Relation between Levinson center, chain recurrent set and center of Birkhoff for compact dissipative dynamical systems

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Abstract. In this paper we prove the analogues of Birkhoff’s theorem for one-sided dynamical systems (both with continuous and discrete times) with noncompact space having a compact global attractor. The relation between Levinson center, chain recurrent set and center of Birkhoff is established for compact dissipative dynamical systems.

Keywords and phrases: Global attractors; Birkhoff’s center; chain recurrent set.

1 Introduction

Let $X$ be a compact metric space, $(X,\mathbb{R},\pi)$ be a flow on $X$, $M \subseteq X$ be a nonempty compact and invariant subset of $X$. Denote $\Omega(M) := \{ x \in M : \text{there exist } \{x_n\} \subseteq M \text{ and } \{t_n\} \subseteq \mathbb{R} \text{ such that } x_n \to x, \ t_n \to +\infty \text{ as } n \to \infty \text{ and } \pi(t_n, x_n) \to x \}$. Recall that the point $x \in X$ is called Poisson stable if $x \in \omega_x \cap \alpha_x$, where by $\omega_x$ (respectively, $\alpha_x$) the $\omega$ (respectively, $\alpha$)-limits set of $x$ is denoted. The following result is well known (see, for example, [1, 14]).

Theorem 1 (Birkhoff’s theorem). The following statements hold:

1. there exists a nonempty, compact and invariant subset $\mathcal{B}(\pi) \subseteq X$ with the properties:
   
   (i) $\Omega(\mathcal{B}(\pi)) = \mathcal{B}(\pi)$;
   
   (ii) $\mathcal{B}(\pi)$ is the maximal compact invariant subset of $J$ with the property (i).

2. $\mathcal{B}(\pi) = \overline{\mathcal{P}(\pi)}$, i.e., the set of all Poisson stable points $\mathcal{P}(\pi)$ of the dynamical system $(X,\mathbb{R},\pi)$ is dense in $\mathcal{B}(\pi)$.

Remark 1. 1. The set $\mathcal{B}(\pi)$ is called the Bikhoff center of dynamical system $(X,\mathbb{R},\pi)$.

2. Note that Birkhoff theorem remains true also for the discrete dynamical systems $(X,\mathbb{Z},\pi)$. This fact was established in the work of V. S. Bondarchuk and V. A. Dobrynisky [1].

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3. The second statement of Theorem 1 remains true if we replace the center of Birkhoff $\mathfrak{B}(\pi)$ by arbitrary compact invariant set $M \subseteq J$ with the property $\Omega(M) = M$. Namely the following equality takes place: $M = \overline{P(\pi)} \cap M$.

The main result of this paper is the proof of the analogues of Birkhoff theorem for the one-sided dynamical systems (both with continuous and discrete times) with noncompact phase space having a compact global attractor.

2 Birkhoff center

**Definition 1.** A dynamical system $(X, T, \pi)$ is said to be:

1. pointwise dissipative if there exists a nonempty compact subset $K \subseteq X$ such that
   \[
   \lim_{t \to +\infty} \rho(\pi(t, x), K) = 0
   \] (1)
   for all $x \in X$;

2. compactly dissipative if there exists a nonempty compact subset $K \subseteq X$ such that (1) holds uniformly with respect to $x$ on every compact subset from $X$.

**Remark 2.** Every compact dissipative dynamical system is pointwise dissipative. The converse, generally speaking, is not true (see, for example, [4, Ch.I]).

**Theorem 2** (see [4, Ch.I]). Suppose that $(X, T, \pi)$ is a compact dissipative dynamical system, then there exists a nonempty, compact, invariant subset $J \subseteq X$ possessing the following properties:

1. $J$ attracts every compact subset $A$ from $X$, i.e.,
   \[
   \lim_{t \to +\infty} \rho(\pi(t, x), J) = 0
   \]
   uniformly with respect to $x \in A$;

2. $J$ is orbitally stable, i.e., for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\rho(\pi(t, x), J) < \varepsilon$ for all $t \geq 0$;

3. $J$ is the maximal compact invariant subset of $X$.

Let $M$ be a positively invariant and closed subset of $X$. Denote by $J^+_x(M) := \{ p \in X : \text{there exist } \{x_n\} \subseteq M \text{ and } t_n \to +\infty \text{ such that } x_n \to x \text{ and } \pi(t_n, x_n) \to p \text{ as } n \to +\infty \}$. 

**Lemma 1.** Let $M$ be a positively invariant and closed subset of $X$. If $p_n \to p$, $x_n \to x$ as $n \to \infty$ and $p_n \in J^+_x(M)$, then $p \in J^+_x(M)$. 
Proof. Let \( \varepsilon \) be an arbitrary positive number, \( p_n \to p \) and \( x_n \to x \) as \( n \to \infty \). Then there exists a number \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) such that

\[
\rho(p_n, p) < \varepsilon/3 \quad \text{and} \quad \rho(x_n, x) < \varepsilon/3
\]

for all \( n \geq n_0 \). Since \( p_n \in J^+_x(M) \) for all \( n \in \mathbb{N} \), then there exist \( \{x^n_m\} \subseteq M \) and \( \{t^m_n\} \) (for all \( m \in \mathbb{N} \)) such that \( x^n_m \to x_n, t^m_n \to +\infty \) and \( \pi(t^m_n, x^n_m) \to p_n \) as \( m \to \infty \). In particular, for given \( \varepsilon \) there exists \( n < m_n = m_n(\varepsilon) \in \mathbb{N} \) such that

\[
\rho(x^n_m, x_n) < \varepsilon/3 \quad \text{and} \quad \rho(\pi(t^m_n, x^n_m), p_n) < \varepsilon/3
\]

for all \( m \geq m_n \). Denote by \( \bar{x}_n := x^n_{m_n} \) and \( \bar{t}_n := t^m_n > n \). Note that \( \{\bar{x}_n\} \subseteq M \), \( \bar{t}_n \to +\infty \) as \( n \to \infty \) and

\[
\rho(\bar{x}_n, x) = \rho(x^n_{m_n}, x) \leq \rho(x^n_{m_n}, x_n) + \rho(x_n, x) < \varepsilon/3 + \varepsilon/3 < \varepsilon
\]

for all \( n \geq n_0(\varepsilon) \), i.e., \( \bar{x}_n \to x \) as \( n \to \infty \). In addition we have

\[
\rho(\pi(\bar{t}_n, \bar{x}_n), p) = \rho(\pi(t^n_{m_n}, x^n_{m_n}), p) \leq \rho(\pi(t^n_{m_n}, x^n_{m_n}), p_n) + \rho(p_n, p) < \varepsilon/3 + \varepsilon/3 < \varepsilon
\]

for all \( n \geq n_0 \). Thus for the point \( p \) we find the sequence \( \{\bar{x}_n\} \subseteq M \) and \( \bar{x}_n \to +\infty \) as \( n \to \infty \) such that \( \bar{x}_n \to x \) and \( \pi(\bar{t}_n, \bar{x}_n) \to p \) as \( n \to \infty \), i.e., \( p \in J^+_x(M) \). Lemma is proved. \( \square \)

**Lemma 2.** Let \( M \) be a positively invariant and closed subset of \( X \) and \( x \in X \). The following statements hold:

1. \( J^+_x(M) \subseteq M \) for all \( x \in M \);
2. the set \( J^+_x(M) \) is closed and positively invariant;
3. if \( M \) is compact, then \( J^+_x(M) \) is invariant.

**Proof.** Let \( p \in J^+_x(M) \) and \( t \in \mathbb{T} \), then there are \( \{x_n\} \) and \( t_n \to +\infty \) such that \( x_n \to x \) and \( \pi(t_n, x_n) \to p \) as \( n \to \infty \). Then we have \( \pi(t, p) = \lim_{n \to \infty} \pi(t_n, x_n) = \lim_{n \to \infty} \pi(t + t_n, x_n) \) and, consequently, \( \pi(t, p) \in J^+_x(M) \) because \( x_n \in M \) and \( M \) is closed and positively invariant. Finally, it is evident that \( J^+_x(M) \subseteq M \) for all \( x \in M \).

Now we will establish the second statement of Lemma. Let \( \{p_n\} \) be a sequence from \( J^+_x(M) \) such that \( p_n \to p \) as \( n \to \infty \), then \( p_n \in J^+_x(M) \) where \( x_n := x \) for all \( n \in \mathbb{N} \). By Lemma 1 \( p \in J^+_x(M) \) because \( p_n \to p \) and \( x_n \to x \) as \( n \to \infty \). Let us show now that the set \( J^+_x(M) \) is positively invariant. Indeed, let \( t \in \mathbb{T} \) and \( p \in J^+_x(M) \), then there are \( \{x_n\} \subseteq M \) and \( t_n \to +\infty \) as \( n \to \infty \) such that \( \pi(t_n, x_n) \to p \) as \( n \to \infty \). Note that \( \pi(t, p) = \lim_{n \to \infty} \pi(t + t_n, x_n) \) and, consequently, \( \pi(t, p) \in J^+_x(M) \).

