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On Soft Trees

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Abstract. In this paper, we introduce the notions of soft trees, soft cycles, soft bridges, soft cutnodes, and describe various methods of construction of soft trees. We investigate some of their fundamental properties.

Mathematics subject classification: 05C99, 03E72.
Keywords and phrases: Soft sets, soft cycle, soft bridge, soft cutnode, soft trees.

1 Introduction and Preliminaries

In 1975, Rosenfeld [11] first discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffmann [8] in 1973. Rosenfeld also proposed the fuzzy relations between fuzzy sets and developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Moreover, Bhattacharya [5] gave some remarks on fuzzy graphs. Bhutani and Rosenfeld [6] introduced the concept of $M-$ strong fuzzy graphs and described some of their properties. Recently, Thumbakara and George [13] discussed the concept of soft graphs in the specific way. On the other hand, Akram and Nawaz [4] have introduced the concepts of soft graphs and vertex-induced soft graphs in broad spectrum. In this paper, we introduce the concepts of soft trees, soft cycles, soft bridges, soft cutnodes and investigate some of their properties. We discuss some interesting properties of soft trees as a generalization of crisp trees. We also introduce some operations including union, intersection, AND operation and OR operation on soft trees.

Soft sets were proposed by Molodtsov in 1999, which provides a general mechanism for uncertainty modelling in a wide variety of applications [8, 12]. Let U be an initial universe of objects and P be the set of all parameters associated with objects in U, called a *parameter space*. In most cases parameters are considered to be attributes, characteristics or properties of objects in U. The power set of U is denoted by $\mathcal{P}(U)$.

Definition 1 (see [10])**.** A pair $\mathfrak{S} = (F, A)$ is called a *soft set* over U, where $A \subseteq P$ is a parameter set and $F\colon A \to \mathcal{P}(U)$ is a set-valued mapping, called the *approximate function* of the soft set $\mathfrak{S}$.

Let $G^* = (V, E)$ be a crisp graph and A be any nonempty set. Let subset R of $A \times V$ be an arbitrary relation from A to V. A mapping (or set-valued function) from A to $\mathcal{P}(V)$ written as $F : A \to \mathcal{P}(V)$ can be defined as $F(x) = \{y \in V : xRy\}$. The pair (F, A) is a soft set over V.

Definition 2 (see [3])**.** A *soft graph* $G = (G^*, F, K, A)$ is a 4-tuple such that

(a) $G^* = (V, E)$ is a simple graph,

(b) A is a non-empty set of parameters,

(c) (F, A) is a soft set over V,

(d) (K, A) is a soft set over E,

(e) $H(x) = (F(x), K(x))$ is a subgraph of G^* for all $x \in A$.

In what follows, we will use G^* for a simple graph, G for a soft graph and $H(x)$ for subgraph.

Definition 3 (see [3])**.** Let G be a soft graph of G^*. Then G is said to be a *complete soft graph* if every $H(x)$ is a complete graph for all $x \in A$.

2 Soft Trees

Definition 4. Let G be a soft graph of G^*. Then G is said to be a *soft tree* if every $H(x)$ is a tree for all $x \in A$.

Example 1. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_3, v_3v_5, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$. Let $A = \{v_2, v_6\} \subseteq V$. We define an approximate function

$$F : A \to \mathcal{P}(V) \text{ by } F(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq \mathit{1}\}.$$

That is, $F(v_2) = \{v_1, v_2, v_3\}$, $F(v_6) = \{v_1, v_5, v_6\}$. Thus, $(F, A) = \{F(v_2), F(v_6)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_2) = \{v_1v_2, v_2v_3\}$, $K(v_6) = \{v_5v_6, v_6v_1\}$. Thus, $(K, A) = \{K(v_2), K(v_6)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_2) = (F(v_2), K(v_2))$, $H(v_6) = (F(v_6), K(v_6))$ are connected subgraphs of G^* and also trees as shown in Fig. 1.

Hence, $G = \{H(v_2) = (F(v_2), K(v_2)), H(v_6) = (F(v_6), K(v_6))\}$ is a soft tree of G^*.

