$ given by

$$D^i f(z) = p^i z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda-i}}z^{p+1} \tag{33}$$

or by

$$f(z) = z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda}}z^{p+1}. \tag{34}$$

Corollary 5. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{c,\lambda}(p,\alpha,\beta)$. Then we have*

$$|z|^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda}}|z|^{p+1}$$
$$\leq |f(z)|$$
$$\leq |z|^p + \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](p+1)^{\lambda}}|z|^{p+1} \tag{35}$$

for $z \in \mathbb{U}$. The result is sharp, for the function $f(z)$ given by (34).

Theorem 6. *Let the function $f(z)$ defined by (5) be in the class $f(z) \in S^*_{sc,\lambda}(p,\alpha,\beta)$. Then we have*

$$p^i|z|^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left(\beta(1+\alpha) + (1-\beta)(-1)^{p+1}\right)\right](p+1)^{\lambda-i}}|z|^{p+1}$$
$$\leq \left|D^i f(z)\right|$$
$$\leq p^i|z|^p + \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[(1+\alpha\beta) + p\left(\beta(1+\alpha) + (1-\beta)(-1)^{p+1}\right)\right](p+1)^{\lambda-i}}|z|^{p+1} \tag{36}$$

for $z \in \mathbb{U}$, where $0 \leq i \leq n$. The result is sharp.

4 Extreme points

Theorem 7. *Let $f_p(z) = z^p$ and*

$$f_{k+p}(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} z^{k+p}, \tag{37}$$

where $k \geq 1$. Then $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \sigma_{k+p} f_{k+p}(z), \tag{38}$$

where $\sigma_{k+p} \geq 0$ $(k \geq 1)$ and $\sum_{k=0}^{\infty} \sigma_{k+p} = 1$.

Proof. Suppose

$$f(z) = \sum_{k=0}^{\infty} \sigma_{k+p} f_{k+p}(z)$$
$$= z^p - \sum_{k=1}^{\infty} \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} \sigma_{k+p} z^{k+p}. \tag{39}$$

Then we get

$$\sum_{k=1}^{\infty} \frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}$$
$$\bullet \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} \sigma_{k+p}$$
$$= \sum_{k=1}^{\infty} \sigma_{k+p} = 1 - \sigma_p \leq 1. \tag{40}$$

It follows from Theorem 1 that the function $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$.

Conversely, suppose that $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Again, by using Theorem 1, we can show that

$$|a_{p+k}| \leq \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}} \; k \geq 1, \quad (41)$$

where $k \geq 1,\ \lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ p \in \mathbb{N},\ 0 \leq \alpha \leq 1,\ 0 < \beta \leq 1$ and $z \in \mathbb{U}$.

Setting

$$\sigma_{p+k} \leq \frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]} \; (k \geq 1) \quad (42)$$

and

$$\sigma_p = 1 - \sum_{k=1}^{\infty} \sigma_{k+p}, \quad (43)$$

we can see that $f(z)$ can be expressed in the form (38). This completes the proof of Theorem 7.

Corollary 6. *The extreme points of the class $S^*_{s,\lambda}(p,\alpha,\beta)$ are functions $f_{k+p}(z)$ $(k \geq 1,\ p \in \mathbb{N})$ given by Theorem 7.*

Similar to Theorem 7, we can easily prove the following theorems for $S^*_{c,\lambda}(p,\alpha,\beta)$ and $S^*_{sc,\lambda}(p,\alpha,\beta)$ classes.

Theorem 8. *Let $f_p(z) = z^p$ and*

$$f_{k+p}(z) = z^p - \frac{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}{\left[k(1+\alpha\beta) + (\beta(\alpha+2)-1)\right](k+p)^{\lambda+1}} z^{k+p} \; (k \geq 1), \quad (44)$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$. Then $f(z) \in S^*_{c,\lambda}(p,\alpha,\beta)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \sigma_{k+p} f_{k+p}(z)$, where $\sigma_{k+p} \geq 0$ $(k \geq 1)$ and $\sum_{k=1}^{\infty} \sigma_{k+p} = 1$.