Suppose that the set \( M \) is compact and \( p \in J^+_x(M) \), then there are \( \{x_n\} \subseteq M \) and \( t_n \to +\infty \) as \( n \to \infty \) such that \( \pi(t_n, x_n) \to p \) as \( n \to \infty \). Let \( t \in \mathbb{T} \) be an arbitrary number, then for sufficiently large \( n \in \mathbb{N} \) we have \( t_n - t \in \mathbb{T} \) because \( t_n \to +\infty \) as \( n \to \infty \). Since the set \( M \) is positively invariant and compact, then without loss of
generality we can suppose that the sequence \( \{\pi(t_n - t, x_n)\} \) is convergent. Denote by \( p_1 \) its limit, then we obtain \( p = \lim_{n \to \infty} \pi(t_n - t + t, x_n) = \lim_{n \to \infty} \pi(t_n - t, x_n) = \pi(t, p_1) \) and, consequently, \( p \in \pi(t, J^+_{\pi}(M)) \), i.e., \( J^+_{\pi}(M) \subseteq \pi(t, J^+_{\pi}(M)) \) for all \( t \in \mathbb{T} \). Thus \( J^+_{\pi}(M) \) is positively and negatively invariant, i.e., it is invariant.

**Definition 2.** Let \( M \) be a subset of \( X \). A point \( x \in X \) is said to be non-wandering with respect to \( M \) if \( x \in J^+_{\pi}(M) \).

Denote by \( \Omega(M) := \{x \in M : x \in J^+_{\pi}(M)\} \) the set of all non-wandering points of \( M \) with respect to \( M \).

**Remark 3.** Let \( A \) and \( B \) be two closed and positively invariant subsets of \( X \), then \( \Omega(A) \subseteq \Omega(B) \).

**Definition 3.** A point \( p \in X \) is said to be:

- Poisson stable in the positive direction if \( x \in \omega_{\pi} \);
- Poisson stable in the negative direction if there exists an entire trajectory \( \gamma_p \in \Phi_{\pi} \) such that \( x \in \alpha_{\gamma_p} \), where \( \alpha_{\gamma_p} := \{q \in X : \text{there exists } t_n \to -\infty \text{ such that } \gamma_p(t_n) \to q \text{ as } n \to \infty\} \);
- Poisson stable if it is Poisson stable in the both directions.

**Lemma 3.** Let \( M \) be a nonempty, closed and positively invariant set, then the following statements hold:

1. the set \( \Omega(M) \) is closed;
2. if \( p \in M \) is Poisson stable in the positive direction, then \( p \in \Omega(M) \);
3. if the point \( p \in M \) and \( \gamma \in \Phi_p \) is an entire trajectory such that \( \gamma(S) \subseteq M \) and \( p \in \alpha_{\gamma} \), then \( p \in \Omega(M) \).

**Proof.** The first statement directly follows from Lemma 1 and definition of \( \Omega(M) \).

Let \( p \in M \) and \( p \in \omega_{\pi} \); then there exists a sequence \( t_n \to +\infty \) such that \( \pi(t_n, p) \to p \) as \( n \to \infty \). Let \( p_n := p \) for all \( n \in \mathbb{N} \), then \( p_n \to p \) and \( \pi(t_n, p_n) \to p \) as \( n \to \infty \). This means that \( p \in J^+_{\pi}(M) \), i.e., \( p \in \Omega(M) \).

Let \( p \in M \), \( \gamma \in \Phi_p \), \( \gamma(S) \subseteq M \) and \( p \in \alpha_{\gamma} \). Then there exists a sequence \( t_n \to +\infty \) such that \( \gamma(-t_n) \to p \) as \( n \to \infty \). Denote by \( p_n := \gamma(-t_n) \), then \( p_n \to p \) and \( p = \pi(t_n, p_n) \to p \) as \( n \to \infty \). Thus \( p \in J^+_{\pi}(M) \) and, consequently, \( p \in \Omega(M) \).

**Lemma 4.** Suppose that \( M \) is a nonempty, compact positively invariant set and \( \mathcal{M} \) is a nonempty, compact minimal subset of \( M \), then \( \mathcal{M} \subseteq \Omega(M) \).

**Proof.** Let \( p \in \mathcal{M} \) and \( \gamma \in \Phi_p \) be an entire trajectory of \((X, \mathbb{T}, \pi)\) passing through \( p \) at the initial moment such that \( \gamma(S) \subseteq M \). Since \( \mathcal{M} \) is minimal, \( \omega_{\pi} \) and \( \alpha_{\gamma} \) are nonempty, compact and invariant we have \( \alpha_{\gamma} = \omega_{\pi} = \mathcal{M} \). In particular there exists a sequence \( \tau_n \to +\infty \) such that \( p_n := \gamma(-\tau_n) \to p \) as \( n \to \infty \). Note that \( \pi(\tau_n, p_n) = p \) for all \( n \in \mathbb{N} \) and, consequently, \( p \in \Omega(\mathcal{M}) \subseteq \Omega(M) \).
Corollary 1. If $M$ is a nonempty, compact positively invariant set, then $\Omega(M) \neq \emptyset$.

Proof. Let $M$ be a nonempty, compact and positively invariant set of $(X, \mathbb{T}, \pi)$. By Birkhoff theorem there exists a nonempty minimal subset $\mathcal{M} \subseteq M$ and by Lemma 4 we have $\mathcal{M} \subseteq \Omega(M)$.

Denote by $\Phi_x$ the set of all entire trajectories $\gamma_x$ of $(X, \mathbb{T}, \pi)$ passing through the point $x$ at the initial moment $t = 0$.

Lemma 5. Suppose that $M$ is a nonempty, compact and positively invariant set. Then $\Omega(M)$ is a nonempty, compact and positively invariant subset of $M$.

Proof. By Corollary 1 the set $\Omega(M)$ is a nonempty subset. By Lemma 1 the set $\Omega(M)$ is closed. Since $\Omega(M) \subseteq M$ and $M$ is compact, then $\Omega(M)$ is so. Let now $p \in \Omega(M)$ and $t \in \mathbb{T}$, then there are $p_n \to p$ ($p_n \in M$) and $t_n \to +\infty$ as $n \to \infty$ such that $p = \lim_{n \to \infty} \pi(t_n, p_n)$. Note that $\pi(t, p) = \lim_{n \to \infty} \pi(t, \pi(t_n, p_n)) = \lim_{n \to \infty} \pi(t_n, \pi(t, p_n))$ and, consequently, $\pi(t, p) \in J^+_p(M)$ because $\lim_{n \to \infty} \pi(t_n, p_n) = \pi(t, p)$ and $\{\pi(t_n, p_n)\} \subseteq M$. This means that $\pi(t, p) \in \Omega(M)$, i.e., $\Omega(M)$ is positively invariant.

Lemma 6. Let $M$ be a nonempty positively invariant subset of $X$, then the following statements hold:

1. if $(X, \mathbb{T}, \pi)$ is pointwise dissipative, then $\Omega(M)$ is nonempty, closed and positively invariant;

2. if the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative and $J$ is its Levinson center, then the set $\Omega(M)$ is nonempty, compact, positively invariant and $\Omega(M) \subseteq J$;

3. if the dynamically system $(X, \mathbb{T}, \pi)$ is point dissipative (but not compactly dissipative), then the set $\Omega(X)$, generally speaking, is not compact.

Proof. Since $(X, \mathbb{T}, \pi)$ is pointwise dissipative, then $\Omega_M := \bigcup\{\omega_x : x \in M\} \subseteq X$ is a nonempty compact invariant subset of $(X, \mathbb{T}, \pi)$ and by Birkhoff’s theorem in $\Omega_M$ there exists at least one compact minimal subset $\mathcal{M} \subseteq \Omega \subseteq X$. By Corollary 1 $\Omega(M) \neq \emptyset$. Let us show that $\Omega(M)$ is closed. If $p = \lim_{n \to \infty} p_n$ and $p_n \in \Omega(M)$, then $p_n \in J^+_p(M)$. By Lemma 1 we have $p \in J^+_p(M)$, i.e., $p \in \Omega(M)$. If $p \in \Omega(M)$ and $t \in \mathbb{T}$, then there are $p_n \in M$ and $t_n \to +\infty$ such that $p = \lim_{n \to \infty} \pi(t_n, p_n)$ and, consequently, $\pi(t, p) = \lim_{n \to \infty} \pi(t, \pi(t_n, p_n)) = \lim_{n \to \infty} \pi(t_n, \pi(t, p_n))$, i.e., $\pi(t, p) \in J^+_{\pi(t, p)}(M)$ because $\lim_{n \to \infty} \pi(t_n, p_n) = \pi(t, p)$. This means that $\pi(t, p) \in \Omega(M)$, i.e., $\Omega(M)$ is positively invariant.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and $x \in \Omega(M)$, then there exist $\{x_n\} \subseteq M$ and $t_n \to +\infty$ such that $x_n \to x$ and $\pi(t_n, x_n) \to x$ as $n \to \infty$. Denote $K_0 := \{x_n\}$, where by bar the closure in $X$ is denoted. Then we have

$$\rho(\pi(t_n, x_n), J) \leq \sup_{p \in K_0} \rho(\pi(t_n, p), J), \quad (2)$$
where $J$ is Levinson center of $(X, T, \pi)$. Passing to limit in (2) we obtain $x \in J$. By the first item the set $\Omega(X)$ is nonempty, compact and positively invariant.