Theorem 1. *Let $H(x)$ be subgraph with $n \geq 3$ vertices of G^* and G a soft tree of G^*. Then G is not a complete soft graph.*

Proof. Suppose on contrary that G is a complete soft graph, then every subgraph $H(x)$, for all $x \in A$ is complete. Suppose u, v be arbitrary nodes of $H(x)$ and they are connected by an edge uv. Since $H(x)$ is a graph with $n \geq 3$ vertices, then we can always find at least one vertex w which is connected to v by an edge wv and to u by an edge wu, because $H(x)$ is a complete graph. Then there exists a cycle uvw. Therefore, $H(x)$ cannot be a tree which contradicts the fact that $H(x)$ is a connected subgraph of soft tree G. Hence, G can not be a complete soft graph. $\square$

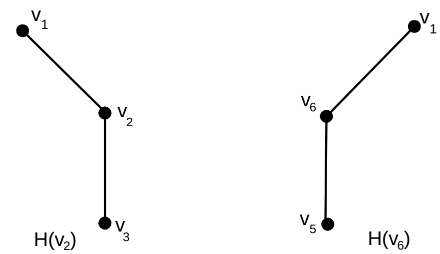

Figure 1. Subtrees

Definition 5. Let G be a soft graph of G^*. Then G is said to be a *soft cycle* if $H(x)$ is a cycle for all $x \in A$.

Example 2. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_1v_4, v_2v_4, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $A = \{v_3, v_5\} \subseteq V$. We define an approximate function $F : A \to \mathcal{P}(V)$ by

$$F(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\}.$$

That is, $F(v_3) = \{v_2, v_3, v_4\}$, $F(v_5) = \{v_1, v_4, v_5\}$. Thus, $(F, A) = \{F(v_3), F(v_5)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_3) = \{v_2v_3, v_3v_4, v_4v_2\}$, $K(v_5) = \{v_1v_4, v_4v_5, v_5v_1\}$. Thus, $(K, A) = \{K(v_3), K(v_5)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_3) = (F(v_3), K(v_3))$, $H(v_5) = (F(v_5), K(v_5))$ are connected subgraphs of G^* and also cycles as shown in the Fig. 2. Hence, $G = \{H(v_3) = (F(v_3), K(v_3)), H(v_5) =$

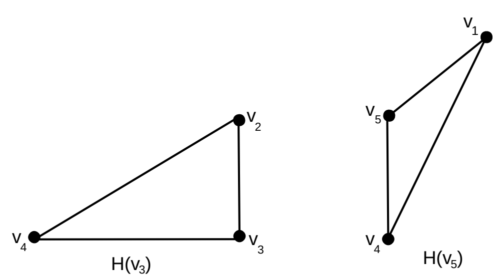

Figure 2. Subcycles

$(F(v_5), K(v_5))\}$ is a soft cycle of G^*.

Definition 6. Let G be a soft graph of G^*. Let u, v be two nodes and $H(x)$ a subgraph of G^*, then an edge $uv \in H(x)$ is called a *soft bridge* of G if removal of uv disconnects the $H(x)$.

Definition 7. Let G be a soft graph of G^*. Let u be a node of G^*, then u is called a *soft cutnode* of G if deletion of it from some $H(x)$, a subgraph of G, disconnects the $H(x)$.

In other words, we can say that u is a *soft cutnode* if it is a cutnode of some $H(x)$, $x \in A$.

Example 3. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6\}$. Let $A = \{v_1, v_4\} \subseteq V$. We define an approximate function $F : A \to \mathcal{P}(V)$ by

$$F(x) = \{y \in V : xRy \Leftrightarrow e(y) \leq e(x)\},$$

where $e(v_1) = e(v_3) = e(v_6) = 3$, $e(v_2) = 2$, $e(v_4) = e(v_5) = 4$,., That is, $F(v_1) = \{v_1, v_2, v_3, v_6\}$, $F(v_4) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Thus, $(F, A) = \{F(v_1), F(v_4)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_1) = \{v_1v_2, v_2v_6, v_2v_3\}$, $K(v_4) = \{v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6\}$. Thus, $(K, A) = \{K(v_1), K(v_4)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_1) = (F(v_1), K(v_1))$ and $H(v_4) = (F(v_4), K(v_4))$ are connected subgraphs of G^* as shown in Fig. 3. Therefore, $G = \{H(v_1) =$

