BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(78), 2015, Pages 96–113
ISSN 1024–7696

On LCA groups with locally compact rings of continuous endomorphisms. II

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Abstract. For certain classes $\mathcal{S}$ of locally compact abelian groups, we determine the groups $X \in \mathcal{S}$ with the property that the ring $E(X)$ of continuous endomorphisms of X is locally compact in the compact-open topology.

Mathematics subject classification: 16W80.
Keywords and phrases: LCA groups, rings of continuous endomorphisms, compact-open topology.

1 Introduction

Let $\mathcal{L}$ be the class of locally compact abelian groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of continuous endomorphisms of X, taken with the compact-open topology.

In the present paper, we continue our work begun in [17] concerning the problem of characterizing the groups $X \in \mathcal{L}$ for which $E(X)$ is locally compact. Our main results are as follows. We establish some necessary conditions and, respectively, some sufficient conditions on X in order for $E(X)$ be locally compact. For groups in $\mathcal{L}$ containing a lattice and for densely divisible torsion-free groups in $\mathcal{L}$, we give a complete solution to the considered problem. We also determine the topological torsion groups $X \in \mathcal{L}$ with the property that $E(A/B)$ is locally compact for all closed subgroups A, B of X such that $A \supset B$.

2 Notation

We will follow the notation used in [17]. In addition, for $X, Y \in \mathcal{L}$ and $f \in H(X, Y)$, we denote by f^* the transpose of f, i.e. the homomorphism $f^* \in H(Y^*, X^*)$ defined by the rule $f^*(\gamma) = \gamma \circ f$ for all $\gamma \in Y^*$. If C is a closed subgroup of X and $n \in \mathbb{N}_0$, we set $\frac{1}{n}C = \{x \in X \mid nx \in C\}$. We will also make use of the discrete group $\mathbb{Z}$ of integers, and of the groups of reals $\mathbb{R}$ and of p-adic numbers $\mathbb{Q}_p$, where $p \in \mathbb{P}$, all taken with their usual topologies. Finally, if $(X_i)_{i\in I}$ is a family of topological groups (rings) such that, for each $i \in I$, X_i admits an open subgroup (subring) U_i, then $\prod_{i\in I}(X_i; U_i)$ stands for the local direct product of $(X_i)_{i\in I}$ with respect to $(U_i)_{i\in I}$. Recall that $\prod_{i\in I}(X_i; U_i)$ is the subgroup (subring) of $\prod_{i\in I} X_i$ consisting of all families $(x_i)_{i\in I}$ such that $x_i \in U_i$ for all but finitely many $i \in I$, topologized by declaring all neighborhoods of zero in the topological group (ring) $\prod_{i\in I} U_i$ to be a fundamental system of neighborhoods of zero in $\prod_{i\in I}(X_i; U_i)$.

3 Local compactness of some homomorphism groups

In this preparatory section, we determine the groups $X \in \mathcal{L}$ with the property that the topological groups $H(X,\mathbb{R})$, $H(\mathbb{R},X)$, $H(X,\mathbb{Q})$, $H(\mathbb{Q},X)$, $H(X,\mathbb{Q}^*)$, and $H(\mathbb{Q}^*,X)$ are locally compact.

We first recall the following definition, due to V. Charin [4].

Definition 1. A topological group X is said to be a group of finite (special) rank if there exists a natural number r such that every finite subset F of X topologically generates a subgroup with no more than r topological generators, i.e. $\overline{\langle F\rangle} = \overline{\langle x_1,\dots,x_k\rangle}$ for some $x_1,\dots,x_k \in X$ and $k \le r$. The smallest r with this property is called the special rank of X. In case no such r exists, X is said to have infinite special rank.

As is well known, a discrete torsion-free group $X \in \mathcal{L}$ has finite special rank r if and only if its torsion-free rank is equal to r. It is also known that if $X \in \mathcal{L}$ is a topologically p-primary group for some $p \in \mathbb{P}$, then X has finite special rank r if and only if

$$X \cong G_1 \times \cdots \times G_r,$$

where every $G_i, 1 \le i \le r$, is topologically isomorphic with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}_0$ [5, Theorem 5].

We now begin the study of local compactness of the mentioned homomorphism groups. For $H(X,\mathbb{R})$ and $H(X,\mathbb{Q})$, we have

Theorem 1. *Let X be a group in $\mathcal{L}$ containing a compact open subgroup. The following conditions are equivalent:*

(i) *$H(X,\mathbb{R})$ is locally compact.*

(ii) *$H(X,\mathbb{Q})$ is locally compact.*

(iii) *$X/k(X)$ has finite rank.*

Proof. The fact that (i) and (iii) are equivalent follows from [15, Lemma 3.2]. Let us establish the equivalence of (ii) and (iii). Assume (ii), and let Ω be a compact neighborhood of zero in $H(X,\mathbb{Q})$. By the definition of the compact-open topology, there is a compact subset K of X such that $\Omega_{X,\mathbb{Q}}(K,\{0\}) \subset \Omega$. Let $\pi : X \to X/k(X)$ be the canonical projection. Since X has a compact open subgroup, $X/k(X)$ is discrete, and hence $\pi(K)$ is finite. Let $G = \langle \pi(K)\rangle_*$. It is clear that G has finite rank [12, p. 41]. We shall show that $G = X/k(X)$. Assume the contrary, and pick an arbitrary non-zero $b \in \big(X/k(X)\big)/G$. Since G is pure in $X/k(X)$, the quotient group $\big(X/k(X)\big)/G$ is torsion-free, so $o(b) = \infty$. Letting $\varphi : X/k(X) \to \big(X/k(X)\big)/G$ denote the canonical projection, write $b = \varphi(b')$ for some $b' \in X/k(X)$. Now, given any

$r \in \mathbb{Q}$, let $\xi_r : \big(X/k(X)\big)/G \to \mathbb{Q}$ be the extension of the group homomorphism from $\langle b \rangle$ to $\mathbb{Q}$ which carries b to r [8, Theorem 21.1]. Then $\xi_r \circ \varphi \circ \pi \in \Omega_{X,\mathbb{Q}}(K, \{0\})$, so $r \in \Omega b'$. Since $r \in \mathbb{Q}$ was chosen arbitrarily, we get $\mathbb{Q} \subset \Omega b'$, which is a contradiction because $\Omega b'$ is finite and $\mathbb{Q}$ is infinite. This proves that $G = X/k(X)$, so (i) implies (iii).