To prove the third item it is sufficient to construct an example with the corresponding properties. To this end we note that in the works [5] and [8] a dynamical system $(X, T, \pi)$ with the following properties was constructed:

1. $(X, T, \pi)$ is point dissipative, but it is not compactly dissipative;
2. $\Omega(X)$ is an unbounded set and, consequently, it is not compact.

Lemma is proved.

Let $(X, T, \pi)$ be a compact dissipative dynamical system and $J$ be its Levinson center and $M \subseteq X$ be a nonempty, closed and positively invariant subset from $X$. Denote by $M_1 := \Omega(M)$ the set of all non-wandering (with respect to $M$) points of $(X, T, \pi)$. By Lemma 6 the set $M_1$ is a nonempty, compact and positively invariant subset of $J$. We denote by $M_2 := \Omega(M_1) \subseteq M_1$ the set of all non-wandering (with respect to $M_1$) points. By Corollary 1 and Lemma 5 the set $M_2$ is nonempty, compact and positively invariant. Analogously we define the set $M_3 := \Omega(M_2) \subseteq M_2$ which is also a nonempty, compact and positively invariant set. We can continue this process and we will obtain $M_n := \Omega(M_{n-1})$ for all $n \in \mathbb{N}$. Thus we have a sequence $\{M_n\}_{n \in \mathbb{N}}$ possessing the following properties:

1. for all $n \in \mathbb{N}$ the set $M_n$ is nonempty, compact and positively invariant;
2. $J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$.

Denote by $M_\lambda := \bigcap_{n=1}^{\infty} M_n$, then $M_\lambda$ is a nonempty, compact (since the set $J$ is compact) and invariant subset of $J$. Now we define the set $M_{\lambda+1} := \Omega(M_\lambda)$ and we can continue this process to obtain the following sequence

$$J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq M_{\lambda+1} \supseteq \ldots \supseteq M_{\lambda+k} \supseteq \ldots.$$ 

Now construct the set $M_\mu := \bigcap_{k=1}^{\infty} M_{\mu+k}$ and we denote by $M_{\mu+1} := \Omega(M_\mu)$ and so on. Thus we will obtain a transfinite sequence of nonempty, compact and positively invariant subsets

$$J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\mu \supseteq \ldots.$$ 

Since $J$ is a nonempty compact set, then in the sequence (3) there is at most a countable family of different elements, i.e., there exists a $\gamma$ such that $M_{\mu+1} = M_\mu$.

**Definition 4.** The set $\mathfrak{B}(M) := M_\mu$ is said to be the center of Birkhoff for the closed and positively invariant set $M$. If $M = X$, then the set $\mathfrak{B}(\pi) := \mathfrak{B}(X)$ is said to be the Birkhoff center of compact dissipative dynamical system $(X, T, \pi)$. 

Lemma 7. Let \((X, T, \pi)\) be a compact dissipative dynamical system and \(\mathcal{B}(\pi)\) be its Birkhoff center. Then the following statements hold:

1. \(\mathcal{B}(\pi)\) is a nonempty, compact and invariant set;
2. \(\mathcal{B}(\pi)\) is a maximal compact invariant subset \(M\) of \(X\) such that \(\Omega(M) = M\).

Proof. By Lemma 6 \(\mathcal{B}(\pi)\) is a nonempty, compact and positively invariant set. To finish the proof of the first statement it is sufficient to establish that the set \(\mathcal{B}(\pi)\) is negatively invariant, i.e., \(\mathcal{B}(\pi) \subset \pi(t, \mathcal{B}(\pi))\) for all \(t \in T\). To this end it is sufficient to show that for all \(x \in \mathcal{B}(\pi)\) the set of all entire trajectories \(\gamma_x\) of \((X, T, \pi)\) passing through the point \(x\) at the initial moment with the condition \(\gamma_x(0) \subset \mathcal{B}(\pi)\) is nonempty. Let \(x \in \mathcal{B}(\pi)\). Since \(\Omega(\mathcal{B}(\pi)) = \mathcal{B}(\pi)\), then there are \(\{x_n\} \subset \mathcal{B}(\pi)\) and \(\{\tau_n\} \subset T\) such that \(x_n \to x\), \(\tau_n \to +\infty\) and \(\pi(\tau_n, x_n) \to x\). Denote by \(\gamma_n\) the function from \(C(S, \mathcal{B}(\pi))\) defined by the equality \(\gamma_n(t) = \pi(t + \tau_n, x_n)\) for all \(t \geq -\tau_n\) and \(\gamma_n(t) = x_n\) for all \(t \leq -\tau_n\). We will show that the sequence \(\{\gamma_n\}\) is relatively compact in \(C(S, \mathcal{B}(\pi))\). Let \(l > 0\). Since the set \(\mathcal{B}(\pi)\) is compact, then it is sufficient to check that the sequence \(\{\gamma_n\}\) is equi-continuous on the interval \([-l, l]\).

If we suppose that it is not true then there exist \(\varepsilon_0 > 0\), \(\delta_n \to 0\) and \(t^1_n, t^2_n \in [-l, l]\) such that

\[
|t^1_n - t^2_n| < \delta_n \quad \text{and} \quad \rho(\gamma_n(t^1_n), \gamma_n(t^2_n)) \geq \varepsilon_0
\]  

(4)

for all \(n \in \mathbb{N}\). Without loss of generality we may consider that the sequence \(\{\gamma_n(-l)\}\) is convergent and denote its limit by \(\bar{x}\). From inequality (4) we have

\[
\varepsilon_0 \leq \rho(\gamma_n(t^1_n), \gamma_n(t^2_n)) = \rho(\pi(l + t^1_n, \gamma_n(-l)), \pi(l + t^2_n, \gamma_n(-l)))).
\]  

(5)

Passing to limit in inequality (5) as \(n \to \infty\) and taking into consideration (4), we obtain \(\varepsilon_0 \leq \rho(\pi(l + \bar{t}, \bar{x}), \pi(l + \bar{t}, \bar{x})) = 0\), where \(\bar{t} := \lim_{n \to \infty} t^1_n = \lim_{n \to \infty} t^2_n\). The obtained contradiction proves our statement. Thus the sequence \(\{\gamma_n\}\) is equi-continuous on \([-l, l]\) and the set \(\cup_{n=1}^{\infty} \gamma_n([-l, l]) \subset \mathcal{B}(\pi)\) is relatively compact. Taking into account that \(l\) is an arbitrary positive number we conclude that the sequence \(\{\gamma_n\}\) is relatively compact in \(C(S, \mathcal{B}(\pi))\). We may suppose that the sequence \(\{\gamma_n\}\) is convergent. Denote by \(\gamma := \lim_{n \to \infty} \gamma_n\), then \(\gamma(0) = x := \lim_{n \to \infty} \pi(\tau_n, x_n)\) and \(\gamma \in \mathcal{B}(\pi)\) such that \(\gamma(S) \subset \mathcal{B}(\pi) = \Omega(\mathcal{B}(\pi))\), because by construction \(\gamma_n(S) \subset \mathcal{B}(\pi)\) for all \(n \in \mathbb{N}\).

Let now \(M \subset X\) be an arbitrary nonempty, compact and invariant subset of \(X\) with the property \(\Omega(M) = M\). Then by construction of \(\mathcal{B}(M)\) we have \(\mathcal{B}(M) = M\). On the other hand \(M \subset J\), where \(J\) is the Levinson center of the compact dissipative dynamical system \((X, T, \pi)\) and, consequently, \(\mathcal{B}(M) \subset \mathcal{B}(X) = \mathcal{B}(\pi)\). Lemma is completely proved. \(\Box\)
Definition 5. Recall that the mapping \( f : X \mapsto X \) is said to be open if for all \( p \in X \) and \( \delta > 0 \) the set \( f(B(p, \delta)) \) is open.