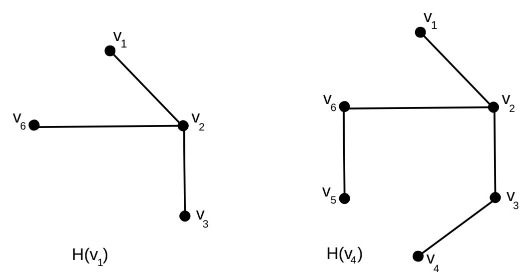

Figure 3. Connected subgraphs

$(F(v_1), K(v_1)), H(v_4) = (F(v_4), K(v_4))\}$ is a soft graph. In $H(v_1)$, all edges v_2v_6, v_1v_2, v_2v_3 are bridges because removal of any edge from $H(v_1)$ disconnects it as shown in Fig. 4. In $H(v_4)$, $v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6$ are bridges, because removal of any edge from $H(v_4)$ disconnects it as shown in Fig. 5. Therefore, $v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6$ are soft bridges of G. v_2 is a cutnode of $H(v_1)$ because deletion of it from $H(v_1)$ disconnects the $H(v_1)$ as shown in Fig. 6. Here v_2, v_3, v_6 are cutnodes of $H(v_4)$ because deletion of each of them from $H(v_4)$ disconnects the $H(v_4)$ as shown in Fig. 7. Therefore, v_2, v_3, v_6 are soft cutnodes of G.

Theorem 2. *If w is a common node of at least two soft bridges, then w is a soft cutnode.*

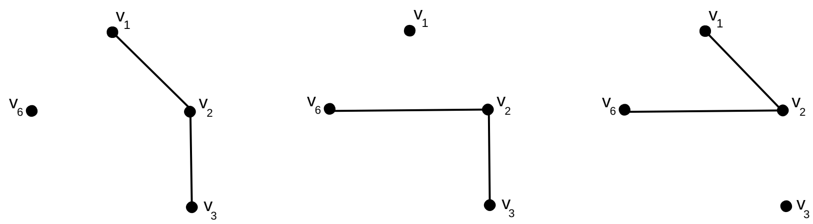

Figure 4. Disconnected subgraphs of $H(v_1)$

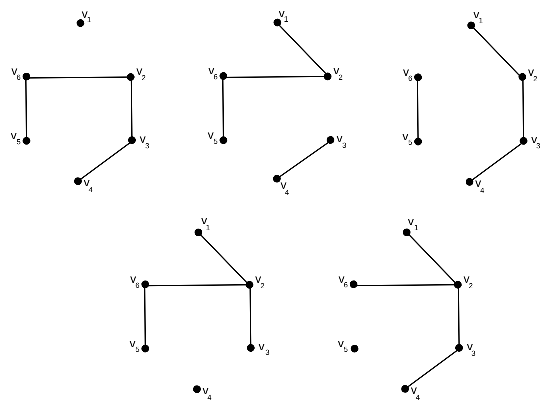

Figure 5. Disconnected subgraph of $H(v_4)$

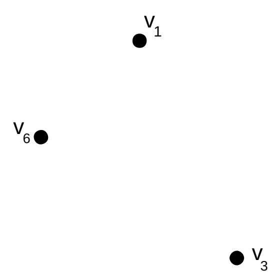

Figure 6. Disconnected subgraphs of $H(v_1)$

Proof. Let v_1w and wv_2 be two soft bridges of G. Then v_1w and wv_2 are bridges of some $H(x)$, that is, there exist some u, v such that v_1w is on every $u - v$ path. Clearly, if we delete w, then all the edges associated with it get removed. Then every $u - v$ path is disconnected. Thus, $H(x)$ is disconnected and w is a cutnode. Hence, w is a soft cutnode. □

Remark 1. The converse statement of Theorem 2 is not true as it can be seen in the following example.

