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On certain subclasses of analytic functions associated with generalized struve functions

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Abstract. The goal of the present paper is to investigate some characterization for generalized Struve functions of first kind to be in the new subclasses of β uniformly starlike and β uniformly convex functions of order α. Further we point out some consequences of our main results.

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1 Introduction

Denote by $\mathcal{A}$ the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1}$$

Also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \le \alpha < 1)$ if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \le \alpha < 1)$ if and only if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. We remark that, according to the Alexander duality theorem [1] the function $f : \mathbb{U} \to \mathbb{C}$ is convex of order α, where $0 = \alpha < 1$, if and only if $z \to zf'(z)$ is starlike of order α. We note that every starlike (and hence convex) function of the form (1) is in fact close-to-convex, and every close-to-convex function is univalent. However, if a function is starlike then it is not necessary that it will be close-to-convex with respect to a particular convex function.

The class $\beta - \mathcal{UCV}$ was introduced by Kanas et al. [14], where its geometric definition and connections with the conic domains were considered. The class $\beta - \mathcal{UCV}$ was defined purely by geometrically as a subclass of univalent functions that map each circular arc contained in the unit disk $\mathbb{U}$ with a center $\xi, |\xi| \le \beta$ $(0 \le$

$\beta < 1$), onto a convex arc. The notion of $\beta-$ uniformly convex function is a natural extension of the classical convexity. Observe that, if $\beta = 0$ then the center ξ is the origin and the class $\beta - \mathcal{UCV}$ reduces to the class of convex univalent functions $\mathcal{K}$. Moreover for $\beta = 1$, the class $\beta - \mathcal{UCV}$ corresponds to the class $\mathcal{UCV}$ introduced by Goodman [12,13] and studied extensively by Rønning [21,22]. The class $\beta - \mathcal{S}_P$ is related to the class $\beta - \mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex $\mathcal{K}$ and starlike $\mathcal{S}^*$ functions. Further the analytic criteria for functions in these classes are given below.

For $-1 < \alpha \leq 1$ and $\beta \geq 0$, a function $f \in \mathcal{A}$ is said to be in the class
(i) $\beta-$ uniformly starlike functions of order α, denoted by $\mathcal{S}_P(\alpha, \beta)$, if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{U} \tag{2}$$

and
(ii) $\beta-$ uniformly convex functions of order α, denoted by $\mathcal{UCV}(\alpha, \beta)$, if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathbb{U}. \tag{3}$$

Indeed it follows from (2) and (3) that

$$f \in \mathcal{UCV}(\alpha, \beta) \Leftrightarrow zf' \in \mathcal{S}_P(\alpha, \beta). \tag{4}$$

Remark 1. It is of interest to state that $\mathcal{UCV}(\alpha, 0) = \mathcal{K}(\alpha)$ and $\mathcal{S}_P(\alpha, 0) = \mathcal{S}^*(\alpha)$.

Motivated by the above definitions we define the following subclasses of $\mathcal{A}$ due to Murugusundaramoorthy and Magesh [18].

For $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, we let $\mathcal{S}_P(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right|, \quad z \in \mathbb{U}, \tag{5}$$

and also, let $\mathcal{UCV}(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\left(\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - \alpha\right) > \beta\left|\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - 1\right|, \quad z \in \mathbb{U}. \tag{6}$$

We further let $\mathcal{TS}_P(\lambda, \alpha, \beta) = \mathcal{S}_P(\lambda, \alpha, \beta) \cap \mathcal{T}$ and $\mathcal{UCT}(\lambda, \alpha, \beta) = \mathcal{UCV}(\lambda, \alpha, \beta) \cap \mathcal{T}$ where $\mathcal{T}$ denotes the subclass of $\mathcal{A}$ consisting of functions whose nonzero coefficients from second on, is given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n. \tag{7}$$

$\mathcal{S}_P(0, \alpha, 0) \equiv \mathcal{T}^*(\alpha)$ and $\mathcal{UCT}(0, \alpha, 0) \equiv \mathcal{C}(\alpha)$ are the class of starlike and convex functions of order α ($0 \leq \alpha < 1$), introduced and studied by Silverman [23]. Suitably specializing the parameters one can define various subclasses defined in [2, 7, 23, 27, 28]. Now we recall the following necessary and sufficient conditions for functions f to be in the function classes $\mathcal{S}_P(\lambda,\alpha,\beta)$, $\mathcal{TS}_P(\lambda,\alpha,\beta)$, $\mathcal{UCV}(\lambda,\alpha,\beta)$ and $\mathcal{UCT}(\lambda,\alpha,\beta)$ due to Murugusundaramoorthy and Magesh [18].