Let \( p \in \mathcal{B}(\pi) \) and \( \varepsilon > 0 \). Denote by \( \tilde{B}(p, \varepsilon) := B(p, \varepsilon) \cap \mathcal{B}(\pi) \).

Lemma 8. Let \((X, \mathbb{T}, \pi)\) be a compact dissipative dynamical system and \( \mathcal{B}(\pi) \) be its Birkhoff center. Then the following statements hold:

1. for all \( p \in \mathcal{B}(\pi), \varepsilon > 0 \) and \( t_0 \in \mathbb{T} \) there exists a number \( t = t(p, \varepsilon, t_0) > t_0 \) such that \( \pi(t, \tilde{B}(p, \varepsilon)) \cap \tilde{B}(p, \varepsilon) \neq \emptyset \);

2. for all \( \varepsilon > 0, L > 0 \) and \( p \in \mathcal{B}(\pi) \) there are \( q \in \tilde{B}(p, \varepsilon) \), \( \delta = \delta(L, \varepsilon) > 0 \) and \( t > L \) such that
\[
\tilde{B}(q, \delta) \cup \pi(t, \tilde{B}(q, \delta)) \subset \tilde{B}(p, \varepsilon).
\]

Proof. Suppose that under the conditions of Lemma the first statement is not true. Then there exist \( p_0 \in \mathcal{B}(\pi), \varepsilon_0 > 0 \) and \( t_0 \in \mathbb{T} \) such that
\[
\pi(t, \tilde{B}(p_0, \varepsilon_0)) \cap \tilde{B}(p_0, \varepsilon_0) = \emptyset \tag{6}
\]
for all \( t \geq t_0 \). On the other hand since \( p_0 \in \mathcal{B}(\pi) \), then there exist \( \{p_n\} \subseteq \mathcal{B}(\pi) \) and \( t_n \to +\infty \) such that \( \pi(t_n, p_n) \to p \) as \( n \to \infty \) and, consequently,
\[
\pi(t_n, \tilde{B}(p_n, \varepsilon_0)) \cap \tilde{B}(p_0, \varepsilon_0) \neq \emptyset \tag{7}
\]
for all \( n \in \mathbb{N} \). Conditions (6) and (7) are contradictory. The obtained contradiction proves our statement.

Now we will establish the second statement. Let \( \varepsilon > 0, L > 0 \) and \( p \in \mathcal{B}(\pi) \). Since \( p \in J_J^+(\mathcal{B}(\pi)) \), then there are \( q \in \tilde{B}(p, \varepsilon) \) and \( t > L \) such that \( \pi(t, q) \in \tilde{B}(p, \varepsilon) \).

Let \( \mu \) be a positive number such that \( \tilde{B}(\pi(t, q), \mu) \subset \tilde{B}(p, \varepsilon) \). By continuity of the map \( \pi(t, \cdot) : \mathcal{B}(\pi) \mapsto \mathcal{B}(\pi) \) there exists a positive number \( \delta = \delta(t, q, \varepsilon) \) such that
\[
\tilde{B}(q, \delta) \subset \tilde{B}(p, \varepsilon) \text{ and } \pi(t, \tilde{B}(q, \delta)) \subset \tilde{B}(\pi(t, q), \mu) \subset \tilde{B}(p, \varepsilon). \tag*{\square}
\]

Lemma 9. Suppose that \((X, \mathbb{T}, \pi)\) is a dynamical system and the following conditions hold:

1. the space \( X \) is compact;

2. \( X \) is an invariant set, i. e., \( \pi(t, X) = X \) for all \( t \in \mathbb{T} \);

3. \( \Omega(X) = X \).

Then for all \( x \in X, \varepsilon > 0 \) and \( l > 0 \) there exists a number \( t > l \) such that
\[
\pi^{-t}B(x, \varepsilon) \cap B(x, \varepsilon) \neq \emptyset.
\]
Proof. Let $x \in X$ and $l, \varepsilon$ be two arbitrary positive numbers. Since $x \in J_\pi^+$, then there are sequences $\{x_n\} \subseteq X$ and $\{t_n\} \subseteq \mathbb{T}$ such that

$$x_n \to x, \ t_n \to +\infty \text{ and } \pi(t_n, x_n) \to x$$

as $n \to \infty$. For the sufficiently large $n \in \mathbb{N}$ we have

$$t_n > l \text{ and } x_n, \pi(t_n, x_n) \in B(x, \varepsilon).$$

Let $\gamma_n \in \Phi_{\pi(t_n, x_n)}$ be a full trajectory of $(X, \mathbb{T}, \pi)$ passing through $\pi(t_n, x_n)$ at the initial moment $t = 0$ such that $\gamma_n(s) = \pi(s + t_n, x_n)$ for all $s \geq -t_n$. Then $\gamma_n(-t_n) = x_n \in B(x, \varepsilon)$ and $x_n = \gamma_n(-t_n) \in \pi^{-t_n}(x_n) \subseteq \pi^{-t_n}B(x, \varepsilon)$. Thus we will have

$$x_n \in \pi^{-t_n}B(x, \varepsilon) \bigcap B(x, \varepsilon) \neq \emptyset$$

for all sufficiently large $n \in \mathbb{N}$. □

Corollary 2. Under the conditions of Lemma 9 for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists $t > 1$ such that $B(x, \varepsilon) \bigcap \pi^tB(x, \varepsilon) \neq \emptyset$.

Proof. By Lemma 9 for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists $t > l$ such that $\pi^{-t}B(x, \varepsilon) \bigcap B(x, \varepsilon) \neq \emptyset$ and, consequently,

$$\pi^{t}B(x, \varepsilon) \bigcap B(x, \varepsilon) \subseteq B(x, \varepsilon) \bigcap \pi^{t}B(x, \varepsilon) \neq \emptyset.$$ □

Corollary 3. Suppose that the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and $\mathfrak{B}(\pi)$ is its Birkhoff's center, then for all $x \in \mathfrak{B}(\pi)$, $\varepsilon > 0$ and $l > 0$ there exists a number $t > l$ such that $\pi^{-t}\overline{B}(x, \varepsilon) \bigcap \overline{B}(x, \varepsilon) \neq \emptyset$.

Proof. This statement directly follows from Lemmas 7 and 9. □

Theorem 3. Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system, for all $t > 0$ the mapping $\tilde{\pi}(t, \cdot) := \pi(t, \cdot)\big|_{\mathfrak{B}(\pi)}$ is open, then the set of all Poisson stable in the positive direction points of $(X, \mathbb{T}, \pi)$ is dense in $\mathfrak{B}(\pi)$, i.e., $\mathfrak{B}(\pi) = \overline{P(\pi)}$.

Proof. By Lemma 3 we have $P(\pi) \subseteq \mathfrak{B}(\pi)$ and, consequently, $P(\pi) \subseteq \mathfrak{B}(\pi)$. To finish the proof of Theorem it is sufficiently to show that $P(\pi) \supseteq \mathfrak{B}(\pi)$.

Let $p \in \mathfrak{B}(\pi)$ and $\varepsilon$ be an arbitrary (sufficient small) positive number. Let $\{t_n\}$ be an increasing sequence such that $t_n \to +\infty$. By Lemma 8 (item 2) there exists $t_1 > t_1$ such that

$$\tilde{B}[x_1, \varepsilon_1] \subseteq \tilde{B}[p, \varepsilon] \text{ and } \pi(t_1, \tilde{B}[x_1, \varepsilon_1]) \subseteq \tilde{B}[p, \varepsilon].$$

Since the mapping $\pi(t_1, \cdot)$ is open, then we can choose $x_1 \in \mathfrak{B}(\pi)$ and $\varepsilon_1 > 0$ such that

$$\tilde{B}[x_1, \varepsilon_1] \subseteq \pi(t_1, \tilde{B}[p, \varepsilon]) \subseteq \tilde{B}[p, \varepsilon].$$
By Lemma 8 there is $t_2 > \tau_2$ such that we will have
\[ \bar{B}[x_2, \varepsilon_2] \subseteq \bar{B}[x_1, \varepsilon_1] \quad \text{and} \quad \pi(t_2, \bar{B}[x_2, \varepsilon_2]) \subseteq \bar{B}[x_1, \varepsilon_1]. \]

Since the mapping $\pi(t_2, \cdot)$ is open we can again choose $x_2 \in \mathcal{B}(\pi)$ and $0 < \varepsilon_2 < \varepsilon_1/2$ such that
\[ \bar{B}[x_3, \varepsilon_3] \subseteq \bar{B}[x_2, \varepsilon_2] \quad \text{and} \quad \pi(t_3, \bar{B}[x_3, \varepsilon_3]) \subseteq \bar{B}[x_2, \varepsilon_2]. \]