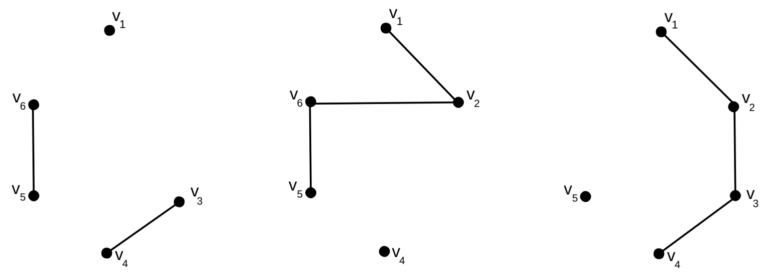

Figure 7. Disconnected subgraphs of $H(v_4)$

Example 4. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_1, v_3v_4, v_4v_5\}$. Let $A = \{v_1, v_2\} \subseteq V$. We define an approximate function $F : A \to \mathcal{P}(V)$ by

$$F(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 0\}.$$

That is, $F(v_1) = \{v_1, v_2, v_3, v_4, v_5\} = F(v_2)$. Thus, $(F, A) = \{F(v_1) = F(v_2)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_1) = \{v_1v_2, v_2v_3, v_1v_3, v_3v_4, v_4v_5\} = K(v_2)$. Thus, $(K, A) = \{K(v_1) = K(v_2)\}$ is a soft set over E.

Thus, $H(v_1) = (F(v_1), K(v_1))$ and $H(v_2) = (F(v_2), K(v_2))$ are connected subgraphs of G^* as shown in Fig. 8. In both $H(v_1)$ and $H(v_2)$, v_3v_4, v_4v_5 are bridges as

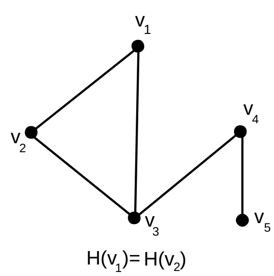

Figure 8. Connected subgraphs

shown in Fig. 9. Therefore, v_3v_4, v_4v_5 are soft bridges of G. v_3, v_4 are cutnodes as shown in Fig. 10. Therefore, v_3, v_4 are soft cutnodes of G. Here v_3 is a soft cutnode but it is not a common node of two soft bridges.

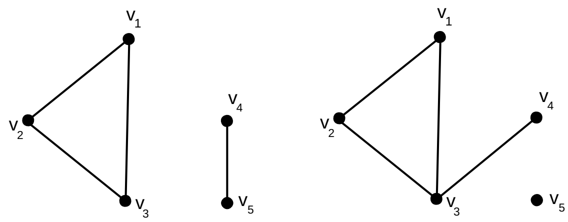

Figure 9. Disconnected subgraphs of $H(v_1) = H(v_2)$

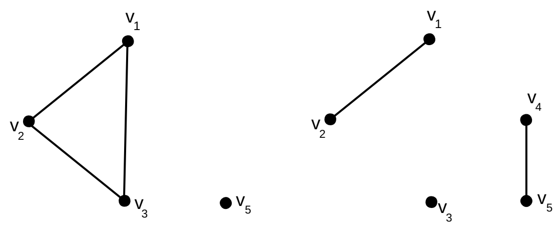

Figure 10. Disconnected subgraphs of $H(v_1) = H(v_2)$

We state the following theorems without their proofs.

Theorem 3. *A complete soft graph has no soft cutnodes.*

Theorem 4. *If G is a soft tree, then all edges of G are the soft bridges of G.*

Theorem 5. *If G is a soft tree, then internal nodes of G are the soft cutnodes of G.*

2.1 Operations on soft trees

Definition 8. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *union of two soft trees* G_1 and G_2 is a soft graph and defined as $G = G_1 \cup G_2 = (G^*, F, K, C)$ if $H(x) = (F(x), K(x))$ for all $x \in C$ is a subgraph, where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cup F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A - B, \\ K_2(x) & \text{if } x \in B - A, \\ K_1(x) \cup K_2(x) & \text{if } x \in A \cap B. \end{cases}$$

Theorem 6. *Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^* such that $A \cap B = \emptyset$, then $G_1 \cup G_2$ is a soft tree.*

Proof. The union of $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ is defined as

$$G = G_1 \cup G_2 = (G^*, F, K, C),$$

where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cup F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A - B, \\ K_2(x) & \text{if } x \in B - A, \\ K_1(x) \cup K_2(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, then $A - B = A$ and $B - A = B$. Thus,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A, \\ F_2(x) & \text{if } x \in B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A, \\ K_2(x) & \text{if } x \in B. \end{cases}$$

$H_1(x) = (F_1(x), K_1(x))$ and $H_2(x) = (F_2(x), K_2(x))$ are trees, since G_1 and G_2 are soft trees. Therefore, $H = (F(x), K(x))$ is a tree and $G = (G^*, F, K, C)$ is a soft tree. $\square$

Remark 2. If $A \cap B \neq \emptyset$, then union of two soft trees may not be a soft tree as it can be seen in the following example.