Theorem 1 (see [18]). *A function $f(z)$ of the form (1) is in $\mathcal{S}_P(\lambda,\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n|\leq 1-\alpha. \tag{8}$$

Theorem 2 (see [18]). *A function $f(z)$ of the form (1) is in $\mathcal{UCV}(\lambda,\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n|\leq 1-\alpha. \tag{9}$$

Theorem 3 (see [18]). *A function $f(z)$ of the form (7) is in $\mathcal{TS}_P(\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n|\leq 1-\alpha. \tag{10}$$

Theorem 4 (see [18]). *A function $f(z)$ of the form (7) is in $\mathcal{UCT}(\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ |a_n|\leq 1-\alpha. \tag{11}$$

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges [10] of the famous Bieberbach conjecture.The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [9, 15, 17, 24, 29] and the Bessel functions [3–6, 16].

We recall here the Struve function of order p (see [19, 30]), denoted by $\mathcal{H}_p$, is given by

$$\mathcal{H}_p(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+p+1},\forall z\in\mathbb{C} \tag{12}$$

which is the particular solution of the second order non-homogeneous differential equation

$$z^2\omega''(z)+z\omega'(z)+(z^2-p^2)\omega(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{13}$$

where p is unrestricted real (or complex) number. The solution of the non-homogeneous differential equation

$$z^2\omega''(z)+z\omega'(z)-(z^2+p^2)\omega(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{14}$$

is called the modified Struve function of order p and is defined by the formula

$$\mathcal{L}_p(z) = -ie^{-ip\pi/2}\mathcal{H}_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C}.$$

Let the second order non-homogeneous linear differential equation [30] (also see [19] and references cited therein),

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{b}{2})} \tag{15}$$

where $b, p, c \in \mathbb{C}$, which is natural generalization of Struve equation. It is of interest to note that when $b = c = 1$, then we get the Struve function (12) and for $c = -1,\ b = 1$ the modified Struve function (14). This permits us to study the Struve and modified Struve functions. Now, denote by $w_{p,b,c}(z)$ the generalized Struve function of order p given by

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C},$$

which is the particular solution of the differential equation (15). Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma\left(p + \frac{b+2}{2}\right)\ z^{\frac{-p-1}{2}}\ \omega_{p,b,c}\ (\sqrt{z}), \quad \sqrt{1} = 1.$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1, 2, 3, \ldots\}) \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$\begin{aligned} u_{p,b,c}(z) &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m)_n\ (3/2)_n}\ z^n \\ &= b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots, \end{aligned}$$

where $m = \left(p + \frac{b+2}{2}\right) \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order inhomogeneous linear differential equation

$$4z^2u''(z) + 2(2p+b+3)zu'(z) + (cz + 2p + b)u(z) = 2p + b.$$

For convenience throughout in the sequel, we use the following notations

$$w_{p,b,c}(z) = w_p(z) \qquad u_{p,b,c}(z) = u_p(z), \qquad m = p + \frac{b+2}{2}$$

and for if $c < 0, m > 0$ $(m \neq 0, -1, -2, \dots)$ let

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n = z + \sum_{n=2}^{\infty} b_{n-1} z^n \tag{16}$$

and

$$\Psi(z) = z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n \tag{17}$$

Mapping properties of various subclasses of analytic and univalent functions are potentially useful in a number of widespread areas of the mathematical, physical and engineering sciences. In particular, in order to solve such applied problems that are expressible in terms of functions of a complex variable, but that exhibit inconvenient geometrical shapes, we can appropriately choose one or the other of such mappings and thereby transform the inconvenient geometrical shape into a much more convenient and easy-to-handle geometrical shape. Several mapping properties of the function classes $\beta{-}\mathcal{UST}$ and $\beta{-}\mathcal{UCV}$ involving hypergeometric functions were studied by Srivastava et al [26] (also see [9,15,17,24,29]) and references cited therein. Recently Yagmur and Orhan [30] (see [19]) have determined various sufficient conditions for the parameters p, b and c such that the functions $u_{p,b,c}(z)$ or $z \to zu_{p,b,c}(z)$ to be univalent, starlike, convex and close to convex in the open unit disk. Motivated essentially by the aforementioned works and by work of Baricz [3–6], in our present investigation, we determined sufficient conditions for the family of Struve functions $(zu_p(z))$ in order to belong to the classes $\mathcal{TS}_P(\lambda, \alpha, \beta)$ and $\mathcal{UCT}(\lambda, \alpha, \beta)$ in the open unit disk $\mathbb{U}$. We also proved that those sufficient conditions are necessary for functions of the form (17). Further we deduce several interesting corollaries and consequences by suitably applying our main results.