To see the converse, assume (iii), and pick any elements $a_1, \dots, a_m \in X$ such that $a_1 + k(X), \dots, a_m + k(X)$ form a basis in $X/k(X)$. We claim that

$$\Omega_{X,\mathbb{Q}}\big(\{a_1, \dots, a_m\}, \{0\}\big) = \{0\},$$

which means that $H(X, \mathbb{Q})$ is discrete. To see this, fix any $a \in X \setminus k(X)$. Then there exist $n \in \mathbb{N}_0$ and $l_1, \dots, l_m \in \mathbb{Z}$ such that

$$n(a + k(X)) = \sum_{i=1}^{m} l_i(a_i + k(X)),$$

and hence

$$na - \sum_{i=1}^{m} l_i a_i \in k(X).$$

Pick any $f \in \Omega_{X,\mathbb{Q}}\big(\{a_1, \dots, a_m\}, \{0\}\big)$. Since $k(\mathbb{Q}) = \{0\}$, we have $k(X) \subset \ker(f)$. It follows that

$$nf(a) = \sum_{i=1}^{m} l_i f(a_i) = 0,$$

so $f(a) = 0$. Since $a \in X \setminus k(X)$ was chosen arbitrarily, it follows that $f = 0$, and hence (iii) implies (ii). □

As a direct consequence, we derive the following:

Corollary 1. *Let X be a group in $\mathcal{L}$ containing a compact open subgroup. The following conditions are equivalent:*

(i) *$H(\mathbb{R}, X)$ is locally compact.*

(ii) *$H(\mathbb{Q}^*, X)$ is locally compact.*

(iii) *$c(X)$ has finite dimension.*

Proof. Since $H(\mathbb{R}, X) \cong H(X^*, \mathbb{R})$ and $H(\mathbb{Q}^*, X) \cong H(X^*, \mathbb{Q})$ [11, Ch. IV, Theorem 4.2, Corollary 2], the assertion follows from Theorem 1 and duality. □

For $H(\mathbb{Q}, X)$, we have:

Theorem 2. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$H(\mathbb{Q}, X)$ is locally compact.*

(ii) *There is a symmetric open neighborhood V of zero in X such that $(\frac{1}{n}V)\cap d(X)$ is relatively compact for all $n \in \mathbb{N}_0$.*

(iii) *There is an open subgroup F of $\overline{d(X)}$ such that $(\frac{1}{n}F) \cap \overline{d(X)}$ is compactly generated for all $n \in \mathbb{N}_0$.*

Proof. Assume (i), and let Ω be a compact neighborhood of zero in $H(\mathbb{Q}, X)$. Then there is a finite subset $K = \{a_1, \dots, a_k\}$ of $\mathbb{Q}$ and an open neighborhood U of zero in X such that $\Omega_{\mathbb{Q},X}(K, U) \subset \Omega$. As is well known, the finitely generated subgroups of $\mathbb{Q}$ are cyclic [8, p. 17], so $\langle K\rangle = \langle a\rangle$ for some $a \in \mathbb{Q}$. Write $a_i = m_i a$ with $m_i \in \mathbb{Z}$ for all $i = 1, \dots, k$. Further, set $m = \max_{1\le i\le k} |m_i|$, and choose a symmetric open neighborhood V of zero in X such that

$$\underbrace{V + \cdots + V}_{m} \subset U.$$

We claim that $(\frac{1}{n}V) \cap d(X)$ is relatively compact for all $n \in \mathbb{N}_0$. Indeed, given any $f \in \Omega_{\mathbb{Q},X}(\{a\}, V)$, we have

$$f(a_i) = m_i f(a) \in \underbrace{V + \cdots + V}_{m} \subset U$$

for all $i = 1, \dots, k$. Consequently,

$$\Omega_{\mathbb{Q},X}(\{a\}, V) \subset \Omega_{\mathbb{Q},X}(K, U) \subset \Omega,$$

proving that $\Omega_{\mathbb{Q},X}(\{a\}, V)$ has compact closure in $H(\mathbb{Q}, X)$. It follows from the Ascoli's theorem that for each $q \in \mathbb{Q}$, the orbit $\Omega_{\mathbb{Q},X}(\{a\}, V)q$ is relatively compact in X. Now, fix any $n \in \mathbb{N}_0$ and any $x \in (\frac{1}{n}V) \cap d(X)$. Then $nx \in V$. Define $h \in H(\langle \frac{a}{n}\rangle, d(X))$ by setting $h(\frac{a}{n}) = x$. Since $d(X)$ is divisible, h extends to a homomorphism $\widehat{h} \in H(\mathbb{Q}, d(X))$ [8, Theorem 21.1]. Let j be the canonical injection of $d(X)$ into X. We have $\widehat{h}(a) = n\widehat{h}(\frac{1}{n}a) = nx \in V$, so $j \circ \widehat{h} \in \Omega_{\mathbb{Q},X}(\{a\}, V)$, and hence

$$x \in \Omega_{\mathbb{Q},X}(\{a\}, V)\frac{a}{n}.$$

Since $x \in \left(\frac{1}{n}V\right) \cap d(X)$ was chosen arbitrarily, we get

$$\left(\frac{1}{n}V\right) \cap d(X) \subset \Omega_{\mathbb{Q},X}(\{a\}, V)\frac{a}{n},$$

proving that $\left(\frac{1}{n}V\right) \cap d(X)$ is relatively compact in X. So (i) implies (ii).

Now assume (ii), and fix an arbitrary $n \in \mathbb{N}_0$. It follows from [7, Exercise 1.3.D(a)] that

$$\overline{\left(\frac{1}{n}V\right) \cap \overline{d(X)}} = \overline{\left(\frac{1}{n}V\right) \cap d(X)},$$

so $\overline{\left(\frac{1}{n}V\right) \cap \overline{d(X)}}$ is compact, and hence the subgroup $\left\langle\left(\frac{1}{n}V\right) \cap \overline{d(X)}\right\rangle$ is compactly generated in $\overline{d(X)}$ [9, (5.13)]. Since $\left(\frac{1}{n}V\right) \cap \overline{d(X)}$ is open in $\overline{d(X)}$, it also follows that

$\big\langle\big(\frac{1}{n}V\big)\cap\overline{d(X)}\big\rangle$ is closed in X [9, (5.5)]. In a similar manner, $\langle V\cap\overline{d(X)}\rangle$ is open in $\overline{d(X)}$, so closed in X, and hence $\frac{1}{n}\langle V\cap\overline{d(X)}\rangle$ is closed in X because multiplication by n is continuous. We assert that

$$\Big\langle\Big(\frac{1}{n}V\Big)\cap\overline{d(X)}\Big\rangle=\frac{1}{n}\Big\langle V\cap\overline{d(X)}\Big\rangle\cap\overline{d(X)}. \tag{1}$$

Indeed, if $x\in\big(\frac{1}{n}V\big)\cap\overline{d(X)}$, then $nx\in V\cap\overline{d(X)}$, so $x\in\big\langle\big(\frac{1}{n}V\big)\cap\overline{d(X)}\big\rangle\cap\overline{d(X)}$, proving that

$$\Big\langle\Big(\frac{1}{n}V\Big)\cap\overline{d(X)}\Big\rangle\subset\frac{1}{n}\Big\langle V\cap\overline{d(X)}\Big\rangle\cap\overline{d(X)}.$$