Reasoning analogously we can construct sequences \{${x_n}$\} $\subseteq \mathcal{B}(\pi)$ and \{${\varepsilon_n}$\} such that $\varepsilon_n < \varepsilon_{n-1}/2$, $\bar{B}[x_n, \varepsilon_n] \subseteq \bar{B}[x_{n-1}, \varepsilon_{n-1}]$ and $\pi(t_n, \bar{B}[x_n, \varepsilon_n]) \subseteq \bar{B}[x_{n-1}, \varepsilon_{n-1}]$ for all $n \in \mathbb{N}$, where $\varepsilon_0 := \varepsilon$ and $x_0 := p$. Since $\mathcal{B}(\pi)$ is a nonempty compact set, then
\[ \Lambda := \bigcap_{n=0}^{\infty} \bar{B}(x_n, \varepsilon_n) \neq \emptyset \quad \text{and it consists of a unique point. Let } \{x\} = \Lambda. \]
We will show that the point $x$ is Poisson stable in the positive direction. In fact, if $L > 0$ is a sufficiently large number and $\delta > 0$, respectively, sufficiently small number, then we choose a natural number $m \in \mathbb{N}$ with the condition that $t_m > L$ and $\varepsilon_m < \delta$, then $\pi(t_n, \bar{B}[x_n, \varepsilon_n]) \subseteq \bar{B}[x_m, \varepsilon_m] \subseteq \bar{B}[x, \delta]$ for all $n > m$. In particular $\pi(t_n, x) \in \bar{B}[x, \delta]$ for all $n > m$, i.e., $x \in \omega_x$. Thus $x \in \bar{B}(p, \varepsilon)$ and, consequently, $\mathcal{B}(\pi) \subseteq \bar{P}(\pi)$. Theorem is proved.

Remark 4. 1. Note that the mappings $\bar{\pi}(t, \cdot)$ ($t \in \mathbb{T}$) are open, if on $\mathcal{B}(\pi)$ the dynamical system $(X, \mathbb{T}, \pi)$ is invertible, i.e., for all $t \in \mathbb{T}$ the mapping $\bar{\pi}(t, \cdot) : \mathcal{B}(\pi) \to \mathcal{B}(\pi)$ is a homeomorphism.

2. If the dynamical system $(X, \mathbb{T}, \pi)$ is invertible on $\mathcal{B}(\pi)$, then by Theorem 1.14 [14, Ch.III] (see also Proposal 1.1 from [1], where the analogue of Theorem 1.4 for the discrete dynamical systems was proved) in the set $\mathcal{B}(\pi)$ the set of all Poisson stable (both in the positive and negative directions) points from $X$ is dense.

Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system. Recall that a compact set $M \subseteq X$ is called a weak attractor of the dynamical system $(X, \mathbb{T}, \pi)$ if $\omega_x \cap M \neq \emptyset$ for all $x \in X$. In this section we establish the relationship between weak attractors of the dynamical system $(X, \mathbb{T}, \pi)$ and its Levinson center.

**Theorem 4** (see [4, Ch.I]). Let $(X, \mathbb{T}, \pi)$ be compactly dissipative, $J$ be its Levinson center and $M$ be a compact weak attractor of the dynamical system $(X, \mathbb{T}, \pi)$. Then $J = J^+(M)$.

Denote by $J^+_x := \{p \in X : \text{there exist the sequences } x_n \to x \text{ and } t_n \to +\infty \text{ such that } \pi(t_n, x_n) \to p \text{ as } n \to \infty\}$ and $J^+(M) := \bigcup\{J^+_x : x \in M\}$.

**Lemma 10.** Let $M \subseteq X$ be a nonempty, compact, positively invariant and minimal subset of $X$. Then the following statements hold:

1. the set $M$ is invariant, i.e., $\pi(t, M) = M$ for all $t \in \mathbb{T}$;

2. for every $x \in M$ each full trajectory $\gamma \in \Phi_x$ is Poisson stable, i.e., $x \in \omega_x = \alpha\gamma$. 


Proof. Let \( t_0 \in \mathbb{T} \) and \( M := \pi(t_0, M) \), then \( M' \subseteq M \) and \( \pi(t, M') = \pi(t + t_0, M) \subseteq M \). Since \( M \) is a nonempty, compact and positively invariant set, then the set \( M' \) is so. Taking into consideration that \( M \) is a minimal set we conclude that \( M = \pi(t_0, M) \) for all \( t_0 \in \mathbb{T} \) and, consequently, it is invariant.

Let now \( x \in M \) be an arbitrary point from \( M \), then \( \omega_x \) is a nonempty, compact and positively invariant subset of \( M \). Since the set \( M \) is minimal, then we have \( \omega_x = M \). Let now \( \gamma \in \Phi_x \) be an arbitrary full trajectory of \((X, \mathbb{T}, \pi)\) with the properties: \( \gamma(0) = x \) and \( \gamma(S) \subseteq M \), then its \( \alpha \)-limit set \( \alpha_{\gamma} \subseteq M \) is a nonempty and compact subset of \( \omega_x = M \). If \( p \in \alpha_{\gamma} \), then there exists a sequence \( s_n \to -\infty \) such that \( p = \lim_{n\to\infty} \gamma(s_n) \). For all \( t \in \mathbb{T} \) the sequence \( \{ \gamma(t+s_n) \} \subseteq M \) is relatively compact and, consequently, without loss of generality, we may suppose that \( \{ \gamma(t+s_n) \} \) converges. Denote by \( p_t \) its limit, i.e., \( p_t := \lim_{n\to\infty} \gamma(t+s_n) \). Note that

\[
\pi(t, p) = \lim_{n\to\infty} \pi(t, \gamma(s_n)) = \lim_{n\to\infty} \gamma(t+s_n) \in \alpha_{\gamma} \subseteq M
\]

for all \( t \in \mathbb{T} \) and, consequently, \( \omega_p \) is a nonempty, compact, positively invariant subset of \( M \). On the other hand we have \( \omega_p \subseteq \alpha_{\gamma} \subseteq M \). Since the set \( M \) is minimal, then we obtain \( M = \omega_p \subseteq \alpha_{\gamma} \subseteq M \) and, consequently, \( \alpha_{\gamma} = M \). Thus we have \( x \in \omega_x = \alpha_{\gamma} = M \). Lemma is completely proved.

Theorem 5. Let \((X, \mathbb{T}, \pi)\) be a compact dissipative dynamical system, \( J \) be its Levinson center and \( \mathfrak{B}(\pi) \) be the Birkhoff center of \((X, \mathbb{T}, \pi)\). Then the following equality takes place: \( J = J^+(\mathfrak{B}(\pi)) \).

Proof. By Lemmas 3 and 6 we have \( \overline{\mathcal{P}(\pi)} \subseteq \mathfrak{B}(\pi) \subseteq J \) and \( \overline{\mathcal{P}(\pi)} \) is a nonempty and compact subset of \( J \). It is not difficult to show that the set \( \mathcal{P}(\pi) \) is a weak attractor for \((X, \mathbb{T}, \pi)\). In fact, let \( x \in X \) be an arbitrary point of \( X \). Since the dynamical system \((X, \mathbb{T}, \pi)\) is compact dissipative, then the \( \omega \)-limit set \( \omega_x \) of the point \( x \) is a nonempty, compact and positively invariant subset of \( X \). By theorem of Birkhoff in \( \omega_x \) there exists a nonempty, compact, positively invariant and minimal subset \( M \subseteq \omega_x \). By Lemma 10 every point \( p \) from \( M \) is Poisson stable and, consequently, \( M \subseteq \mathcal{P}(\pi) \subseteq \overline{\mathcal{P}(\pi)} \subseteq \mathfrak{B}(\pi) \). Thus we have \( M \subseteq \omega_x \cap \mathfrak{B}(\pi) \) for each \( x \in X \), i.e., \( \mathfrak{B}(\pi) \) is a weak attractor of \((X, \mathbb{T}, \pi)\). Now to finish the proof of Theorem it is sufficient to apply Theorem 4.

3 Chain recurrent motions

Let \( \Sigma \subseteq X \) be a compact positively invariant set, \( \varepsilon > 0 \) and \( t > 0 \).

Definition 6. The collection \( \{ x = x_0, x_1, x_2, \ldots, x_k = y; t_0, t_1, \ldots, t_k \} \) of the points \( x_i \in \Sigma \) and the numbers \( t_i \in \mathbb{T} \) such that \( t_i \geq t \) and \( \rho(x_i x_{i+1}) < \varepsilon \) \( (i = 0, 1, \ldots, k-1) \) is called (see, for example, [2, 3, 6, 7, 12] and the bibliography therein) a \((\varepsilon, t, \pi)\)-chain joining the points \( x \) and \( y \).