2 Main results and their consequences

Lemma 1 (see [19]). *If $b, p, c \in \mathbb{C}$ and $m \neq 0, -1, -2, ...,$ then the function u_p satisfies the recursive relation*

$$2zu_p'(z) + u_p(z) + \frac{cz}{2m} u_{p+1}(z) = 1$$

for all $z \in \mathbb{C}$.

Theorem 5. *If $c < 0, m > 0$ $(m \neq 0, -1, -2, ...)$, then the sufficient condition for $zu_p(z) \in \mathcal{TS}_P(\lambda, \alpha, \beta)$ is*

$$[1 + \beta - \lambda(\alpha + \beta)]u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha). \tag{18}$$

Moreover (18) is necessary and sufficient for $\Psi(z)$, given by (17) to be in $\mathcal{TS}_P(\lambda, \alpha, \beta)$.

Proof. According to Theorem 3, we must show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \leq (1-\alpha). \tag{19}$$

Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\
&=\sum_{n=2}^{\infty}[(n-1)\{1+\beta-\lambda(\alpha+\beta)\}+(1-\alpha)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\
&=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=2}^{\infty}\frac{(n-1)((-c/4))^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}+(1-\alpha)\sum_{n=2}^{\infty}\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\
&=[1+\beta-\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)[u_p(1)-1].
\end{aligned}$$

But the last expression is bounded from above by $1-\alpha$ if and only if (18) holds. Since

$$z(2-u_p(z)) = z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n \tag{20}$$

the necessity of (18) for $z(2-u_p(z))$ to be in $\mathcal{TS}_P(\lambda,\alpha,\beta)$ follows from Theorem 3. □

Theorem 6. *If $c<0, m>0$ ($m\neq 0,-1,-2,...,$ then the sufficient condition for $zu_p(z)\in\mathcal{UCT}(\lambda,\alpha,\beta)$ is*

$$[1+\beta-\lambda(\alpha+\beta)]u_p''(1)+[3+2\beta-\alpha-2\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha). \tag{21}$$

Moreover (21) is necessary and sufficient for $\Psi(z)$, given by (17) to be in $\mathcal{UCT}(\lambda,\alpha,\beta)$.

Proof. In view of Theorem 4, we need to show that

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \leq (1-\alpha).$$

If we let $g(z)=zu_p(z)$, then we have $g'(1)=u_p'(1)+u_p(1)$ and $g''(1)=u_p''(1)+2u_p'(1)$. Further we notice that

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\
&=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=2}^{\infty}n^2\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}
\end{aligned}$$

$$
\begin{aligned}
&-(\alpha+\beta)(1-\lambda)\sum_{n=2}^{\infty} n\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\\
&=[1+\beta-\lambda(\alpha+\beta)]\left\{\sum_{n=2}^{\infty} n(n-1)\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right\}\\
&\quad+[1+\beta-\lambda(\alpha+\beta)-(\alpha+\beta)(1-\lambda)]\left\{\sum_{n=2}^{\infty} n\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right\}\\
&=[1+\beta-\lambda(\alpha+\beta)]g''(z)+(1-\alpha)[g'(z)-1],
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\\
&=[1+\beta-\lambda(\alpha+\beta)](u_p''(1)+u_p'(1))+(1-\alpha)(u_p'(1)+u_p(1)-1)\\
&=[1+\beta-\lambda(\alpha+\beta)]u_p''(1)+[3+2\beta-2\lambda(\alpha+\beta)-\alpha]u_p'(1)+(1-\alpha)[u_p(1)-1].
\end{aligned}
$$

The last expression is bounded from above by $(1-\alpha)$ if and only if (21) holds. By Theorem 4, the condition (21) is also necessary for $z(2-u_p(z))=\Psi(z)\in \mathcal{UCT}(\lambda,\alpha,\beta)$. □

Remark 2. In particular when $\lambda=0$ and $\beta=0$ the conditions given in (18) and (21) yield the results obtained in [30].

By taking $\lambda=0$ and $\alpha=0$, we state the following results for the function classes $\mathcal{TS}_P(0,0,\beta)\equiv\mathcal{TS}_P(\beta)$ and $\mathcal{UCT}(0,0,\beta)\equiv\mathcal{UCT}(\beta)$ defined in [27].

Corollary 1. *If* $c<0, m>0$ $(m\neq 0,-1,-2,...,$ *then*
(i) the sufficient condition for $zu_p(z)\in\mathcal{TS}_P(\beta)$ *is*

$$(1+\beta)u_p'(1)+u_p(1)\le 2,$$

moreover it is necessary and sufficient for functions $\Psi(z)=z(2-u_p(z))$ *to be in* $\mathcal{TS}_P(\beta)$
(ii) the sufficient condition for $zu_p(z)\in\mathcal{UCT}(\beta)$ *is*

$$(1+\beta)u_p''(1)+(3+2\beta)u_p'(1)+u_p(1)\le 2,$$

moreover it is necessary and sufficient for functions $\Psi(z)=z(2-u_p(z))$ *to be in* $\mathcal{UCT}(\beta)$.