To see the inverse inclusion, pick an arbitrary $x\in\frac{1}{n}\big\langle V\cap\overline{d(X)}\big\rangle\cap\overline{d(X)}$. Since

$$\begin{aligned}\big\langle V\cap\overline{d(X)}\big\rangle&=\big\langle\overline{V\cap\overline{\overline{d(X)}}}\big\rangle\\&=\big\langle\overline{V\cap d(X)}\big\rangle=\overline{\big\langle V\cap d(X)\big\rangle},\end{aligned}$$

we conclude that there exist $m\in\mathbb{N}_0$, $l_1,\dots,l_m\in\mathbb{Z}$, and $a_1,\dots,a_m\in V\cap d(X)$ such that

$$nx-\sum_{i=1}^{m}l_ia_i\in V.$$

Further, since $d(X)$ is divisible, we can write $a_i=nb_i$ with $b_i\in d(X)$ for all $i=1,\dots,m$. It follows that

$$n(x-\sum_{i=1}^{m}l_ib_i)\in V,$$

so

$$x-\sum_{i=1}^{m}l_ib_i\in\frac{1}{n}V,$$

and hence

$$x-\sum_{i=1}^{m}l_ib_i\in\Big(\frac{1}{n}V\Big)\cap\overline{d(X)}.$$

As $b_1,\dots,b_m\in\big(\frac{1}{n}V\big)\cap\overline{d(X)}$, this proves that $x\in\Big\langle\Big(\frac{1}{n}V\Big)\cap\overline{d(X)}\Big\rangle$, so

$$\frac{1}{n}\Big\langle V\cap\overline{d(X)}\Big\rangle\cap\overline{d(X)}\subset\Big\langle\Big(\frac{1}{n}V\Big)\cap\overline{d(X)}\Big\rangle,$$

proving (1). Finally, taking $F=\big\langle V\cap\overline{d(X)}\big\rangle$, we conclude that (ii) implies (iii).

Next assume (iii), and let U be a symmetric open neighborhood of zero in X such that $\overline{U}$ is compact and $F=\langle U\cap\overline{d(X)}\rangle$ [9, (5.13)]. We shall show that $\Omega_{\mathbb{Q},X}(\{1\},U)$ is relatively compact in $H(\mathbb{Q},X)$. Since $\mathbb{Q}$ is discrete, it is clear that $\Omega_{\mathbb{Q},X}(\{1\},U)$

is equicontinuous. Fix any $l, n \in \mathbb{N}_0$. To show that $\Omega_{\mathbb{Q},X}(\{1\},U)\frac{l}{n}$ is relatively compact in X, observe first that

$$\Omega_{\mathbb{Q},X}(\{1\},U)\frac{l}{n} \subset l\Big(\big(\frac{1}{n}U\big) \cap \overline{d(X)}\Big). \tag{2}$$

Indeed, for any $f \in \Omega_{\mathbb{Q},X}(\{1\},U)$, we have $nf(\frac{1}{n}) = f(1) \in U$, so $f(\frac{1}{n}) \in \big(\frac{1}{n}U\big) \cap \overline{d(X)}$ because $\mathbb{Q}$ is divisible, and hence

$$f(\frac{l}{n}) = lf(\frac{1}{n}) \in l\Big(\big(\frac{1}{n}U\big) \cap \overline{d(X)}\Big).$$

Since

$$\big(\frac{1}{n}U\big) \cap \overline{d(X)} = \frac{1}{n}\big(U \cap \overline{d(X)}\big) \cap \overline{d(X)}$$

it is clear that the inclusion (2) will assure the compactness of $\overline{\Omega_{\mathbb{Q},X}(\{1\},U)\frac{l}{n}}$ if we show that $\frac{1}{n}\big(U \cap \overline{d(X)}\big)$ has compact closure. Now, since $G = \big(\frac{1}{n}F\big) \cap \overline{d(X)}$ is compactly generated, we can write $G = A \oplus B \oplus C$, where $A \cong \mathbb{R}^d$ and $B \cong \mathbb{Z}^s$ for some $d, s \in \mathbb{N}$, and C is a compact subgroup of G [9, (9.8)]. Let $\pi_A, \pi_B, \pi_C \in E(G)$ be the canonical projections of G onto A, B, and C, respectively. Since $\big(\frac{1}{n}V\big) \cap \overline{d(X)} \subset G$ and $1_G = \pi_A + \pi_B + \pi_C$, where 1_G is the identity mapping on G, we have

$$\frac{1}{n}\big(U \cap \overline{d(X)}\big) \subset \pi_A\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big) + \pi_B\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big) + \pi_C\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big).$$

But

$$\pi_A\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big) \subset \frac{1}{n}\pi_A(U \cap \overline{d(X)}) \cap A,$$

$$\pi_B\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big) \subset \frac{1}{n}\pi_B(U \cap \overline{d(X)}) \cap B$$

and

$$\pi_C\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big) \subset \frac{1}{n}\pi_C(U \cap \overline{d(X)}) \cap C,$$

so

$$\frac{1}{n}\big(U \cap \overline{d(X)}\big) \subset \frac{1}{n}\pi_A(U \cap \overline{d(X)}) \cap A + \frac{1}{n}\pi_B(U \cap \overline{d(X)}) \cap B + \frac{1}{n}\pi_C(U \cap \overline{d(X)}) \cap C,$$

proving that $\frac{1}{n}\big(U \cap \overline{d(X)}\big)$ has compact closure in X. It follows by the Ascoli's theorem that $\Omega_{\mathbb{Q},X}(\{1\},U)$ is relatively compact in $H(\mathbb{Q},X)$, and hence (iii) implies (i). □

In order to dualize the preceding theorem, we will need the following lemma.

Lemma 1. *Let $X \in \mathcal{L}$. For every closed subgroup C of X and every $n \in \mathbb{N}_0$, $A(X^*, nC) = \frac{1}{n}A(X^*, C)$.*

Proof. We have

$$\begin{aligned} A(X^*, nC) &= \{\gamma \in X^* \mid \gamma(nx) = 0 \text{ for all } x \in C\} \\ &= \{\gamma \in X^* \mid n\gamma(x) = 0 \text{ for all } x \in C\} \\ &= \{\gamma \in X^* \mid n\gamma \in A(X, C)\} \\ &= \frac{1}{n} A(X^*, C). \end{aligned}$$

□

Corollary 2. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) $H(X, \mathbb{Q}^*)$ *is locally compact.*

(ii) *There is a closed subgroup C of X such that $m(X) \subset C$, $C/m(X)$ is compact, and $X/\overline{\overline{nC} + m(X)}$ has no small subgroups for all $n \in \mathbb{N}_0$.*

Proof. Assume (i). Since $H(\mathbb{Q}, X^*) \cong H(X, \mathbb{Q}^*)$ [11, Ch. IV, Theorem 4.2, Corollary 2], it follows from Theorem 2 that there is an open subgroup F of $\overline{d(X^*)}$ such that $\left(\frac{1}{n}F\right) \cap \overline{d(X^*)}$ is compactly generated for all $n \in \mathbb{N}_0$. Set $C = A(X, F)$. Clearly, $m(X^*) \subset C$ and $C/m(X^*) \cong \left(\overline{d(X^*)}/F\right)^*$ [6, Exercise 3.8.7], so $C/m(X^*)$ is compact [9, (5.21) and (23,17)]. By Lemma 1, we have

$$A(X^*, nC) = \frac{1}{n} A(X^*, C) = \frac{1}{n} F,$$

so $\overline{nC} = A(X, \frac{1}{n}F)$, and hence

$$\begin{aligned} A\big(X, (\frac{1}{n}F) \cap \overline{d(X^*)}\big) &= \overline{A(X, \frac{1}{n}F) + A(X, \overline{d(X^*)})} \\ &= \overline{\overline{nC} + m(X)} \end{aligned}$$

for all $n \in \mathbb{N}_0$. It follows from [9, (23.25)] that

$$\left(X/\overline{\overline{nC} + m(X)}\right)^* \cong \big(\frac{1}{n}F\big) \cap \overline{d(X^*)},$$

so $X/\overline{\overline{nC} + m(X)}$ has no small subgroups [1, Proposition 7.9] for all $n \in \mathbb{N}_0$. Consequently, (i) implies (ii).