Remark 5. Without loss of generality we can suppose always that \( t_i \leq 2t \), where \( t_i \) and \( t \) the numbers figuring in Definition 6 (see, for example, [2, Ch.I]).
We denote by $P(\Sigma)$ the set \( \{ (x, y) : x, y \in \Sigma, \forall \varepsilon > 0 \forall t > 0 \exists (\varepsilon, t, \pi) \text{-chain joining } x \text{ and } y \} \). The relation $P(\Sigma)$ is closed, invariant and transitive \([2, 6, 10–12]\).

**Definition 7.** The point $x \in \Sigma$ is called chain recurrent (in $\Sigma$) if $(x, x) \in P(\Sigma)$.

We denote by $R(\Sigma)$ the set of all chain recurrent (in $\Sigma$) points from $\Sigma$.

**Remark 6.** Note that if $\Sigma_1$ and $\Sigma_2$ are two positively invariant subsets of $(X, T, \pi)$ with condition $\Sigma_1 \subseteq \Sigma_2$, then $R(\Sigma_1) \subseteq R(\Sigma_2)$.

**Definition 8.** Let $A \subseteq X$ be a nonempty positively invariant set. The set $A$ is called (see, for example, \([9]\)) internally chain recurrent if $R(A) = A$, and internally chain transitive if the following stronger condition holds: for any $a, b \in A$ and any $\varepsilon > 0$ and $t > 0$, there is an $(\varepsilon, t, \pi)$-chain in $A$ connecting $a$ and $b$.

The set of all chain recurrent points for $(X, T, \pi)$ is denoted by $R(\Sigma)$, i.e., $R(\Sigma) := \{ x \in \Sigma : (x, x) \in P(\Sigma) \}$. On $R(\Sigma)$ we will introduce a relation $\sim$ as follows: $x \sim y$ if and only if $(x, y) \in P(\Sigma)$ and $(y, x) \in P(\Sigma)$. It is easy to check that the introduced relation $\sim$ on $R(\Sigma)$ is a relation of equivalence and, consequently, it is easy to decompose it into the classes of equivalence \( \{ R_\lambda : \lambda \in \mathcal{L} \} \) (internally chain transitive subsets), i.e., $R(\Sigma) = \sqcup\{ R_\lambda : \lambda \in \mathcal{L} \}$. By Proposal 2.6 from \([2]\) (see also \([6]\) and \([10–12]\) for the semi-group dynamical systems) the defined above components of the decomposition of the set $R(\Sigma)$ are closed and positively invariant.

**Lemma 11** (see \([9]\)). Let $x \in X$ and $\gamma \in \Phi_x$. The $\omega$-limit (respectively, $\alpha$-limit) set of positive (respectively, negative) pre-compact orbit of the point $x$ is internally chain transitive, i.e., $R(\omega_x) = \omega_x$ (respectively, $R(\alpha_\gamma) = \alpha_\gamma$).

Let $(X, T, \pi)$ be a compact dissipative dynamical system and $J$ be its Levinson center. Denote by $R(\pi) := R(J)$.

**Problem.** Suppose that $(X, T, \pi)$ is a compact dissipative dynamical system and $J$ is its Levinson center. To prove that $R(\pi) = R(X)$ or to construct a corresponding counterexample.

**Remark 7.** In the connection with the Problem formulated above it is interesting to note that in the works \([5,8]\) an example of dynamical system $(X, T, \pi)$ is constructed which posses the following properties:

1. $(X, T, \pi)$ is point dissipative;
2. $(X, T, \pi)$ is asymptotically compact;
3. $(X, T, \pi)$ is not compact dissipative;
4. $R(X)$ is an unbounded subset of $X$.

Denote by $C(T \times X, X)$ the set of all continuous functions $\pi : T \times X \mapsto X$ equipped with the compact-open topology. If $K \subset X$ is a compact subset from $X$, then we denote by

$$d_K(f, g) := \sup_{L>0} \min \{ \sup_{0 \leq t \leq L} \rho(f(t, x), g(t, x)), L^{-1} \}$$

(11)
and \( \mathcal{D} := \{ d_K : K \in C(X) \} \) a family of pseudo-metrics which generates the compact-open topology on \( C(\mathbb{T} \times X, X) \), where \( C(X) \) is the family of all compact subsets from \( X \).

**Remark 8.** Note that for all \( \varepsilon > 0 \) the inequality \( d_K(f, g) < \varepsilon \) is equivalent to 
\[
\sup_{0 \leq t \leq \varepsilon, x \in K} \rho(f(t, x), g(t, x)) < \varepsilon \text{ (see, for example, [13, Ch.I] or [14, Ch.IV]).}
\]

**Definition 9.** Recall [2, Ch.I] that the collection \([x_1, x_2, \ldots, x_k := y; t_1, t_2, \ldots, t_{k-1}]\) is called a generalized chain joining \( x \) and \( y \) if the following conditions are fulfilled:

1. \( t_i \geq t \);
2. \( \rho(x, x_1) < \varepsilon \);
3. \( \rho(\pi(t_i, x_i), x_{i+1}) < \varepsilon \) \((1 = 1, \ldots, k - 1)\).

**Remark 9.** In the book [2, Ch.I] it is shown that in the definition of chain recurrence the \((\varepsilon, t, f)\)-chains can be replaced by generalized \((\varepsilon, t, f)\)-chains.

**Theorem 6.** Suppose that the following conditions hold:

1. \( \mathcal{M} \subset C(\mathbb{T} \times X, X) \) is a compact subset from \( C(\mathbb{T} \times X, X) \);
2. for all \( \pi \in \mathcal{M} \) the dynamical system \((X, \mathbb{T}, \pi)\) is compact dissipative and \( J_\pi \) is its Levinson center;
3. the set \( J := \bigcup \{ J_\pi : \pi \in \mathcal{M} \} \) is compact.

Then the mapping \( F : \mathcal{M} \mapsto 2^J \) defined by equality \( F(\pi) := \mathcal{R}(\pi) \) is upper semi-continuous, where by \( 2^J \) the space of all compact subsets from \( J \) equipped with the Hausdorff metric is denoted.

**Proof.** Let \( \pi_n, \pi \in \mathcal{M} \) and \( d_f(\pi_n, \pi) \to 0 \), \( a_n \in \mathcal{R}(\pi_n) \) and \( a_n \to a \) as \( n \to \infty \). We need to show that \( a \in \mathcal{R}(\pi) \). Let \( \varepsilon \) be an arbitrary positive number and \( 0 < \delta < \varepsilon / 4 \). There exists a number \( n_0 \in \mathbb{N} \) such that \( \rho(a_n, a) < \delta \) and \( d_f(\pi_n, \pi) < \delta \) for all \( n \geq n_0 \). Since \( a_n \in \mathcal{R}(\pi_n) \), then there is a \((\delta, \varepsilon^{-1}, \pi_n)\)-chain from \( a_n \) to \( a_n \), i.e., there exists a collection \( \{ x_0 = a_n, x_1, \ldots, x_{k-1}, x_k = a_n; t_0, \ldots, t_{k-1} \} \) such that
\[
\rho(\pi_n(t_i, x_i), x_{i+1}) < \delta, \quad \varepsilon^{-1} \leq t_i \leq 2 \varepsilon^{-1} \text{ (} i = 0, 1, \ldots, k - 1 \text{).}
\]

Thus the collection \([x_0, x_1, \ldots, x_{k-1}; a; t_0, t_1, \ldots, t_{k-1}]\) is a generalized \((2\delta, \varepsilon^{-1}, \pi_n)\)-chain joining \( a \) with \( a \). From the inequality \( d_f(\pi_n, \pi) < \delta \) it follows that
\[
\rho(\pi_n(t, x), \pi(t, x)) < \delta \text{ (} x \in J, \ 0 \leq t \leq \delta^{-1} < 4 \varepsilon^{-1}\)
\]
and, consequently, the above indicated generalized \((2\delta, \varepsilon^{-1}, \pi_n)\)-chain is also a generalized \((\varepsilon, \varepsilon^{-1}, \pi)\) chain from \( a \) to \( a \). Since \( \varepsilon \) is an arbitrary positive number, then \( a \in \mathcal{R}(\pi) \). \( \Box \)
Lemma 12. Suppose that \((X, T, \pi)\) is compact dissipative and \(J\) if its Levinson center, then \(\omega_x \subseteq \mathcal{R}(J) = \mathcal{R}(\pi)\) for all \(x \in X\).