By taking $\lambda=0$, we deduce results for the function class defined in [7].

Corollary 2. *If* $c<0, m>0$ $(m\neq 0,-1,-2,...,$ *then*
(i) the sufficient condition for $zu_p(z)\in\mathcal{TS}_P(\alpha,\beta)$ *is*

$$(1+\beta)u_p'(1)+(1-\alpha)u_p(1)\le 2(1-\alpha),$$

(ii) the sufficient condition for $zu_p(z) \in \mathcal{UCT}(\alpha, \beta)$ *is*

$$(1+\beta)u_p''(1) + (3+2\beta-\alpha)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha).$$

Further the above conditions are necessary and sufficient for functions of the form (17).

3 Inclusion Properties

For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}. \tag{22}$$

Now, we considered the linear operator

$$\mathcal{I}(c, m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(c,m)f(z) = zu_{p,b,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} a_n\ z^n \tag{23}$$

where $m = p + \frac{(b+2)}{2} \neq 0$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A, B)$ $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \leq B < A \leq 1)$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathcal{R}^{\tau}(A, B)$ was introduced earlier by Dixit and Pal [11]. If we put

$$\tau = 1,\ A = \beta \text{ and } \ B = -\beta \ (0 < \beta \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in \mathbb{U}; 0 < \beta \leq 1),$$

which was studied by (among others) Padmanabhan [20] and Caplinger and Causey [8]. Making use of the following lemma, we will study the action of the Struve function on the class $\mathcal{UCT}(\lambda, \alpha, \beta)$.

Lemma 2 (see [11]). *If* $f \in \mathcal{R}^{\tau}(A, B)$ *is of form* (1), *then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{24}$$

The bound given in (24) is sharp.

Theorem 7. *Let $c < 0, m > 0$ $(m \neq 0, -1, -2, ...)$. If $f \in \mathcal{R}^{\tau}(A, B)$ and if the inequality*

$$(A-B)|\tau|\left\{[1+\beta-\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)[u_p(1)-1]\right\} \leq 1-\alpha \qquad (25)$$

is satisfied, then $\mathcal{I}(c, m)(f) \in \mathcal{UCT}(\lambda, \alpha, \beta)$.

Proof. Let f of the form (1) belong to the class $\mathcal{R}^{\tau}(A, B)$. By virtue of Theorem 4, it suffices to show that

$$L(\alpha,\beta,\lambda)=\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n| \leq 1-\alpha.$$

Since $f \in \mathcal{R}^{\tau}(A, B)$ then by Lemma 2 we have,

$$|a_n| \leq (A-B)\frac{|\tau|}{n}.$$

Hence

$$\begin{aligned} L(\alpha,\beta,\lambda) &= \sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n| \\ &\leq (A-B)|\tau|\left[\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right]. \qquad (26) \end{aligned}$$

Further, proceeding as in Theorem 5, we get

$$L(\alpha,\beta,\lambda) \qquad \leq (A-B)|\tau|\left\{[1+\beta-\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)[u_p(1)-1]\right\}.$$

But this last expression is bounded above by $1-\alpha$ if and only if (25) holds. □

Theorem 8. *Let $c < 0, m > 0$ $(m \neq 0, -1, -2, ...)$ then*

$$\mathcal{L}(m, c, z) = \int_0^z (2-u_p(t))dt$$

is in $\mathcal{UCT}(\lambda, \alpha, \beta)$ if and only if

$$[1+\beta-\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)u_p(1) \leq 2(1-\alpha). \qquad (27)$$

Proof. Since

$$\mathcal{L}(m, c, z) = z - \sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\frac{z^n}{n}$$

then by Theorem 4 we need only to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{n(m)_{n-1}\ (3/2)_{n-1}} \leq 1-\alpha.$$

That is, let

$$\mathcal{P}(m,c,z)=\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}$$

Now by proceeding as in Theorem 5, we get

$$\mathcal{P}(m,c,z)=[1+\beta-\lambda(\alpha+\beta)]u_p'(1)+(1-\alpha)[u_p(1)-1].$$

which is bounded from above by $1-\alpha$ if and only if (27) holds. □

Remarks. If we put $c=-1$ and $b=1$ in above theorems we obtain results analogous to ones discussed in this paper. Further by taking $\beta=0$ and specializing the parameter λ we can state various interesting results (as proved in above theorems) for the subclasses studied in the literature [2, 7, 23, 27, 28].

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