Now assume (ii), and set $F = A(X^*, C)$. Since $m(X) \subset C$, we clearly have $F \subset \overline{d(X^*)}$. Further, since $\overline{d(X^*)}/F \cong \big(C/m(X)\big)^*$, it is also clear that F is open in $\overline{d(X^*)}$. Finally, given any $n \in \mathbb{N}_0$. we have

$$\left(\big(\frac{1}{n}F\big) \cap \overline{d(X^*)}\right)^* \cong X/\overline{\overline{nC} + m(X)},$$

so $\left(\frac{1}{n}F\right) \cap \overline{d(X^*)}$ is compactly generated [1, Proposition 7.9]. It follows from Theorem 2 that $H(\mathbb{Q}, X^*)$, and hence $H(X, \mathbb{Q}^*)$, is locally compact, proving that (ii) implies (i). □

4 Some necessary and some sufficient conditions

In this section, we reduce the study of local compactness of the ring $E(X)$ for general groups $X \in \mathcal{L}$ to some more special groups. We also establish some sufficient conditions for local compactness of $E(X)$.

Definition 2. A group $X \in \mathcal{L}$ is caled residual if $d(X) \subset k(X)$ and $c(X) \subset m(X)$.

Theorem 3. *Let $X \in \mathcal{L}$. If $E(X)$ is locally compact, then*

$$X \cong \mathbb{R}^d \times \mathbb{Q}^r \times (\mathbb{Q}^*)^s \times T,$$

where $d, r, s \in \mathbb{N}$ and T is a residual group in $\mathcal{L}$ such that $E(T)$ is locally compact.

In addition, if $d \neq 0$, then $T/k(T)$ is of finite rank and $c(T)$ is of finite dimension.

If $r \neq 0$, then $T/k(T)$ is of finite rank and $\overline{d(T)}$ admits an open subgroup F such that $(\frac{1}{n}F) \cap \overline{d(T)}$ is compactly generated for all $n \in \mathbb{N}_0$.

If $s \neq 0$, then $c(T)$ is of finite dimension and T admits a compact subgroup C such that $m(T) \subset C$, $C/m(T)$ is compact, and $T/\overline{\overline{nC} + m(T)}$ has no small subgroups for all $n \in \mathbb{N}_0$.

Proof. By [1, Theorem 9.3], we can write $X = C \oplus D \oplus S \oplus T$, where $C \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $D \cong \mathbb{Q}^{(r)}$ and $S \cong (\mathbb{Q}^*)^s$ for some cardinal numbers r and s, and T is a residual group in $\mathcal{L}$. Since D, S, and T are topological direct summands of X, we conclude from [17, Lemma 2] that $E(D)$, $E(S)$, and $E(T)$ are locally compact. Further, r and s must be finite by virtue of [17, Corollary 2 and Corollary 4]. Taking account of [9, (23,34)(c) and (23,34)(d)], the remaining assertions follow from the results of Section 3. ◻

We also have

Theorem 4. *Let X be a residual group in $\mathcal{L}$. If $E(X)$ is locally compact, then X satisfies one of the following conditions:*

(i) *$X/k(X)$ is of finite rank and $c(X)$ is of finite dimension.*

(ii) *$X/k(X)$ is of finite rank, $c(X)$ is of infinite dimension, and $m(x) = k(X)$.*

(iii) *$X/k(X)$ is of infinite rank, $c(X)$ is of finite dimension, and $d(X) = c(X)$.*

(iv) *$X/k(X)$ is of infinite rank, $c(X)$ is of infinite dimension, $d(X) = c(X)$, and $m(x) = k(X)$.*

Proof. Let $E(X)$ be locally compact. We show first that if $X/k(X)$ is of infinite rank, then $d(X) = c(X)$. Indeed, assume $X/k(X)$ is of infinite rank. By the local compactness of $E(X)$, there exist a compact subset K of X and an open neighborhood U of zero in X such that $U \subset K$ and $\Omega_X(K, U)$ is relatively compact in $E(X)$. Since X is residual and $\langle K \rangle$ is compactly generated, we can write $\langle K \rangle = A \oplus B$, where A is compact and $B \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}_0$. Clearly, $A \subset k(X)$ and $k(X) \cap B = \{0\}$.

Let $\pi : X \to X/k(X)$ be the canonical projection. Since $B \cong \pi(B)$, the pure subgroup $\pi(B)_*$ has finite rank in $X/k(X)$ [12, p. 41], so $X/k(X) \neq \pi(B)_*$, and hence $\big(X/k(X)\big)/\pi(B)_*$ is a non-zero torsion-free group. Fix an arbitrary $c \in X$ such that $\pi(c) \notin \pi(B)_*$. It follows by the Ascoli's theorem that $\Omega_X(K,U)c$ is relatively compact in X. Our goal is to show that $d(X) \subset \Omega_X(K,U)c$. To this end, pick any $z \in d(X)$, and define $\xi_z \in H(\langle\varphi(\pi(c))\rangle, d(X))$ by setting $\xi(\varphi(\pi(c))) = z$, where $\varphi : X/k(X) \to \big(X/k(X)\big)/\pi(B)_*$ is the canonical projection. Let us denote by $\widehat{\xi_z} \in H\big(\big(X/k(X)\big)/\pi(B)_*, d(X)\big)$ the extension of ξ_z to $\big(X/k(X)\big)/\pi(B)_*$ and by j the canonical injection of $d(X)$ into X. We have $j \circ \widehat{\xi_z} \circ \varphi \circ \pi \in \Omega_X(K,U)$ and $z = (j \circ \widehat{\xi_z} \circ \varphi \circ \pi)(c)$, so $z \in \Omega_X(K,U)c$. Since $z \in d(X)$ was picked arbitrarily, we deduce that $d(X) \subset \Omega_X(K,U)c$, so $\overline{d(X)}$ is compact, and hence $\overline{d(X)} = c(X)$ by [9, (24.24)]. Consequently, if $X/k(X)$ is of infinite rank, then $d(X) = c(X)$ [9, (24.25)]. Now, since $E(X^*)$ is locally compact too [17, Lemma 1], we conclude as above for X that if $X^*/k(X^*)$ is of infinite rank, then $d(X^*) = c(X^*)$. It follows by duality that if $c(X)$ is of infinite dimension, then $m(X) = k(X)$.

We further combine these facts, to get the conclusion. First suppose that $X/k(X)$ is of finite rank. If $X^*/k(X^*)$ is of finite rank too, then $c(X)$ is of finite dimension, and hence we have (i). On the other hand, if $X^*/k(X^*)$ is of infinite rank, then $c(X)$ is of infinite dimension and, as we know from the above, also $m(X) = k(X)$, so in this case we have (ii). Next suppose that $X/k(X)$ is of infinite rank. Then we know from the above that $d(X) = c(X)$. Thus, if $X^*/k(X^*)$ is of finite rank, then $c(X)$ is of finite dimension, and in this case we are led to (iii). Finally, if $X^*/k(X^*)$ is of infinite rank, we are led to (iv). □

We will need the following lemma, which is an adaption of Lemma 3 from [10].

