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On the absence of finite approximation relative to model completeness in propositional provability logic

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Abstract. In the present paper we consider the expressibility of formulas in the provability logic GL and related to it questions of the model completeness of system of formulas. We prove the absence of a finite approximation relative to model completeness in GL.

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1 Introduction

Artificial Intelligence (AI) systems simulating human behavior are often called intelligent agents. These intelligent agents exhibit somehow human-like intelligence. Intelligent agents typically represent human cognitive states using underlying beliefs and knowledge modeled in a knowledge representation language, specifically in the context of decision making [1]. In the present paper we investigate some functional properties of the underlying knowledge representation language of intelligent agents which are based on the provability logic GL [2].

The notion of model completeness of systems of formulas was proposed in [6, 7]. In the present paper we prove the propositional provability logic of Gödel-Löb (GL) is not finitely approximable relative to model completeness.

2 Definitions and notations

Provability logic. We consider the propositional provability logic GL whose formulas are based on propositional variables $p, q, r, \dots$ and logical connectives $\&, \vee, \supset, \neg, \Delta$, its axioms are the classical ones together with the following Δ-formulas:

$$\Delta(p \supset q) \supset (\Delta p \supset \Delta q), \quad \Delta(\Delta p \supset p) \supset \Delta p, \quad \Delta p \supset \Delta\Delta p,$$

and the rules of inference are the rules of: 1) substitution; 2) the modus ponens, and 3) the necessity, which allows to get formula ΔA if we already get formula A. The normal extensions of the propositional provability logic GL are defined as usual [2].

Diagonalizable algebras. A diagonalizable algebra [4] is a universal algebra of the form $\mathfrak{A} =< M; \&, \vee, \supset, \neg, \Delta >$, where $< M; \&, \vee, \supset, \neg >$ is a boolean algebra, and the unary operation Δ satisfies the relations

$$\Delta(\Delta x \supset x) = \Delta x,\ \Delta(x\&y) = (\Delta x\&\Delta y),\ \Delta 1_{\mathfrak{A}} = 1_{\mathfrak{A}},$$

where $1_{\mathfrak{A}}$ is the unit of $\mathfrak{A}$, which is denoted also by 1 in case the confusion is avoided.

Diagonalizable algebras are known to be algebraic models for provability logic and its extensions [5]. Obviously we can interpret any formula of the calculus of GL on any diagonalizable algebra $\mathfrak{A}$. As usual a formula F is said to be valid on $\mathfrak{A}$ if for any evaluation of variables of F with elements of $\mathfrak{A}$ the value of the formula on $\mathfrak{A}$ is $1_{\mathfrak{A}}$. The set of all valid formulas on $\mathfrak{A}$, denoted by $L\mathfrak{A}$ and referred to as the logic of the algebra $\mathfrak{A}$, forms an extension $L\mathfrak{A}$ of the provability logic GL [5].

An extension L of GL is called tabular if there is a finite diagonalizable algebra $\mathfrak{A}$ such that $L = L\mathfrak{A}$.

Expressibility and model completeness. The formula $F(p_1, \dots, p_n)$ is a model for the Boolean function $f(x_1, \dots, x_n)$ if for any ordered set $(\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \{0, 1\}, i = 1, \dots, n$, we have $F(\alpha_1, \dots, \alpha_n) = f(\alpha_1, \dots, \alpha_n)$, where logical connectors from F are interpreted in a natural way on the two-valued Boolean algebra [6, 7].

They say the formula F is expressible in the logic L via a system of formulas Σ if F can be obtained from variables and Σ applying finitely many times 2 kinds of rules: a) the rule of weak substitution, b) the rule of passing to equivalent formula in L [3].

The system of formulas Σ is called model complete in the logic L if at least a model for every Boolean function is expressible via Σ in the logic L. System Σ is model pre-complete in L if Σ is not model complete in L, but for any formula F which is not expressible in L via Σ the system $\Sigma \cup \{F\}$ is already model complete in L [8].

The logic L is finitely approximable with respect to model completeness if for any system of formulas Σ which is not model complete in L there is a tabular extension of L in which Σ is also model incomplete.

3 Preliminary results

First let mention an obvious fact:

Proposition. *If a system of formulas Σ is complete with respect to expressibility of formulas in the logic GL then it is also model complete in GL.*

Let us consider the following system of formulas:

$$\{p\&\neg q, \Delta p\}. \tag{1}$$

Lemma 1. *The system of formulas* (1) *is model complete in any tabular extension of the propositional provability logic GL.*

Proof. Note that for any finite diagonalizable algebra $\mathfrak{A}$ there exists a positive integer k such that the following equivalence is valid in the logic $L\mathfrak{A}$

$$\Delta^k(p\&\neg p) \sim (p \supset p),$$

which shows the tautology $p \supset p$ is expressible in the logic $L\mathfrak{A}$ via system of formulas (1). It remains to observe the system

$$\{(p \supset p)\} \cup \{\Delta p, p\&\neg q\}$$

is complete in the logic $L\mathfrak{A}$, so by Proposition it is also model complete in $L\mathfrak{A}$. □

Let $\mathfrak{M}^*$ the diagonalizable algebra of sequences of the form $\alpha = (\mu_1, \mu_2, \ldots)$, where $\mu_i \in \{0, 1\}$ $(i = 1, 2, \ldots)$ and the operations $\&, \vee, \supset, \neg$ made term by term as Boolean functions on the set of $\{0, 1\}$, and $\Delta\alpha$ is a sequence $(\nu_1, \nu_2, \ldots)$, where $\nu_i = (\mu_1\& \cdots \&\mu_i)$ $(i = 1, 2, \ldots)$. The logic $L\mathfrak{M}^*$ coincides with the extension of provability logic generated by the formula

$$\Delta(\Box p \supset q) \vee \Delta(\Box q \supset p),$$

where $\Box p$ means $p\&\Delta p$.

Lemma 2. *Let L be any intermediate logic between GL and $L\mathfrak{M}^*$. The system of formulas* (1) *is not model complete in the propositional provability logic L.*

Proof. Realy, the system of formulas (1) is not model complete in $L\mathfrak{M}^*$ since formulas of the system (1) conserves the relation $x \neq 1$ on the algebra $\mathfrak{M}^*$, and the formula $(p \supset p)$ does not. □

4 Main result

Now we are able to formulate the main result of the present work.

Theorem. *Let L be any intermediate logic between GL and $L\mathfrak{M}^*$. The propositional provability logic L is not finitely approximable with respect to model completeness.*

Proof. The proof results from the above Lemmas 1 and 2. □

Taking into account our previous result [9] together with these new findings we can conclude that traditional algorithm for determining model completeness of systems of formulas in GL is impossible to find out.

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