Example 5. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*.

Let $A = \{v_3, v_4\} \subseteq V$ and $B = \{v_4\} \subseteq V$. We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\} \ \forall x \in A,$$

i.e., $F_1(v_3) = \{v_2, v_3, v_4\}$, $F_1(v_4) = \{v_3, v_4, v_5\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 1\} \ \forall x \in B,$$

i.e., $F_2(v_4) = \{v_1, v_2, v_3, v_5\}$, respectively. Thus, $F(v_3) = F_1(v_3) = \{v_2, v_3, v_4\}$, $F(v_4) = F_1(v_4) \cup F_2(v_4) = \{v_1, v_2, v_3, v_4, v_5\}$. Thus, $(F, A) = \{F(v_3), F(v_4)\}$ is a soft set over V.

We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\} \ \forall x \in A,$$

i. e., $K_1(v_3) = \{v_2v_3, v_3v_4\}$, $K_1(v_4) = \{v_3v_4, v_4v_5\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\}\ \forall x \in B,$$

i. e., $K_2(v_4) = \{v_1v_2, v_2v_3, v_5v_1\}$, respectively. Thus, $K(v_3) = K_1(v_3) = \{v_2v_3, v_3v_4\}$, $K(v_4) = K_1(v_4) \cup K_2(v_4) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Thus, $(K, A) = \{K(v_3), K(v_4)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_3) = (F(v_3), K(v_3))$ and $H(v_4) = (F(v_4), K(v_4))$ are connected subgraphs of G^*. $H(v_3) = (F(v_3), K(v_3))$ is a tree but $H(v_4) = (F(v_4), K(v_4))$ is a cycle as shown in Fig. 11. Hence, $G = \{H(v_3) = (F(v_3), K(v_3)), H(v_4) = (F(v_4), K(v_4))\}$

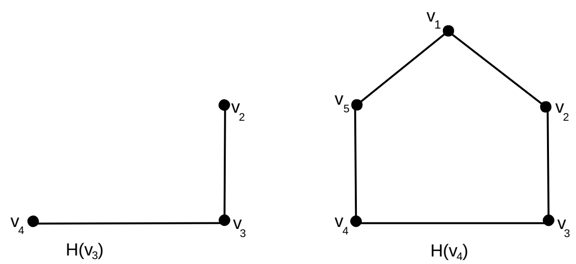

Figure 11. Connected subgraphs

is not a soft tree of G^*.

Definition 9. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *intersection of two soft trees* G_1 and G_2 is a soft graph and defined as $G = G_1 \cap G_2 = (G^*, F, K, C)$ if $H(x) = (F(x), K(x))$ for all $x \in C$ is a subgraph, where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cap F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x), & \text{if } x \in A - B, \\ K_2(x), & \text{if } x \in B - A, \\ K_1(x) \cap K_2(x), & \text{if } x \in A \cap B. \end{cases}$$

Theorem 7. *Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^* such that $A \cap B = \emptyset$, then $G_1 \cap G_2$ is a soft tree.*

Proof. The intersection of $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ is defined as

$$G = G_1 \cap G_2 = (G^*, F, K, C),$$

where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cap F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A - B, \\ K_2(x) & \text{if } x \in B - A, \\ K_1(x) \cap K_2(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, then $A - B = A$ and $B - A = B$. Thus,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A, \\ F_2(x) & \text{if } x \in B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A, \\ K_2(x) & \text{if } x \in B. \end{cases}$$

$H_1(x) = (F_1(x), K_1(x))$ and $H_2(x) = (F_2(x), K_2(x))$ are trees, since G_1 and G_2 are soft trees. Therefore, $H = (F(x), K(x))$ is a tree and $G = (G^*, F, K, C)$ is a soft tree. $\square$

Remark 3. If $A \cap B \neq \emptyset$, then intersection of two soft trees may not be a soft tree as it can be seen in the following example.