Lemma 2. *For any groups $X, Y \in \mathcal{L}$, the following statements are equivalent:*

(i) *There is a neighborhood Ω of zero in $H(X,Y)$ such that Ωx is compact in Y for all $x \in X$.*

(ii) *There is a neighborhood Ω of zero in $H(Y^*, X^*)$ which operates equicontinuously on Y^*.*

Proof. Assume (i). By the definition of the compact-open topology, there exist a compact subset K of X and an open neighborhood U of zero in Y such that $\Omega_{X,Y}(K,U) \subset \Omega$. Since X and Y are locally compact, we can choose an open neighborhood V of zero in X and an open neighborhood W of zero in Y such that $\overline{V}$ and $\overline{W}$ are compact. Let $K_0 = K \cup \overline{V}$ and $U_0 = U \cap W$. It is clear that $\Omega_{X,Y}(K_0,U_0) \subset \Omega_{X,Y}(K,U)$, so $\Omega_{X,Y}(K_0,U_0)$ has compact closure in $H(X,Y)$. Moreover, for any compact subset C of X, the set

$$\Omega_{X,Y}(K_0,U_0)C = \{f(x) \mid f \in \Omega_{X,Y}(K_0,U_0) \text{ and } x \in C\}$$

has compact closure in Y. Indeed, by the compactness of C, there exist elements $x_1, \dots, x_m \in C$ such that $C \subset \cup_{i=1}^m (x_i + V)$. Given any $x \in C$, we then have

$x - x_{i_0} \in V$ for some $i_0 \in \{1, \ldots, m\}$, whence

$$f(x) \in f(x_{i_0}) + f(V) \subset \Omega_{X,Y}(K_0, U_0)x_i + U_0$$

for all $f \in \Omega_{X,Y}(K_0, U_0)$. Consequently,

$$\Omega_{X,Y}(K_0, U_0)C \subset \bigcup_{i=1}^{m} \overline{\Omega_{X,Y}(K_0, U_0)x_i} + \overline{U_0},$$

proving that $\Omega_{X,Y}(K_0, U_0)C$ has compact closure in Y. We shall show that the set

$$\Omega_{X,Y}(K_0, U_0)^* = \{f^* \in H(Y^*, X^*) \mid f \in \Omega_{X,Y}(K_0, U_0)\}$$

is equicontinuous in $H(Y^*, X^*)$. Let O be an arbitrary neighborhood of zero in X^*. We may assume that $O = \Omega_{X,\mathbb{T}}(C, D)$, where C is a compact subset of X and D is an open neighborhood of zero in $\mathbb{T}$. For this C, let $C' = \overline{\Omega_{X,Y}(K_0, U_0)C}$. Then C' is a compact subset of Y, so $O' = \Omega_{Y,\mathbb{T}}(C', D)$ is a neighborhood of zero in Y^*. Now, it is easily seen that $f^*(O') \subset O$ for all $f \in \Omega_{X,Y}(K_0, U_0)$, so $\Omega_{X,Y}(K_0, U_0)^*$ is equicontinuous at zero, and hence on Y^*. This proves that (i) implies (ii).

Now assume (ii), and let Φ be the neighborhood of zero in $H(X, Y)$ such that $\Omega = \{f^* \mid f \in \Phi\}$ [11, Ch. IV, Theorem 4.2, Corollary 2]. We claim that Φ operates with relatively compact orbits. Pick any $a \in X$. It suffices to show that $\xi_Y(\Phi a)$ is relatively compact in Y^{**}, where $\xi_Y : Y \to Y^{**}$ is the canonical topological isomorphism of Y, i.e. $\xi_Y(y)(\gamma) = \gamma(y)$ for all $y \in Y$ and $\gamma \in Y^*$. Observe that

$$\xi_Y(\Phi a) = \big\{\xi_X(a) \circ f^* \mid f \in \Phi\big\},$$

where $\xi_X : X \to X^{**}$ is the canonical topological isomorphism of X. To see that $\xi_Y(\Phi a)$ is equicontinuous, pick an arbitrary neighborhood D of zero in $\mathbb{T}$, and set $O = \{\gamma \in X^* \mid \xi(a)(\gamma) \in D\}$. Since $\xi(a)$ is continuous, O is a neighborhood of zero in X^*. Further, since $\Phi^* = \Omega$ is equicontinuous, there is a neighborhood W of zero in Y^* such that $f^*(W) \subset O$ for all $f \in \Phi^*$. It follows that $(\xi(a) \circ f^*)(W) \subset D$ for all $f \in \Phi^*$, proving that $\xi_Y(\Phi a)$ is equicontinuous. Finally, since $\mathbb{T}$ is compact, it is also clear that $\xi_Y(\Phi a)$ operates with relatively compact orbits. Consequently, $\xi_Y(\Phi a)$ is relatively compact in Y^{**} by the Ascoli's theorem. □

We now establish some sufficient conditions for the local compactness of $E(X)$.

Theorem 5. *Let X be a group in $\mathcal{L}$ satisfying the following conditions:*

i) *$c(X) \cap k(X)$ has finite dimension.*

ii) *For each $p \in S(X)$, $\Big(k(X)/\big(c(X) \cap k(X)\big)\Big)_p$ has finite rank.*

iii) *$X/\big(c(X) + k(X)\big)$ has finite rank.*

Then $E(X)$ is locally compact.

Proof. We can write $X = C \oplus Y$, where $C \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$ and Y contains a compact open subgroup. Then

$$E(X) \cong \begin{pmatrix} E(\mathbb{R}^d) & H(Y,\mathbb{R}^d) \\ H(\mathbb{R}^d, Y) & E(Y) \end{pmatrix}.$$

Now, since $H(Y,\mathbb{R}^d) \cong H(Y,\mathbb{R})^d$ and $H(\mathbb{R}^d, Y) \cong H(\mathbb{R}, Y)^d$ [9, (23.34)(c) and (23.34)(d)], we conclude from Theorem 1 and Theorem 2 that $H(Y,\mathbb{R}^d)$ and $H(\mathbb{R}^d, Y)$ are locally compact. As $E(\mathbb{R}^d)$ is locally compact too, it suffices to show that $E(Y)$ is locally compact. To this purpose, pick any elements $a_1, \dots, a_m$ of Y such that $a_1 + k(Y), \dots, a_m + k(Y)$ form a basis in $Y/k(Y)$, and a compact open subgroup U of Y. We claim that

$$\Omega = \Omega_Y(\{a_1, \dots, a_m\} \cup U, U)$$

is relatively compact in $E(Y)$. Let a be an arbitrary element in Y. Then there exist $n \in \mathbb{N}_0$, $l_1, \dots, l_m \in \mathbb{Z}$, and $b \in k(Y)$ such that $na = b + \sum_{i=1}^m l_i a_i$. Moreover, by multiplying the above equation through by the order of $b+U$ in $k(Y)/U$, if necessary, we may assume that $b \in U$. Now, given any $f \in \Omega$, we have

$$nf(a) = f(b) + \sum_{i=1}^m l_i f(a_i) \subset U,$$

so $f(a) \in \frac{1}{n}U$. Consequently, to conclude that Ω operates with relatively compact orbits, it suffices to show that $\frac{1}{n}U$ is compact. It is clear that $\left(\frac{1}{n}U\right)/U$ is a torsion group of bounded order, so $\frac{1}{n}U \subset k(Y)$, and hence $\left(\frac{1}{n}U\right)/U$ is a subgroup of bounded order of $k(Y)/U$. Since

$$k(Y)/U \cong \big(k(Y)/c(Y)\big)/\big(U/c(Y)\big),$$

we deduce from condition (ii) that the primary components of $k(Y)/U$ have finite rank. Further, since $\left(\frac{1}{n}U\right)/U$ is a subgroup of bounded order of $k(Y)/U$, we conclude that $\left(\frac{1}{n}U\right)/U$ is finite, so $\frac{1}{n}U$ is compact. Consequently, Ω operates with relatively compact orbits.

