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Estimates of stability radius of multicriteria Boolean problem with Hölder metrics in parameter spaces

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Abstract. We consider multiple objective combinatorial linear problem in the situation where parameters of objective functions are exposed to perturbations. We study quantitative characteristic of stability (stability radius) of the problem assuming that there are Hölder metrics in the space of solutions and the criteria space.

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1 Introduction

The main difficulty while studying stability of discrete optimization problems is their combinatorial complexity. Small changes of initial data make a model behave in an unpredictable manner. In addition, in the case of several conflicting objectives the problem complexity may only be increased (see e.g. [1, 2]).

There are a lot of papers devoted to different approaches dealing with uncertainty in discrete models, both in single and multicriteria cases (see e.g. [3–5]). One of such approaches is known as quantitative approach. This approach aims to derive quantitative bounds for feasible initial data changes preserving a given property of the solution set (or of a single solution) or/and create algorithms for the bounds calculation. The limit level of perturbations of problem parameters which preserve the property of invariance is called stability radius. The present work continues a line of investigations initiated in [6–9] that focuses on studying the stability radius of multicriteria Boolean optimization problems with various types of metrics in the parameter space. We have obtained the lower and upper bounds for the stability radius of the multicriteria combinatorial linear problem on the assumption that Hölder norms are specified in the space of solutions and in the space of criteria.

2 Problem statement and basic definitions

Let $\mathbf{R}^m$ be the space of criteria, $\mathbf{R}^n$ be the space of solutions, C be an $m \times n$ matrix with the rows $C_i = (c_{i1}, c_{i2}, \ldots, c_{in}) \in \mathbf{R}^n$, $i \in N_m = \{1, 2, \ldots, m\}$, $x = (x_1, x_2, \ldots, x_n)^T \in X \subseteq \mathbf{E}^n$, $n \geq 2$, $\mathbf{E} = \{0, 1\}$, $|X| \geq 2$. Let a linear vector criterion

$$Cx = (C_1x, C_2x, \dots, C_mx)^T \to \min_{x\in X}$$

be specified on the set of Boolean vectors (solutions) X.

Under a m-criterion problem Boolean problem $Z^m(C)$, $C \in \mathbf{R}^{m\times n}$, we understand the problem of finding the Pareto set, i.e. the set of efficient (Pareto optimal) solutions

$$P^m(C) = \{x \in X : X(x, C) = \emptyset\},$$

where

$$X(x, C) = \{x' \in X : Cx' \leq Cx \ \&\ Cx' \neq Cx\}.$$

Since X is finite, the set $P^m(C)$ is not empty for any matrix $C \in \mathbf{R}^{m\times n}$.

We will perturb elements of the matrix C by adding matrices C' from $\mathbf{R}^{m\times n}$ to it. Thus, the perturbed problem $Z^m(C + C')$ has the form

$$(C + C')x \to \min_{x\in X}$$

and the Pareto set of such a problem has the form $P^m(C + C')$.

For any natural number d in the real space $\mathbf{R}^d$, we specify the Hölder norm l_p, $p \in [1, \infty]$, i. e., the norm of a vector $y = (y_1, y_2, \dots, y_d)$ is understood to be the number

$$\|y\|_p = \begin{cases} \Big(\sum\limits_{i\in N_d} |y_i|^p\Big)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\limits_{i\in N_d} |y_i| & \text{if } p = \infty. \end{cases}$$

For any $p, q \in [1, \infty]$, let us define the Hölder norm l_p in the space of solutions $\mathbf{R}^n$ and the Hölder norm l_q in the criteria space $\mathbf{R}^m$. Thereby, the norm $\|C\|_{pq}$ of the matrix $C \in \mathbf{R}^{m\times n}$ is defined as the norm of the vector whose components are the norms of the matrix rows $C_1, C_2, \dots, C_m$, i.e.

$$\|C\|_{pq} = \big\|\big(\|C_1\|_p, \|C_2\|_p, \dots, \|C_m\|_p\big)^T\big\|_q.$$

It is easy to see that for any $p, q \in [1, \infty]$ the following inequalities hold

$$\|C_i\|_p \leq \|C\|_{pq}, \quad i \in N_m. \tag{1}$$

Obviously, for each $\alpha \geq 0$, $p \in [1, \infty]$ and vector $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$ with components $|a_i| = \alpha$, $i \in N_n$, the following equality holds

$$\|a\|_p = \alpha n^{1/p}. \tag{2}$$

Let $l_{p'}$ be a conjugate norm in the space of solutions $\mathbf{R}^n$ and, as is well known, the numbers p and p' are related by the condition

$$1/p + 1/p' = 1.$$

As usual, we assume that $p' = 1$ if we have $p = \infty$ and that $p' = \infty$ if we have $p = 1$. Thus, henceforth, we assume that the domain of varying the numbers p and p' is the interval $[1, \infty]$ and that the numbers p and p' themselves are related by the above-mentioned condition. In addition, we impose that $1/p = 1$ if $p = \infty$.