Proof. Let \(x \in X\) be an arbitrary point. Since \((X, T, \pi)\) is compact dissipative, then \(\omega_x\) is a nonempty, compact, and invariant subset of \(J\), then \(\mathcal{R}(\omega_x) \subseteq \mathcal{R}(J) = \mathcal{R}(\pi)\). By Lemma 11 we have \(\omega_x = \mathcal{R}(\omega_x)\) and, consequently, \(\omega_x \subseteq \mathcal{R}(\pi)\). \(\square\)

Lemma 13 (see [4, Ch.IV]). If the compact invariant set \(\Sigma\) from \(X\) contains only a finite number of minimal sets, then the relation \(\sim\) decomposes the set \(\mathcal{R}(\Sigma)\) into the finite number of different classes of equivalence (internally chain transitive sets).

Remark 10. 1. Lemma 13 was established in [4, Ch.IV] for the two-sided (group) dynamical systems.
2. For the one-sided (semi-group) dynamical systems this statement may be proved by slight modifications of the arguments from [4, Ch.IV].
3. For two-sided dynamical systems \((T = \mathbb{S})\) with infinite number of compact minimal subsets Lemma 13 remains true if in addition the dynamical system \((X, \mathbb{S}, \pi)\) satisfies some condition of hyperbolicity (see Theorem 4.1 [4, Ch.IV]).

Lemma 14 (see [9]). Let \(M\) be an isolated (local maximal) invariant set and \(\mathcal{R}\) be a compact internally chain transitive set for \((X, T, \pi)\). Assume that \(M \cap \mathcal{R} \neq \emptyset\) and \(M \subseteq \mathcal{R}\).

Then

a. there exists a point \(u \in \mathcal{R} \setminus M\) such that \(\omega_u \subseteq M\);

b. there exists a point \(w \in \mathcal{R} \setminus M\) and an entire trajectory \(\gamma \in \Phi_w\) such that \(\alpha_\gamma \subseteq M\).

Theorem 7. Assume that the following conditions hold:

1. the dynamical system \((X, T, \pi)\) is compactly dissipative and \(J\) is its Levinson center;
2. there exists a finite number \(n\) of compact minimal subsets \(M_i \subseteq J\) \((i = 1, 2, \ldots, k)\) of \((X, T, \pi)\);
3. the collection of subsets \(\{M_1, M_2, \ldots, n\}\) does not admit \(k\)-cycles;
4. for all \(x \in X\) there exists a number \(i \in \{1, 2, \ldots, n\}\) such that \(\omega_x = M_i\).

Then any compact internally chain transitive set \(\mathcal{R}_\lambda(\pi)\) is a minimal set of \((X, T, \pi)\), i.e., there exists \(i \in \{1, 2, \ldots, n\}\) such that \(\mathcal{R}_\lambda = M_i\).

Proof. Let \(\mathcal{R}_\lambda(\pi)\) be a compact internally chain transitive set for \((X, T, \pi)\). Since \(\mathcal{R}_\lambda(\pi)\) is a compact positively invariant set, then by Birkhoff’s theorem in \(\mathcal{R}_\lambda(\pi)\) there exists a nonempty compact minimal set \(M_i \subseteq \mathcal{R}_\lambda(\pi)\) \((i \in \{1, 2, \ldots, n\}\). We will show that \(\mathcal{R}_\lambda(\pi) = M_i\). If we suppose that it is not true, then by Lemma 14 there exists a point \(x_1 \in \mathcal{R}_\lambda(\pi) \setminus M_i\) and an entire trajectory \(\gamma_1 \in \Phi_{x_1}\) such that
α_{x_1} \subseteq M_{i_1}. By conditions of Theorem there exists a number i_2 \in \{1, 2, \ldots, n\} such that \omega_{x_1} = M_{i_2}. Since M_{i_2} \subseteq \mathcal{R}_\lambda(\pi) and \mathcal{R}_\lambda(\pi) \neq M_{i_2} then by Lemma 14 there exists a point x_2 \in \mathcal{R}_\lambda(\pi) \setminus M_{i_2} and an entire trajectory \gamma_2 \in \Phi_{x_2} such that \alpha_{x_2} = M_{i_2} and there exists a number i_3 \in \{1, 2, \ldots, n\} such that \omega_{x_2} = M_{i_3}. Since there is only a finite number of \textit{M}'s, we will eventually arrive at a cyclic chain of some minimal sets of (X, T, \pi), which contradicts our assumption.

**Corollary 4.** Under the conditions of Theorem 7 we have \mathcal{R}(\pi) = \prod_{i=1}^{n} M_i.

**Theorem 8.** Suppose that (X, T, \pi) is a bounded dissipative dynamical system and J is its Levinson center. Then for every \delta > 0 and B \in \mathcal{B}(X) there exists L = L(\delta, B) > 0 \text{ (}L \in \mathbb{T}\text{)} such that

\[ \pi([0, L], x) \cap B(\mathcal{R}(J), \delta) \neq \emptyset \text{ for all } x \in B, \]

i.e., for all x \in B there exists \ell = \ell(x) \in [0, L] such that

\[ \pi(\ell, x) \in B(\mathcal{R}(J), \delta). \]

**Proof.** If we suppose that the statement of Theorem is not true, then there are \delta_0 > 0, B_0 \in \mathcal{B}(X), L_n \geq n and x_n \in B_0 such that

\[ \rho(\pi(t, x_n), \mathcal{R}(J)) \geq \delta_0 \]

for all t \in [0, L_n]. Let \( s_n := [L_n/2] \) and \( \tilde{x}_n := \pi(s_n, x_n) \). Note that

\[ \rho(\tilde{x}_n, J) = \rho(\pi(s_n, x_n), J) \leq \beta(\pi(s_n, B_0), J) \to 0 \]

as \( n \to \infty \), because \( s_n \to \infty \) and \( J \) attracts the bounded subset \( B_0 \) as \( t \to +\infty \).

From (13) it follows that the sequence \{\tilde{x}_n\} is relatively compact. Thus, without loss of generality we can suppose that the sequence \{\tilde{x}_n\} is convergent. Denote \( \hat{x} = \lim_{n \to \infty} \tilde{x}_n \), then by (13) we obtain \( \hat{x} \in J \). On the other hand by (12) we obtain

\[ \rho(\pi(t, \hat{x}_n), \mathcal{R}(J)) = \rho(\pi(t + s_n, x_n), \mathcal{R}(J)) \geq \delta_0 \]

for all t \in [-s_n, s_n]. Let \( \gamma \in \mathcal{F}_{\hat{x}} \) be the full trajectory of dynamical system \( (X, T, \pi) \) passing through \{x\} at the initial moment \( t = 0 \) and defined by equality \( \gamma(t) = \lim_{n \to \infty} \pi(t + s_n, x_n) \) for all \( t \in \mathbb{S} \). Note that \( \gamma(\mathbb{S}) \subseteq J \) because for every \( t \in \mathbb{S} \) we have

\[ \rho(\pi(t + s_n, x_n), J) \leq \rho(\pi(t + s_n, B_0), J) \]

for sufficiently large \( n \), and passing to limit in (15) as \( n \to \infty \) we obtain \( \gamma(t) \in J \) for all \( t \in \mathbb{S} \). By Lemma 12 we have \( \omega_{\hat{x}} \subseteq \mathcal{R}(J) \). But from (14) it follows that \( \gamma(t) \notin \mathcal{R}(J) \) for all \( t \in \mathbb{S} \) and, consequently, \( \omega_{\hat{x}} \cap \mathcal{R}(J) = \emptyset \). The obtained contradiction proves our statement. Theorem is proved.

**Corollary 5.** Suppose that the following conditions hold:
1. \((X, \mathbb{T}, \pi)\) is a bounded dissipative dynamical system and \(J\) is its Levinson center;

2. \((X, \mathbb{T}, \pi)\) is a gradient system;

3. \(\text{Fix}(\pi) = \{p_1, p_2, \ldots, p_m\}\);

4. \(\text{Fix}(\pi)\) does not contain any \(k\)-cycle \((k \geq 1)\).

Then for every \(\delta > 0\) and \(B \in \mathcal{B}(X)\) there exists \(L = L(\delta, B) > 0\) \((L \in \mathbb{T})\) such that

\[
\pi([0, L], B) \cap B(\text{Fix}(\pi), \delta) \neq \emptyset,
\]

i. e., for all \(x \in B\) there exists \(l = l(x) \in [0, L]\) such that

\[
\pi(l, x) \in B(\text{Fix}(\pi), \delta).
\]

Proof. This statement follows from Theorems 7 and 8. \(\square\)

**Theorem 9.** Suppose that the following conditions are fulfilled:

1. the dynamical system \((X, \mathbb{T}, \pi)\) admits a compact global attractor \(J\) which attracts every bounded subset \(B \in \mathcal{B}(X)\);

2. \(\mathcal{R}(J)\) consists of finite number of different classes of equivalence \(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\).