Further, observe that X^* too satisfies the hypotheses of the theorem. Indeed, by [9, (24,17)],[6, Proposition 3.3.3], and [9, (23,25)], we have

$$\begin{aligned} c(X^*) \cap k(X^*) &= A(X^*, c(X) + k(X)) \\ &\cong \Big(X/\big(c(X) + k(X)\big)\Big)^*, \end{aligned}$$

so $c(X^*) \cap k(X^*)$ has finite dimension by (iii) and [9, (24.28)]. Similarly, since

$$\Big(X^*/\big(c(X^*) + k(X^*)\big)\Big)^* \cong c(X) \cap k(X),$$

we deduce from (i) that $X^*/\big(c(X^*) + k(X^*)\big)$ has finite rank. Finally, we see from [6, Exercise 3.8.7] and [9, (6.9)] that

$$\begin{aligned}\Big(k(X^*)/\big(c(X^*)\cap k(X^*)\big)\Big)^* &\cong \big(c(X)+k(X)\big)/c(X)\\ &= \big(C\oplus k(X)\big)/\big(C\oplus \big(c(X)\cap k(X)\big)\big)\\ &\cong k(X)/\big(c(X)\cap k(X)\big).\end{aligned}$$

Given any $p\in S(X)$, we then have

$$\Big(k(X^*)/\big(c(X^*)\cap k(X^*)\big)\Big)_p \cong \Big(k(X)/\big(c(X)\cap k(X)\big)\Big)_p^*,$$

so $\Big(k(X^*)/\big(c(X^*)\cap k(X^*)\big)\Big)_p$ has finite rank by (ii) and [5, Theorem 4]. It follows that X^* too satisfies the hypotheses of the theorem. Consequently, we can conclude by using the same argument as with X that $E(X^*)$ admits a neighborhood of zero, which operates with relatively compact orbits. It follows from Lemma 2 that $E(X)$ admits a neighborhood of zero, which operates equicontinuously on X. It remains to apply the Ascoli's theorem. □

Remark 1. In [10, n° 9], M. Levin has shown that $A\Big(\prod_{n\in\mathbb{N}_0}\big(\mathbb{Z}(p^{2n});p^n\mathbb{Z}(p^{2n})\big)\Big)$ is locally compact although $\prod_{n\in\mathbb{N}_0}\big(\mathbb{Z}(p^{2n});p^n\mathbb{Z}(p^{2n})\big)$ has infinite rank. With similar arguments, it is easy to see that $E\Big(\prod_{n\in\mathbb{N}_0}\big(\mathbb{Z}(p^{2n});p^n\mathbb{Z}(p^{2n})\big)\Big)$ is locally compact as well, so the inverse of Theorem 5 is not valid.

5 Groups containing a lattice

Let X be a group in $\mathcal{L}$. A subgroup L of X is called a lattice in X if L is discrete and X/L is compact. If there exists such a subgroup L in X, then X is said to contain a lattice. If X decomposes as a topological direct sum of a discrete subgroup and a compact one, then it is said to contain a lattice trivially. If X contains a lattice but cannot be decomposed as a topological direct sum of a discrete group and a compact one, it is said to contain a lattice non-trivially.

In the present section, we answer the question of the local compactness of $E(X)$ in the case when X contains a lattice. In preparation for this we first establish a lemma, which introduces a topology, called the Birkhoff topology, on the group of units of a topological ring and shows how this topology is related to the topology of that ring.

Lemma 3. *Let E be a topological ring with identity* 1, *and let $E^\times$ be the group of invertible elements of E.*

(i) *If $\mathcal{B}$ is a filter base of neighborhoods of zero in E, then the set*

$$\mathcal{B}^\times=\big\{\big[(1+B)\cap E^\times\big]\cap\big[(1+B)\cap E^\times\big]^{-1}\mid B\in\mathcal{B}\big\}$$

is a filter base of neighborhoods of 1 *for a group topology on $E^\times$, which we call the Birkhoff topology of $E^\times$.*

(ii) *If E^{op} is the opposite topological ring of E and $E \times E^{op}$ is the topological direct product of topological rings E and E^{op}, then $E^{\times}$ with the Birkhoff topology is topologically isomorphic to a closed subgroup of the multiplicative monoid of $E \times E^{op}$. In particular, if E is locally compact, then $E^{\times}$ with its Birkhoff topology is locally compact too.*

Proof. (i) Since $\mathcal{B} \neq \varnothing$, it is clear that $\mathcal{B}^{\times} \neq \varnothing$ as well. Also, since every $B \in \mathcal{B}$ contains 0, we see that every element of $\mathcal{B}^{\times}$ contains 1, so $\varnothing \notin \mathcal{B}^{\times}$. Further, given any $B_1, B_2 \in \mathcal{B}$, there is $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$. It follows that

$$[(1 + B_3) \cap E^{\times}] \subset [(1 + B_1) \cap E^{\times}] \cap [(1 + B_2) \cap E^{\times}],$$

so

$$[(1 + B_3) \cap E^{\times}]^{-1} \subset [(1 + B_1) \cap E^{\times}]^{-1} \cap [(1 + B_2) \cap E^{\times}]^{-1},$$

and hence $[(1 + B_3) \cap E^{\times}] \cap [(1 + B_3) \cap E^{\times}]^{-1}$ is contained in the set

$$\Big([(1 + B_1) \cap E^{\times}] \cap [(1 + B_1) \cap E^{\times}]^{-1}\Big) \cap \Big([(1 + B_2) \cap E^{\times}] \cap [(1 + B_2) \cap E^{\times}]^{-1}\Big).$$

Consequently, $\mathcal{B}^{\times}$ is a filter base on $E^{\times}$.