We will use the well-known Hölder inequality

$$ab \leq \|a\|_p \|b\|_{p'}, \tag{3}$$

where $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$ and $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$.

As usually (see e. g. [6–9]), by the radius of stability of the problem $Z^m(C)$ we mean the quantity

$$\rho^m(p,q) = \begin{cases} \sup \Xi & \text{if } \Xi \neq \emptyset, \\ 0 & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \big\{\varepsilon > 0 : \forall C' \in \Omega_{pq}(\varepsilon) \ \big(P^m(C + C') \subseteq P^m(C)\big)\big\},$$

$$\Omega_{pq}(\varepsilon) = \{C \in \mathbf{R}^{m \times n} : \|C\|_{pq} < \varepsilon\}.$$

Thus, the stability radius of the problem $Z^m(C)$ is the limiting perturbation of elements of the matrix C in the space $\mathbf{R}^{m\times n}$ that does not produce new efficient solutions. The set $\Omega_{pq}(\varepsilon)$ is called the set of perturbing matrices.

In the trivial case, where $P^m(C) = X$, the inclusion $P^m(C+C') \subseteq P^m(C)$ holds for any perturbing matrix $C' \in \Omega_{pq}(\varepsilon)$, $\varepsilon > 0$. Therefore, no one perturbation of the problem parameters can cause appearance of new efficient solutions, i. e. stability radius of such problem is unbounded above. The problem $Z^m(C)$ for which $P^m(C) \neq X$ will be called non-trivial.

3 Estimates of the stability radius

For a non-trivial problem $Z^m(C)$ and any $p, q \in [1, \infty]$ we assume

$$\varphi^m(p) = \min_{x \in X \setminus P^m(C)} \ \max_{x' \in P^m(x,C)} \ \min_{i \in N_m} \frac{C_i(x - x')}{\|x - x'\|_{p'}},$$

$$\psi^m(p,q) = \min\big\{\sigma^m(p),\ n^{1/p} m^{1/q} \varphi^m(\infty)\big\},$$

where

$$P^m(x, C) = P^m(C) \cap X(x, C),$$

$$\sigma^m(p) = \min\{\|C_i\|_p : i \in N_m\}.$$

Theorem 1. *For any $p, q \in [1, \infty]$ and $m \in \mathbf{N}$, the stability radius $\rho^m(p,q)$ of a non-trivial problem $Z^m(C)$ has the following bounds:*

$$\varphi^m(p) \leq \rho^m(p,q) \leq \psi^m(p,q).$$

Proof. First, let us prove the inequality $\rho^m(p,q) \geq \varphi^m(p)$ which is trivial in the case $\varphi^m(p) = 0$. Assume $\varphi^m(p) > 0$. Let $C' \in \Omega_{pq}(\varphi^m(p))$ be a perturbing matrix with rows C'_i, $i \in N_m$. By the definition of the number $\varphi^m(p)$ and according to (1), for any solution $x \in X \setminus P^m(C)$, there exists an effective solution $x^0 \in P^m(x, C)$ such that

$$\frac{C_i(x - x^0)}{\|x - x^0\|_{p'}} \geq \varphi^m(p) > \|C'\|_{pq} \geq \|C'_i\|_p, \quad i \in N_m.$$

Whence, using the Hölder inequality (3) we find

$$(C_i + C'_i)(x - x^0) \geq C_i(x - x^0) - \|C'_i\|_p \|x - x^0\|_{p'} > 0, \quad i \in N_m.$$

Thus, $x \notin P^m(C + C')$. Therefore, every ineffective solution of the problem $Z^m(C)$ retains its ineffectiveness in perturbed problem $Z^m(C + C')$. Hence $P^m(C + C') \subseteq P^m(C)$ for every perturbing matrix $C' \in \Omega_{pq}(\varphi^m(p))$, i. e. $\rho^m(p,q) \geq \varphi^m(p)$.