Corollary 7. *The extreme points of the class $S^*_{c,\lambda}(p,\alpha,\beta)$ are functions $f_{k+p}(z)$ $(k \geq 1,\ p \in \mathbb{N})$ given by Theorem 8.*

Theorem 9. *Let $f_p(z) = z^p$ and*

$$f_{k+p}(z) = z^p - \frac{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}{\left[k(1+\alpha\beta) + p\left(\beta(1+\alpha) + (1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}} z^{k+p} \; (k \geq 1), \quad (45)$$

where $p \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ and $z \in \mathbb{U}$. Then $f(z) \in S^*_{sc,\lambda}(p,\alpha,\beta)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \sigma_{k+p} f_{k+p}(z)$, where $\sigma_{k+p} \geq 0$ $(k \geq 1)$ and $\sum_{k=1}^{\infty} \sigma_{k+p} = 1$.

Corollary 8. *The extreme points of the class* $S^*_{sc,\lambda}(p,\alpha,\beta)$ *are functions* $f_{k+p}(z)$ $(k \geq 1,\ p \in \mathbb{N})$ *given by Theorem 9.*

Theorem 10. *The class* $S^*_{s,\lambda}(p,\alpha,\beta)$ *is closed under convex linear combination.*

Proof. Let us suppose that the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = z^p - \sum_{k=1}^{\infty} |a_{p+k,j}| z^{p+k} \ (a_{p+k,j} \geq 0,\ j=1,2,\ z \in \mathbb{U}) \tag{46}$$

are in the class $S^*_{s,\lambda}(p,\alpha,\beta)$. Set

$$h(z) = \mu f_1(z) + (1-\mu f_2(z)), \quad (0 \leq \mu \leq 1). \tag{47}$$

Then from (46), we can write

$$f(z) = z^p - \sum_{k=2}^{\infty} \left[\mu|a_{p+k,1}| + (1-\mu)|a_{p+k,1}|\right] z^{p+k} \ (a_{p+k,j} \geq 0,\ j=1,2,\ z \in \mathbb{U}). \tag{48}$$

Thus, in view of Theorem 1, we can have that

$$\sum_{k=2}^{\infty} \left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}|a_{k+p}|\left[\mu|a_{p+k,1}|\right]$$

$$+(1-\mu)|a_{p+k,1}| \leq p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right],$$

which implies that $h(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ and this completes the proof of Theorem 10.

5 Radii of starlikeness and Convexity

Theorem 11. *Let the function* $f(z)$ *of the form (5) be in the class* $S^*_{s,\lambda}(p,\alpha,\beta)$, *then* $f(z)$ *is starlike in the disk* $|z| = r_1 < 1$, *where*

$$r_1 = \inf_{k \geq 1} \left(\frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]} \left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}. \tag{49}$$

Proof. It is ample to show that

$$\left|\tfrac{zf'(z)}{f(z)} - p\right| \leq p, \text{ for } |z| < 1$$

or equivalently,

$$\left|\frac{\sum_{k=1}^{\infty} k\ |a_{k+p}| z^k}{1 - \sum_{k=1}^{\infty} |a_{k+p}| z^k}\right| \leq p$$

which is equivalent to show that

$$\frac{\sum_{k=1}^{\infty}(k+1)|a_{k+p}||z|^k}{p} \leq 1. \tag{50}$$

As $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$ we have from Theorem 1

$$\sum_{k=1}^{\infty} \frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]} |a_{k+p}| \leq 1.$$

Hence, (50) is proven true if

$$\left(\frac{(p+k)|z|}{p}\right) \leq \frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}.$$

That is,

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}\left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}$$

and this ends the proof of Theorem 11.

On similar lines of Theorem 11, we can easily prove the following Theorems for $S^*_{c,\lambda}(p,\alpha,\beta)$ and $S^*_{sc,\lambda}(p,\alpha,\beta)$ classes.