Next we show that $\mathcal{B}^{\times}$ satisfies the conditions (GV'_I), (GV'_{II}), and (GV'_{III}) of [2, Ch. III, §1, n° 2]. Let U be a neighborhood of zero in E. We can choose neighborhoods O and V of zero in E such that $O + O \subset U$, $V + V \subset O$, and $VV \subset O$. Then

$$[(1 + V) \cap E^{\times}][(1 + V) \cap E^{\times}] \subset [(1 + U) \cap E^{\times}],$$

so

$$\begin{aligned}\big([(1 + V) \cap E^{\times}] \cap [(1 + V) \cap E^{\times}]^{-1}\big)\big([(1 + V) \cap E^{\times}] \cap [(1 + V) \cap E^{\times}]^{-1}\big)&\\ \subset [(1 + U) \cap E^{\times}] \cap [(1 + U) \cap E^{\times}]^{-1},&\end{aligned}$$

and hence (GV'_I) holds. Further, since

$$\Big([(1 + U) \cap E^{\times}] \cap [(1 + U) \cap E^{\times}]^{-1}\Big)^{-1} = [(1 + U) \cap E^{\times}]^{-1} \cap [(1 + U) \cap E^{\times}],$$

it is clear that (GV'_{II}) holds too. Finally, given any $a \in E^{\times}$, we can choose neighborhoods Φ and W of zero in E such that $\Phi a \subset U$ and $a^{-1}W \subset \Phi$, whence $a^{-1}Wa \subset U$. But then

$$a^{-1}[(1 + W) \cap E^{\times}]a \subset [(1 + U) \cap E^{\times}],$$

so

$$a^{-1}[(1 + W) \cap E^{\times}]^{-1}a \subset [(1 + U) \cap E^{\times}]^{-1},$$

and hence

$$a^{-1}\Big([(1 + W) \cap E^{\times}] \cap [(1 + W) \cap E^{\times}]^{-1}\Big)a \subset [(1 + U) \cap E^{\times}] \cap [(1 + U) \cap E^{\times}]^{-1}.$$

This proves (GV'_{III}). It follows that there is a unique group topology on $E^\times$, admitting $\mathcal{B}^\times$ as a filter base of neighborhoods of 1.

(ii) Recall that E^{op} is the topological ring in which the underlying set, the additive structure, and the topology are those of E, and whose multiplication is obtained by multiplying in E with reverse order. Consider the topological direct product $E \times E^{op}$. Since the mappings $(u, v) \to u \circ v$ and $(u, v) \to v \circ u$ from $E \times E^{op}$ to E are continuous, the sets

$$S = \{(u, v) \in E \times E^{op} \mid u \circ v = 1\}$$

and

$$T = \{(u, v) \in E \times E^{op} \mid v \circ u = 1\}$$

are closed in $E \times E^{op}$. It follows that $S \cap T$ is closed in $E \times E^{op}$. Clearly,

$$S \cap T = \{(u, u^{-1}) \in E \times E^{op} \mid u \in E^\times\}.$$

Moreover, $S \cap T$ has a group structure with respect to component-wise multiplication. Further, if we endow $S \cap T$ with the induced topology, then $S \cap T$ becomes a topological group. Indeed, the multiplication in $S \cap T$ is the restriction to $S \cap T$ of the multiplication in $E \times E^{op}$, and hence is continuous. Similarly, taking of inverses in $S \cap T$ is the restriction to $S \cap T$ of the mapping $(u, v) \to (v, u)$ from $E \times E^{op}$ onto $E \times E^{op}$, and hence is continuous too. It remains to observe that the mapping $\xi : u \to (u, u^{-1})$ is an isomorphism of topological groups from $E^\times$ onto $S \cap T$. Indeed, ξ is, clearly, an isomorphism of groups. Now, if U is a neighborhood of zero in E, then

$$\xi\Big([(1 + U) \cap E^\times] \cap [(1 + U) \cap E^\times]^{-1}\Big) = \Big((1 + U) \times (1 + U)\Big) \cap (S \cap T),$$

so ξ is bicontinuous. □

Specializing to the case $E = E(X)$, we have the following

Corollary 3. *Let $X \in \mathcal{L}$. Then $A(X)$ coincides with $E(X)^\times$ taken with its Birkhoff topology, and hence $A(X)$ is topologically isomorphic to a closed subgroup of the multiplicative monoid of $E(X) \times E(X)^{op}$.*

We are now prepared to describe all the groups $X \in \mathcal{L}$ containing a lattice for which $E(X)$ is locally compact. First, we consider the case when X contains a lattice non-trivially.

Theorem 6. *Let X be a group in $\mathcal{L}$ containing a lattice non-trivially. The following statements are equivalent:*

(i) *$E(X)$ is locally compact.*

(ii) *$A(X)$ is locally compact.*

(iii) *X satisfies the following conditions:*

1) *$c(X) \cap k(X)$ has finite dimension.*
2) *For each $p \in S(X)$, $\Big(k(X)/\big(c(X) \cap k(X)\big)\Big)_p$ has finite rank.*
3) *$X/\big(c(X) + k(X)\big)$ has finite rank.*

Proof. The fact that (i) implies (ii) follows from Corollary 3, the fact that (ii) implies (iii) follows from [10, Theorem 5], and the fact that (iii) implies (i) follows from Theorem 5. □

For the case of groups containing a lattice trivially, we have:

Theorem 7. *Let X be a group in $\mathcal{L}$ containing a lattice trivially, say $X = L \oplus C$ with L discrete and C compact. Then $E(X)$ is locally compact if and only if $E(L)$ and $E(C)$ are both locally compact.*

Proof. We have
$$E(X) \cong \begin{pmatrix} E(L) & H(C, L) \\ H(L, C) & E(C) \end{pmatrix}.$$
Since L is discrete, $H(L, C)$ is equicontinuous. Since C is compact, $H(L, C)$ operates with relatively compact orbits. Consequently, $H(L, C)$ is compact by the Ascoli's theorem. On the other hand, $H(C, L)$ is discrete because $\Omega_{C,L}(C, \{0\}) = \{0\}$. It follows that $E(X)$ is locally compact if and only if $E(L)$ and $E(C)$ are both locally compact. □

Remark 2. Taking account of the results in [17], the problem of determining the groups $X \in \mathcal{L}$ containing a lattice for which the ring $E(X)$ is locally compact is completely solved. In a similar way, the results of [17] and those of Section 3 can be used to describe the structure of any group $X \in \mathcal{L}$ with locally compact ring $E(X)$, which decomposes as a topological direct product of a finite number of copies of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^*$, and a group containing a lattice trivially. For example, this can be done for compactly generated groups [9, (9.8)], for groups with no small subgroups [1, Proposition 7.9], for groups with open connected component [1, Corollary 6.8], and for groups with compact subgroup of compact elements [1, Corollary 6.10], respectively.

We close this section by transferring to $E(X)$ a result of P. Plaumann for $A(X)$. We need the following definition from [13].

Definition 3. Let $X \in \mathcal{L}$. A factor of X is a quotient of the form A/B, where A and B are closed subgroups of X such that $A \supset B$.

Theorem 8. *For a topological torsion group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(F)$ is locally compact for every factor F of X.*

(ii) *$A(F)$ is locally compact for every factor F of X.*

(iii) *For each $p \in S(X)$, X_p has finite rank.*

Proof. The fact that (i) implies (ii) follows from Corollary 3, the fact that (ii) implies (iii) follows from [13, Theorem 3.6 and Lemma 3.1], and the fact that (iii) implies (i) follows from Theorem 5 because every factor of X has primary components of finite rank [4, 1)]. $\square$

6 Densely divisible torsion-free groups

In this final section, we answer the question of local compactness of the ring $E(X)$ for densely divisible torsion-free groups $X \in \mathcal{L}$. We begin with a special case.