Example 6. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. Let $A = \{v_2, v_4\} \subseteq V$ and $B = \{v_3, v_4\} \subseteq V$. We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 1\} \ \forall x \in A,$$

i.e., $F_1(v_2) = \{v_1, v_3, v_4\}$, $F_1(v_4) = \{v_1, v_2, v_3\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\} \ \forall x \in B,$$

i.e., $F_2(v_3) = \{v_2, v_3, v_4\}$, $F_2(v_4) = \{v_1, v_3, v_4\}$, respectively. Thus, $F(v_2) = F_1(v_2) = \{v_1, v_3, v_4\}$, $F(v_3) = F_2(v_3) = \{v_2, v_3, v_4\}$, $F(v_4) = F_1(v_4) \cap F_2(v_4) = \{v_1, v_3\}$. Thus, $(F, A) = \{F(v_2), F(v_3), F(v_4)\}$ is a soft set over V. We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\} \ \forall x \in A,$$

i.e., $K_1(v_2) = \{v_3v_4, v_4v_1\}$, $K_1(v_4) = \{v_1v_2, v_2v_3\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\} \ \forall x \in B,$$

i.e., $K_2(v_3) = \{v_2v_3, v_3v_4\}$, $K_2(v_4) = \{v_3v_4, v_4v_1\}$, respectively. Thus, $K(v_2) = K_1(v_2) = \{v_3v_4, v_4v_1\}$, $K(v_3) = K_2(v_3) = \{v_2v_3, v_3v_4\}$, $K(v_4) = K_1(v_4) \cap K_2(v_4) = \{\}$. Thus, $(K, A) = \{K(v_2), K(v_3), K(v_4)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_2) = (F(v_2), K(v_2))$ and $H(v_3) = (F(v_3), K(v_3))$ are connected subgraphs of G^* as well as trees as shown in Fig. 12. But $H(v_4) = (F(v_4), K(v_4))$ is not a connected subgraph and hence, not a tree as shown in Fig. 13. Hence, $G = \{H(v_2) = (F(v_2), K(v_2)), H(v_3) = (F(v_3), K(v_3)), H(v_4) = (F(v_4), K(v_4))\}$ is not a soft tree of G^*.

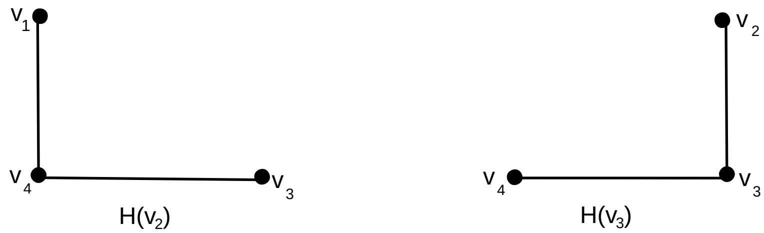

Figure 12. Subtrees

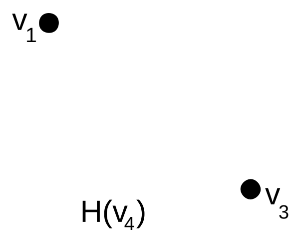

Figure 13. Disconnected subgraph

Definition 10. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *OR operation on two soft trees* G_1 and G_2 is a soft tree and defined as

$$G = G_1 \vee G_2 = (G^*, F, K, C),$$

if the subgraph $H(a, b) = (F(a, b), K(a, b))$, for all $(a, b) \in C$ is a tree, where $C = A \times B$ and for all $(a, b) \in C$, $F(a, b) = F_1(a) \cup F_2(b)$ and $K(a, b) = K_1(a) \cup K_2(b)$.

Example 7. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*.

Let $A = \{v_2, v_5\} \subseteq V$ and $B = \{v_1, v_3\} \subseteq V$. Then $A \times B = \{(v_2, v_1), (v_2, v_3), (v_5, v_1), (v_5, v_3)\}$.