Theorem 12. *Let the function $f(z)$ of the form (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta) + p\left(\beta(\alpha+2)-1\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(\beta(\alpha+2)-1)\right]}\left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}. \tag{51}$$

Theorem 13. *Let the function $f(z)$ of the form (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta) + p\left(\beta(1+\alpha) + (1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}\left(\frac{p}{p+k}\right)\right)^{\frac{1}{k}}. \tag{52}$$

Similarly we can proved the following Results.

Theorem 14. *Let the function $f(z)$ of the form (5) be in the class $S^*_{s,\lambda}(p,\alpha,\beta)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta) + p\left((-1)^{p+k} + \beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\left[(-1)^p + \beta(\alpha+1-(-1)^p)\right]}\left(\frac{p}{p+k}\right)^2\right)^{\frac{1}{k}}. \tag{53}$$

Theorem 15. *Let the function $f(z)$ of the form (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{[k(1+\alpha\beta)+p\,(\beta(\alpha+2)-1)]\,(k+p)^{\lambda+1}}{p^{\lambda+1}\,[(\beta(\alpha+2)-1)]}\left(\frac{p}{p+k}\right)^2\right)^{\frac{1}{k}}. \tag{54}$$

Theorem 16. *Let the function $f(z)$ of the form (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_1 = \inf_{k\geq 1}\left(\frac{\left[k(1+\alpha\beta)+p\left(\beta(1+\alpha)+(1-\beta)(-1)^{p+k}\right)\right](k+p)^{\lambda+1}}{p^{\lambda+1}\,[(-1)^p+\beta(\alpha+1-(-1)^p)]}\left(\frac{p}{p+k}\right)^2\right)^{\frac{1}{k}}. \tag{55}$$

In order to establish the required results in Theorems 14, 15 and 16, it is sufficies to show that

$$\left|1+\tfrac{zf''(z)}{f'(z)}-p\right| \leq p, \text{ for } |z|<1.$$

6 Integral Operator

Definition 4. Let $f \in \mathcal{T}_p$, an integral operator $R_c(f)$ with $c > -p$ is defined by

$$R_c(f) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt, \quad (z \in \mathbb{U}). \tag{56}$$

Theorem 17. *Let the function $f(z)$ of the form (5) be in the class $S^*_{s,\lambda}(p,\alpha,\beta)$, then $R_c(f)$ defined by (56) be also in the class $S^*_{s,\lambda}(p,\alpha,\beta)$.*

Proof. From (56), we get

$$R_c(f) = z^p - \sum_{k=1}^{\infty}\frac{c+p}{c+p+k}|a_{k+p}|z^{k+p}. \tag{57}$$

Therefore by hypothesis

$$\sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}\left(\frac{c+p}{c+p+k}\right)|a_{k+p}|$$
$$\leq \sum_{k=1}^{\infty}\left[k(1+\alpha\beta)+p\left((-1)^{p+k}+\beta(\alpha+1-(-1)^{p+k})\right)\right](k+p)^{\lambda+1}|a_{k+p}|$$
$$\leq p^{\lambda+1}\,[(-1)^p+\beta(\alpha+1-(-1)^p)]$$

since $f(z) \in S^*_{s,\lambda}(p,\alpha,\beta)$. Hence, by Theorem 1, $R_c(f) \in S^*_{s,\lambda}(p,\alpha,\beta)$.

Similar to Theorem 17, we can easily prove the following theorems for $S^*_{c,\lambda}(p,\alpha,\beta)$ and $S^*_{sc,\lambda}(p,\alpha,\beta)$.

Theorem 18. *Let the function $f(z)$ of the form (5) be in the class $S^*_{c,\lambda}(p,\alpha,\beta)$, then $R_c(f)$ defined by (56) be also in the class $S^*_{c,\lambda}(p,\alpha,\beta)$.*

Theorem 19. *Let the function $f(z)$ of the form (5) be in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$, then $R_c(f)$ defined by (56) be also in the class $S^*_{sc,\lambda}(p,\alpha,\beta)$.*

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