Theorem 9. *Let $p \in \mathbb{P}$, and let X be a densely divisible, torsion-free, topological p-primary group in $\mathcal{L}$. The ring $E(X)$ is locally compact if and only if $X \cong \mathbb{Q}_p^r$ for some $r \in \mathbb{N}$.*

Proof. Let $E(X)$ be locally compact. Then $E(X^*)$ is locally compact as well. It is also clear that X^* is densely divisible and torsion-free. Let Ω be a compact neighborhood of zero in $E(X^*)$. By the definition of the compact-open topology, there exist a compact subset K of X^* and an open neighborhood U of zero in X^* such that $\Omega_{X^*}(K,U) \subset \Omega$. Since X^* is totally disconnected [1, Theorem 3.5], there is a compact open subgroup V of X^* such that $V \subset U$ [9, (7.5)], whence $\Omega_{X^*}(K,V) \subset \Omega_{X^*}(K,U)$, and hence $\Omega_{X^*}(K,V)$ is compact in $E(X^*)$.

We claim that $\frac{1}{p^n}V$ is compact for all $n \in \mathbb{N}$. To see this, fix any non-zero character $\alpha \in d(X^*)$, and let D_α be the minimal divisible subgroup of X^* containing α. Then $\overline{D_\alpha} \cong \mathbb{Q}_p$ [14, Lemma 2.4], so $X^* = \overline{D_\alpha} \oplus \Gamma$ for some closed subgroup Γ of X^* [1, Proposition 6.23]. Let $\pi_\alpha, \pi_\Gamma \in E(X^*)$ be the canonical projections of X^* onto $\overline{D_\alpha}$ and Γ, respectively. As $\pi_\alpha(K)$ is compact in $\overline{D_\alpha}$, we have $\pi_\alpha(K) \subset \frac{1}{p^{n_K}}\overline{\langle\alpha\rangle}$ for some $n_K \in \mathbb{N}_0$. Pick any $n \in \mathbb{N}_0$ and any $\beta \in d(X^*) \cap \frac{1}{p^n}V$, and let $\alpha' \in \overline{D_a}$ be the unique element satisfying $p^{n+n_K}\alpha' = \alpha$. Further, define $f \in H(\overline{\langle\alpha'\rangle} \oplus \Gamma, \overline{D_\alpha})$ by setting $f(\alpha') = \beta$ and $f(\gamma) = 0$ for all $\gamma \in \Gamma$. Since $\overline{\langle\alpha'\rangle} \oplus \Gamma$ is open in X^*, f extends to continuous group homomorphism $\hat{f} : X^* \to \overline{D_\alpha}$, so $j \circ \hat{f} \in E(X^*)$, where $j : \overline{D_\alpha} \to X^*$ is the canonical injection. Now, given any $\chi \in K$, we have

$$\begin{aligned}\hat{f}(\chi) = \hat{f}(\pi_\alpha(\chi)) \in \hat{f}\big(\frac{1}{p^{n_K}}\overline{\langle\alpha\rangle}\big) &= \hat{f}\big(\overline{\langle\frac{1}{p^{n_K}}\alpha\rangle}\big)\\ &= \hat{f}\big(\overline{\langle p^n\alpha'\rangle}\big) \subset \overline{\langle p^n\beta\rangle} \subset V,\end{aligned}$$

so $j \circ \hat{f} \in \Omega_{X^*}(K,V)$. Since $\beta \in d(X^*) \cap \frac{1}{p^n}V$ was chosen arbitrarily, it follows from [7, Theorem 1.3.6] that

$$\frac{1}{p^n}V = \overline{d(X^*) \cap \frac{1}{p^n}V} \subset \Omega_{X^*}(K,V)\alpha',$$

so $\frac{1}{p^n}V$ is compact.

Next, let $W = A(X, V)$. Clearly, W is compact and open in X [1, P. 22(e)]. Given any $n \in \mathbb{N}_0$, we deduce from Lemma 1 that

$$A(X^*, p^n W) = \frac{1}{p^n} A(X^*, W),$$

so $p^n W = A(X, \frac{1}{p^n}V)$. It follows that $p^n W$ is open in X, and hence in W. But $W \cong \mathbb{Z}_p^\nu$ for some cardinal number ν [3, Ch. III, §1, Proposition 3]. Consequently, ν must be finite, i.e. $\nu = r$ for some $r \in \mathbb{N}$.

The converse is clear, because $E(\mathbb{Q}_p^r)$ is topologically isomorphic to the matrix ring $M_r(\mathbb{Q}_p)$ over the field of p-adic numbers $\mathbb{Q}_p$, taken with its usual product topology. □

With this preparation, we can prove:

Theorem 10. *Let X be a densely divisible, torsion-free group in $\mathcal{L}$. The ring $E(X)$ is locally compact if and only if*

$$X \cong \mathbb{R}^d \times \mathbb{Q}^r \times (\mathbb{Q}^*)^s \times \prod_{p \in S(X)} (\mathbb{Q}_p^{r_p}; \mathbb{Z}_p^{r_p}),$$

where $d, r, s,$ and the r_p's are natural numbers.

Proof. Assume that $E(X)$ is locally compact. It follows from Theorem 3 that

$$X \cong \mathbb{R}^d \times \mathbb{Q}^r \times (\mathbb{Q}^*)^s \times T,$$

where $d, r, s \in \mathbb{N}$ and T is a residual in $\mathcal{L}$ such that $E(T)$ is locally compact. Now, in view of our hypotheses, $\overline{d(T)} = T$ and $m(T) = \{0\}$, whence $k(T) = T$ and $c(T) = \{0\}$. Consequently, T is a topological torsion group in $\mathcal{L}$, and hence

$$E(T) \cong \prod_{p \in S(X)} (E(T_p); \Omega_{T_p}(U_p, U_p)),$$

where, for each $p \in S(X)$, U_p is a compact open subgroup of T_p [16, (2.2)]. It follows that, for every $p \in S(X)$, $E(T_p)$ is locally compact ([3, p. 9] or [9, (6.16)(c)]), so $T_p \cong \mathbb{Q}_p^{r_p}$ for some $r_p \in \mathbb{N}_0$ by virtue of Theorem 9, and hence $T \cong \prod_{p \in S(X)} (\mathbb{Q}_p^{r_p}; \mathbb{Z}_p^{r_p})$ by [3, Ch. III, Proposition 4].

To show the converse, we write

$$X = A \oplus B \oplus C \oplus D,$$

where $A \cong \mathbb{R}^d$, $B \cong \mathbb{Q}^r$, $C \cong (\mathbb{Q}^*)^s$, and $D \cong \prod_{p \in S(X)} (\mathbb{Q}_p^{r_p}; \mathbb{Z}_p^{r_p})$. It is clear that $c(X) = A \oplus C$ and $k(X) = C \oplus D$, so $c(X) \cap k(X) = C$. We also have

$$c(X) + k(X) = A \oplus C \oplus D,$$

so $X/\big(c(X) + k(X)\big) \cong B$. Finally, given any $p \in S(X)$, we have

$$\Big(k(X)/\big(c(X) \cap k(X)\big)\Big)_p \cong \mathbb{Q}_p^{r_p}.$$

It remains to apply Theorem 5. □

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Received June 12, 2015