We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\} \ \forall x \in A,$$

i. e., $F_1(v_2) = \{v_1, v_2, v_3\}$, $F_1(v_5) = \{v_1, v_4, v_5\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) > 1\} \ \forall x \in B,$$

i. e., $F_2(v_1) = \{v_3, v_4\}$, $F_2(v_3) = \{v_1, v_5\}$, respectively. Thus, $F(v_2, v_1) = F_1(v_2) \cup F_2(v_1) = \{v_1, v_2, v_3, v_4\}$, $F(v_2, v_3) = F_1(v_2) \cup F_2(v_3) = \{v_1, v_2, v_3, v_5\}$, $F(v_5, v_1) = F_1(v_5) \cup F_2(v_1) = \{v_1, v_3, v_4, v_5\}$, $F(v_5, v_3) = F_1(v_5) \cup F_2(v_3) = \{v_1, v_4, v_5\}$. Thus, $(F, A) = \{F(v_2, v_1), F(v_2, v_3), F(v_5, v_1), F(v_5, v_3)\}$ is a soft set over V.

We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\}\ \forall x \in A,$$

i. e., $K_1(v_2) = \{v_1v_2, v_2v_3\}$, $K_1(v_5) = \{v_4v_5, v_5v_1\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\}\ \forall x \in B,$$

i. e., $K_2(v_1) = \{v_3v_4\}$, $K_2(v_3) = \{v_5v_1\}$, respectively. Thus, $K(v_2, v_1) = K_1(v_2) \cup K_2(v_1) = \{v_1v_2, v_2v_3, v_3v_4\}$, $K(v_2, v_3) = K_1(v_2) \cup K_2(v_3) = \{v_1v_2, v_2v_3, v_5v_1\}$, $K(v_5, v_1) = K_1(v_5) \cup K_2(v_1) = \{v_3v_4, v_4v_5, v_5v_1\}$, $K(v_5, v_3) = K_1(v_5) \cup K_2(v_3) = \{v_4v_5, v_5v_1\}$. Thus, $(K, A) = \{K(v_2, v_1), K(v_2, v_3), K(v_5, v_1), K(v_5, v_3)\}$ is a soft set over E. Hence, $H(v_2, v_1) = (F(v_2, v_1), K(v_2, v_1))$, $H(v_2, v_3) = (F(v_2, v_3), K(v_2, v_3))$, $H(v_5, v_1) = (F(v_5, v_1), K(v_5, v_1))$ and $H(v_5, v_3) = (F(v_5, v_3), K(v_5, v_3))$ are connected subgraphs of G^* and are also trees as shown in Fig. 14. Hence, $G =$

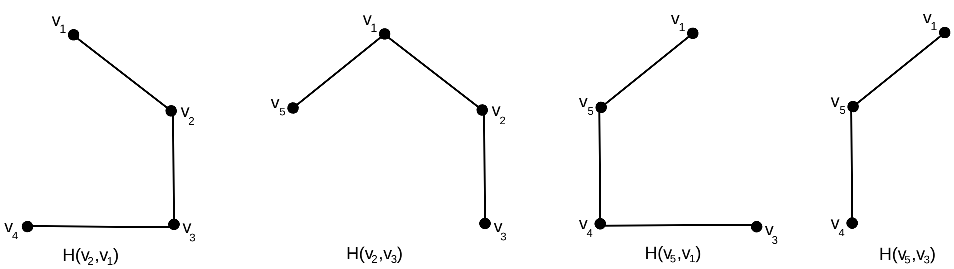

Figure 14. Subtrees

$\{H(v_2, v_1) = (F(v_2, v_1), K(v_2, v_1)), H(v_2, v_3) = (F(v_2, v_3), K(v_2, v_3)), H(v_5, v_1) = (F(v_5, v_1), K(v_5, v_1)), H(v_5, v_3) = (F(v_5, v_3), K(v_5, v_3))\}$ is a soft tree.

Definition 11. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *AND operation on two soft trees* G_1 and G_2 is a soft tree and defined as

$$G = G_1 \wedge G_2 = (G^*, F, K, C)$$

if the subgraph $H(a, b) = (F(a, b), K(a, b))$, for all $(a, b) \in C$, is a tree, where $C = A \times B$ and for all $(a, b) \in C$, $F(a, b) = F_1(a) \cap F_2(b)$ and $K(a, b) = K_1(a) \cap K_2(b)$.