Then for every \(\delta > 0\) there exists \(\delta \in (0, \delta)\) such that for every \(x \in B(\mathcal{R}_i, \delta)\) \((i = \overline{1, k})\) with \(\pi(t, x) \in B(\mathcal{R}_i, \delta)\) for all \(t \in [0, T]\) and \(\pi(T, x) \notin B(\mathcal{R}_i, \delta)\) we have \(\pi(t, x) \notin B(\mathcal{R}_i, \delta)\) for each \(t \geq T\) (i. e., never returns again in \(B(\mathcal{R}_i, \delta)\) for all \(t \geq T\)).

Proof. By Lemma 4.3 [4, Ch.IV] in the collection \(\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\}\) there is no \(r\)-cycles \((r \geq 1)\). We will show that if we suppose that the statement of Theorem is not true, then we will have a contradiction this the fact formulated above. In fact. Suppose that Theorem is wrong, then there are \(\mathcal{R}_{i_0}, B(\mathcal{R}_{i_0}, \delta_0)\) \((\delta_0 > 0), \)

\[T_n \in \mathbb{T}, T_n' > T_n\]

and a sequence \(\{x_n\} \subset B(\mathcal{R}_{i_0}, \delta_0)\) such that

\[
\pi(T_n, x_n) \notin B(\mathcal{R}_{i_0}, \delta_0) \quad \text{and} \quad \pi(T_n', x_n) \in B(\mathcal{R}_{i_0}, 1/n).
\]

Without loss of generality we can suppose that \(\pi(t, x_n) \in B(\mathcal{R}_{i_0}, \delta_0)\) for all \(t \in [0, T_n)\).

Note that \(T_n \to \infty\) as \(n \to \infty\). If we suppose that it is not so, then we can consider that \(\{T_n\}\) is bounded (otherwise we can extract a subsequence \(\{T_{k_n}\}\) which converges to \(+\infty\) as \(n\) goes to \(+\infty\)), i. e., there exists a number \(L > 0\) such that

\[
\pi(t, x_n) \notin B(\mathcal{R}_{i_0}, \delta_0)
\]
for all $t \geq L$ and $n \in \mathbb{N}$. Since $x_n \in B(\mathcal{R}_{i_0}, 1/n)$, then without loss of generality we can suppose that $\{x_n\}$ is convergent. Denote by $p := \lim_{n \to \infty} x_n$, then $p \in \mathcal{R}_{i_0}$ and passing into limit in (16) as $n \to \infty$ we obtain
\[
\pi(t, p) \notin B(\mathcal{R}_{i_0}, \delta_0)
\] (17)
for all $t \geq L$. On the other hand
\[
\pi(t, p) \in \mathcal{R}_{i_0}
\] (18)
for all $t \geq 0$ because the set $\mathcal{R}_{i_0}$ is invariant. Relations (17) and (18) are contradictory. The obtained contradiction proves our statement.

Denote by $\tilde{x}_n := \pi(T_n, x_n)$, then we have

1. $\tilde{x}_n \notin B(\mathcal{R}_{i_0}, \delta_0)$ for all $n \in \mathbb{N}$;
2. $\pi(t, \tilde{x}_n) = \pi(t + T_n, x_n) \in B(\mathcal{R}_{i_0}, \delta_0)$ for all $-T_n \leq t < 0$;
3. $\pi(T'_n, \tilde{x}_n) \in B(\mathcal{R}_{i_0}, 1/n)$ for all $n \in \mathbb{N}$, where $\tilde{T}'_n := T'_n - T_n > 0$.

Since $x_n \in B(\mathcal{R}_{i_0}, 1/n)$, $T_n \to +\infty$ and $(X, T, \pi)$ is compactly dissipative, then the sequence $\{\tilde{x}_n\}$ is relatively compact and without loss of generality we can suppose that it is convergent. Denote by $\tilde{x} := \lim_{n \to \infty} \tilde{x}_n$ and consider $\gamma \in \Phi_{\tilde{x}}$, where $\gamma(t) := \lim_{n \to \infty} \pi(t + T_n, x_n)$ for all $t \in \mathbb{S}$.

Note that $\tilde{T}'_n \to +\infty$ as $n \to \infty$. In fact, if we suppose that it is not true, then without loss of generality we can consider that $\{\tilde{T}'_n\}$ is bounded, for example, $\tilde{T}'_n \in [0, L]$ for all $n \in \mathbb{N}$, where $L$ is some positive number. Let $l := \lim_{n \to \infty} \tilde{T}'_n$, then $l \in [0, L]$ (it is necessary that we can extract a convergent subsequence from $\{\tilde{T}'_n\}$). Then from (iii) we obtain $\pi(l, \tilde{x}) \in \mathcal{R}_{i_0}$ and, consequently, $\tilde{x} \in \mathcal{R}_{i_0}$ because $\mathcal{R}_{i_0}$ is invariant. The obtained contradiction proves our statement.

We will show that $\gamma(t) \in J$ for all $t \in \mathbb{S}$. In fact
\[
\rho(\pi(t + T_n, x_n), J) \leq \beta(\pi(t + T_n, K), J) \to 0
\]
as $n \to \infty$, where $K := \overline{\{x_n\}}$ and by bar the closure in the space $X$ is denoted. Now we note that $\gamma(t) \in B(\mathcal{R}_{i_0}, \delta_0)$ for all $t < 0$. Since the set $\mathcal{R}_{i_0}$ is local maximal, then without loss of generality we can suppose that in $B(\mathcal{R}_{i_0}, \delta_0)$ the invariant set $\mathcal{R}_{i_0}$ is maximal and, consequently, $\alpha_x \subseteq \mathcal{R}_{i_0}$. On the other hand $\omega_x \subseteq \mathcal{R}(J)$ and, consequently, there exists a number $i_1 \in \{1, 2, \ldots, k\}$ such that $\omega_x \subseteq \mathcal{R}_{i_1}$. Since the collection $\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\}$ has not $r$-cycles ($r \geq 1$), then $i_1 \neq i_0$.

Since $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ and $\omega_x \subseteq \mathcal{R}_{i_1}$, then by integral continuity for all $n \in \mathbb{N}$ there exists a number $T'_n > 0$ such that $\pi(T'_n, \tilde{x}_n) \in B(\mathcal{R}_{i_1}, 1/n)$. Taking into account that $\tilde{T}'_n \to +\infty$ as $n \to \infty$ and Theorem 8 we can consider that $T'_n \leq \tilde{T}'_n$.

On the other hand by Theorem 8 for all $n \in \mathbb{N}$ there exists $T^2_n \in (T'_n, \tilde{T}'_n)$ such that $\pi(T^2_n, \tilde{x}_n) \notin B(\mathcal{R}_{i_1}, \delta_0)$. Repeating the reasoning above for the set $\mathcal{R}_{i_1}$ and the
sequence \( \{\tilde{x}_n\} \) we can find a full trajectory \( \gamma_1 \) so that \( \alpha_{\gamma_1} \subseteq R_{i_1} \) and \( \omega_{\tilde{x}_1} \subseteq R_{i_2} \), where \( i_2 \neq i_0, i_1 \) and \( \tilde{x}_1 := \gamma_1(0) \).

Reasoning analogously we will construct a sequence \( \{\gamma, \gamma_1, \ldots, \gamma_p\} \) \((p \leq k - 1)\) so that \( \alpha_{\gamma_p} \subseteq R_{i_p} \) and \( \omega_{\tilde{x}_p} \subseteq R_{i_{p+1}} \) \((\gamma_0 := \gamma)\). Since the family \( \{R_1, R_2, \ldots, R_k\} \) contains a finite number of sets \( R_p \), then after the finite number \( q \) of steps we will have \( R_{i_p} = R_{i_0} \), i.e., we will obtain a \( q \)-cycle. The obtained contradiction proves our Theorem.

\[ \square \]

**Corollary 6.** Suppose that the following conditions hold:

1. \((X, \mathbb{T}, \pi)\) is a bounded dissipative dynamical system and \( J \) its Levinson center;
2. \((X, \mathbb{T}, \pi)\) is a gradient system;
3. \( \text{Fix}(\pi) = \{p_1, p_2, \ldots, p_m\} \);
4. \( \text{Fix}(\pi) \) does not contain any \( k \)-cycle \((k \geq 1)\).

Then for every \( \tilde{\delta} > 0 \) there exists \( \delta \in (0, \tilde{\delta}) \) such that for every \( x \in B(R_i, \delta) \) \((i = 1, k)\) with \( \pi(t, x) \in B(R_i, \delta) \) for all \( t \in [0, T) \) and \( \pi(T, x) \notin B(R_i, \delta) \) we have \( \pi(t, x) \notin B(R_i, \delta) \) for each \( t \geq T \) (i.e., never returns again in \( B(R_i, \delta) \) for all \( t \geq T \)).

**Proof.** This statement follows from Theorems 8 and 9.

\[ \square \]

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**References**


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