Now, let us prove the inequality $\rho^m(p,q) \leq n^{1/p} m^{1/q} \varphi^m(\infty)$. According to the definition of the number $\varphi^m(\infty)$ there exists a solution $x^0 \in X \setminus P^m(C)$ such that for each solution $x \in P^m(x^0, C)$ there exists an index $k = k(x) \in N_m$ satisfying

$$C_k(x^0 - x) \leq \varphi^m(\infty) \|x^0 - x\|_1. \tag{4}$$

Choose an arbitrary number ε that obeys the condition $\varepsilon > n^{1/p} m^{1/q} \varphi^m(\infty)$ and specify elements of the perturbing matrix $C^0 = [c^0_{ij}] \in \mathbf{R}^{m \times n}$ with rows C^0_i, $i \in N_m$, as follows

$$c^0_{ij} = \begin{cases} -\delta & \text{if } i \in N_m,\ x^0_j = 1, \\ \delta & \text{if } i \in N_m,\ x^0_j = 0, \end{cases}$$

where $\varphi^m(\infty) < \delta < \varepsilon / n^{1/p} m^{1/q}$. Using (2) we derive

$$\|C^0_i\|_p = \delta n^{1/p}, \quad i \in N_m,$$

$$\|C^0\|_{pq} = \delta n^{1/p} m^{1/q}, \quad C^0 \in \Omega_{pq}(\varepsilon),$$
$$C^0_i(x^0 - x) = -\delta \|x^0 - x\|_1 < 0, \quad i \in N_m. \tag{5}$$

Therefore, taking into account inequality (4) we obtain

$$(C_k + C^0_k)(x^0 - x) = C_k(x^0 - x) + C^0_k(x^0 - x) \leq (\varphi^m(\infty) - \delta) \|x^0 - x\|_1 < 0.$$

As a result we have

$$\forall x \in P^m(x^0, C) \quad \big(x \notin X(x^0, C + C^0)\big). \tag{6}$$

If $X(x^0, C + C^0) = \emptyset$, then $x^0 \in P^m(C + C^0)$. Assume $X(x^0, C + C^0) \neq \emptyset$. In this case, due to external stability of the Pareto set $P^m(C + C^0)$ (see e. g. p. 34 in [10]) there exists a solution $x^* \in P^m(x^0, C + C^0)$. Let us prove that $x^* \notin P^m(C)$. Assume to the contrary that $x^* \in P^m(C)$. According to (6) this yields $x^* \in P^m(C) \setminus P^m(x^0, C)$. Therefore, there are only two cases: the equality

$Cx^* = Cx^0$ holds or the inequality $Cx^* \leq Cx^0$ does not hold. In the first case, taking into account (5) we have

$$(C_i + C_i^0)(x^0 - x^*) < 0, \quad i \in N_m.$$

In the second case, there exists an index $l \in N_m$ such that $C_l x^* > C_l x^0$. Taking into account (5) again we obtain

$$(C_l + C_l^0)(x^0 - x^*) < 0.$$

In both cases we obtained contradictions with the inclusion $x^* \in P^m(x^0, C + C^0)$.

Summarizing the above we state that for any number $\varepsilon > n^{1/p} m^{1/q} \varphi^m(\infty)$ there exist perturbing matrix $C^0 \in \Omega_{pq}(\varepsilon)$ and an inefficient solution (x^0 or x^*) of the problem $Z^m(C)$ such that it becomes efficient in perturbed problem $Z^m(C + C^0)$. Hence

$$\forall \varepsilon > n^{1/p} m^{1/q} \varphi^m(\infty)\ \exists C^0 \in \Omega_{pq}(\varepsilon) \ \ \big(P^m(C + C^0) \not\subseteq P^m(C)\big),$$

i. e.

$$\rho^m(p, q) \leq n^{1/p} m^{1/q} \varphi^m(\infty).$$

It remains to prove that $\rho^m(p, q) \leq \sigma^m(p)$. Let x^0 be an inefficient solution of the problem $Z^m(C)$ and the index $k \in N_m$ be such that

$$\sigma^m(p) = \|C_k\|_p. \tag{7}$$

Assuming that $\varepsilon > \sigma^m(p)$ we denote a number δ with the conditions

$$0 < \delta n^{1/p} < \varepsilon - \sigma^m(p). \tag{8}$$

We define the vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ by

$$\eta_j = \begin{cases} -\delta & \text{if } x_j^0 = 1, \\ \delta & \text{if } x_j^0 = 0. \end{cases}$$