Example 8. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. Let $A = \{v_1, v_3\} \subseteq V$ and $B = \{v_2, v_4\} \subseteq V$. Then $A \times B = \{(v_1, v_2), (v_1, v_4), (v_3, v_2), (v_3, v_4)\}$. We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\}\ \forall x \in A,$$

i. e., $F_1(v_1) = \{v_1, v_2, v_5\}$, $F_1(v_3) = \{v_2, v_3, v_4\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 1\}\ \forall x \in B,$$

i. e., $F_2(v_2) = \{v_1, v_3, v_4, v_5\}$, $F_2(v_4) = \{v_1, v_2, v_3, v_5\}$, respectively. Thus, $F(v_1, v_2) = F_1(v_1) \cap F_2(v_2) = \{v_1, v_5\}$, $F(v_1, v_4) = F_1(v_1) \cap F_2(v_4) = \{v_1, v_2, v_5\}$, $F(v_3, v_2) = F_1(v_3) \cap F_2(v_2) = \{v_3, v_4\}$, $F(v_3, v_4) = F_1(v_3) \cap F_2(v_4) = \{v_2, v_3\}$. Thus, $(F, A) = \{F(v_1, v_2), F(v_1, v_4), F(v_3, v_2), F(v_3, v_4)\}$ is a soft set over V.

We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\} \ \forall x \in A,$$

i. e., $K_1(v_1) = \{v_1v_2, v_5v_1\}$, $K_1(v_3) = \{v_2v_3, v_3v_4\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\} \ \forall x \in B,$$

i. e., $K_2(v_2) = \{v_3v_4, v_4v_5, v_5v_1\}$, $K_2(v_4) = \{v_1v_2, v_2v_3, v_5v_1\}$. Thus, $K(v_1, v_2) = K_1(v_1) \cap K_2(v_2) = \{v_5v_1\}$, $K(v_1, v_4) = K_1(v_1) \cap K_2(v_4) = \{v_1v_2, v_5v_1\}$, $K(v_3, v_2) = K_1(v_3) \cap K_2(v_2) = \{v_3v_4\}$, $K(v_3, v_4) = K_1(v_3) \cap K_2(v_4) = \{v_2v_3\}$. Thus, $(K, A) = \{K(v_1, v_2), K(v_1, v_4), K(v_3, v_2), K(v_3, v_4)\}$ is a soft set over E. Hence, $H(v_1, v_2) = (F(v_1, v_2), K(v_1, v_2))$, $H(v_1, v_4) = (F(v_1, v_4), K(v_1, v_4))$, $H(v_3, v_2) = (F(v_3, v_2), K(v_3, v_2))$ and $H(v_3, v_4) = (F(v_3, v_4), K(v_3, v_4))$ are connected subgraphs of G^* and are also trees as shown in Fig. 15. Hence, $G = \{H(v_1, v_2) =$

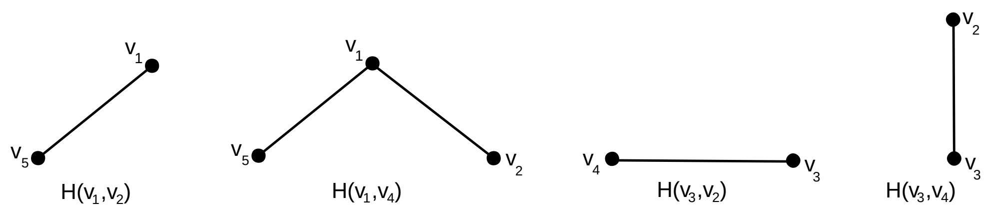

Figure 15. Subtrees

$(F(v_1, v_2),\ K(v_1, v_2))$, $H(v_1, v_4) = (F(v_1, v_4),\ K(v_1, v_4))$, $H(v_3, v_2) = (F(v_3, v_2),$ $K(v_3, v_2))$, $H(v_3, v_4) = (F(v_3, v_4), K(v_3, v_4))\}$ is a soft tree.

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