Then we have

$$\|\eta\|_p = \delta n^{1/p} \tag{9}$$

and for each solution $x \in X \setminus \{x^0\}$ we obtain

$$\eta(x^0 - x) = -\delta \|x^0 - x\|_1 < 0. \tag{10}$$

We specify the rows $C_i^0 \in \mathbf{R}^n$, $i \in N_m$, of the perturbing matrix $C^0 \in \mathbf{R}^{m \times n}$ by the rule

$$C_i^0 = \begin{cases} \eta - C_i & \text{if } i = k, \\ \mathbf{0} & \text{if } i \in N_m \setminus \{k\}. \end{cases}$$

Hence, taking into account (10) we derive

$$C_k^0(x^0 - x) = (\eta - C_k)(x^0 - x) = -\delta \|x^0 - x\|_1 - C_k(x^0 - x).$$

It follows from equalities (7) and (9) and inequality (8) that

$$\|C^0\|_{pq} = \|C_k^0\|_p = \|\eta - C_k\|_p \leq \|\eta\|_p + \|C_k\|_p = \delta n^{1/p} + \sigma^m(p) < \varepsilon.$$

Consequently, for each solution $x \in X \setminus \{x^0\}$ we deduce

$$(C_k + C_k^0)(x^0 - x) = -\delta\|x^0 - x\|_1 < 0,$$

i.e. $x \notin X(x^0, C + C^0)$, where $C^0 \in \Omega_{pq}(\varepsilon)$. Using $x^0 \notin X(x^0, C + C^0)$ we get $X(x^0, C + C^0) = \emptyset$, which implies $x^0 \in P^m(C + C^0)$. Due to $x^0 \notin P^m(C)$ the inequality $\rho^m(p, q) \leq \varepsilon$ is true for any number $\varepsilon > \sigma^m(p)$. Thus we have proved that $\rho^m(p, q) \leq \sigma^m(p)$. This with proved inequality $\rho^m(p, q) \leq n^{1/p}m^{1/q}\varphi^m(\infty)$ implies $\rho^m(p, q) \leq \psi^m(p, q)$. $\square$

4 Corollaries

As corollaries of Theorem 1 we obtain the following results.

Corollary 1 [11]. $\varphi^m(p) \leq \rho^m(p, p) \leq (nm)^{1/p}\varphi^m(\infty)$.

Corollary 2 [6]. $\rho^m(\infty, \infty) = \varphi^m(\infty) = \min\limits_{x \in X \setminus P^m(C)} \max\limits_{x' \in P^m(x,C)} \min\limits_{i \in N_m} \dfrac{C_i(x - x')}{\|x - x'\|_1}$.

Corollary 3 [12]. $\varphi^m(p) \leq \rho^m(p, \infty) \leq n^{1/p}\varphi^m(\infty)$.

Corollary 4 [13]. $\varphi^m(\infty) \leq \rho^m(\infty, q) \leq m^{1/q}\varphi^m(\infty)$.

Note that the paper [13] describes a class of problems $Z^m(C)$ for which the following formula holds

$$\rho^m(\infty, q) = m^{1/q}\varphi^m(\infty), \quad q \in [1, \infty].$$

This means that the upper-bound of Corollary 4 is achievable.

The following known result proves that the lower-bound estimate of the stability radius is achievable.

Theorem 2 [9]. *If $|P^m(C)| = 1$, then for any numbers $p, q \in [1, \infty]$ the stability radius is expressed by the formula*

$$\rho^m(p, q) = \varphi^m(p).$$

We denote the stability radius of scalar problem $Z^1(C)$

$$Cx \to \min_{x \in X}, \quad C \in \mathbf{R}^{1 \times n}, \quad X \subseteq \mathbf{E}^n,$$

by $\rho^1(p)$, $p \in [1, \infty]$.

Corollary 5. $\varphi^1(p) \leq \rho^1(p) \leq n^{1/p}\varphi^1(\infty)$.

The paper [12] describes a class of scalar linear problems $Z^1(C)$ for which the following formula holds

$$\rho^1(p) = n^{1/p}\varphi^1(\infty), \quad p \in [1, \infty].$$

Therefore, the upper-bound of Corollary 5 is achievable.

Corollaries 2 and 5 imply the following known result.

Corollary 6 [14, 15]. $\rho^1(\infty) = \varphi^1(\infty)$.

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