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Solvability of a nonlinear integral equation arising in kinetic theory

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Abstract. In the paper the question of solvability of an Urysohn type nonlinear integral equation arising in kinetic theory of gases has been studied. We prove the existence of a positive and bounded solution and also suggest an approach for the construction of a solution. We also show that there is a qualitative difference between solutions in the linear and nonlinear cases. In the nonlinear case the solution is a positive and bounded function, while the corresponding linear equation has an alternating solution, which possesses linear growth at infinity.

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1 Introduction

The paper is devoted to the study and solution of the following Urysohn nonlinear integral equation

$$F(x) = g(x) + \int_0^\infty W(x,t,F(t))dt, \tag{1.1}$$

with respect to the unknown function $F(x)$, where

$$g(x) = \frac{2\varepsilon c}{3\sqrt{\pi}} \int_0^\infty e^{-xs} e^{-\frac{1}{s^2}} (s^2+1) \frac{ds}{s^4}, \tag{1.2}$$

$$\begin{gathered} W(x,t,F(t)) = \frac{2}{3\sqrt{\pi}} \sqrt{F(t)} \times \\ \times \int_0^\infty \left[e^{-|x-t|s} + (1-\varepsilon) e^{-(x+t)s} \right] e^{-\frac{1}{s^2 F(t)}} \left[\frac{1}{s^2 F(t)} + 1 \right] \frac{ds}{s}. \end{gathered} \tag{1.3}$$

Equation (1.1), as well as its intrinsic mathematical interest, has important applications in kinetic theory of gases (see [1–3]). Equation (1.1) may be derived from the Boltzmann model equation. By equation (1.1) the flow of a rarefied gas in the half-space $x > 0$ bounded by flat plate $x = 0$ is described. The function $F(x)$ represents temperature distribution near the wall. Here x is the distance from the wall, $c = \dfrac{\beta}{\alpha}$

$(0 < c \le 1)$, where α is the mean value of density in the boundary layer and β is the density of particles reflected from the wall. We will assume that c is previously known. ε is the accomodation coefficient $(0 < \varepsilon \le 1)$.

In the present note we prove the existence theorem of a positive and bounded solution of equation (1.1) and also suggest the approach for the construction of a solution. We also show that there is a qualitative difference between solutions in the nonlinear and linear cases. In the nonlinear case the solution is a positive and bounded function, while the corresponding linear equation has an alternating solution, which possesses linear growth at infinity.

2 The existence of a bounded solution for an Urysohn type nonlinear integral equation

Below we formulate the theorem of global solvability of equation (1.1) in the space of bounded functions for arbitrary values of $c > 0$ and $\alpha > 0$.

We consider the following function

$$\xi(t) = t^4 - ct^3 - 1, \quad t \in \mathbb{R}^+ \equiv [0, +\infty). \tag{2.1}$$

We note that $\xi(0) = -1$, $\xi'(t) = 4t^3 - 3ct^2 \ge 0$ if $t \in \left[\frac{3c}{4}, +\infty\right)$ and $\xi'(t) \le 0$ if $t \in \left[0, \frac{3c}{4}\right]$, $\xi(c) < 0$, $\lim\limits_{t\to\infty} \xi(t) = +\infty$, then there exists a unique point $t_0 > c$ such that $\xi(t_0) = 0$, moreover, for $t > t_0$, $\xi(t) > 0$.

We introduce the following iterations for equation (1.1):

$$F_{n+1}(x) = g(x) + \int\limits_0^\infty W(x, t, F_n(t))dt, \tag{2.2}$$

$$F_0(x) = t_0^2 = c_0. \tag{2.3}$$

It is easy to verify that the function W defined by (1.3) is monotone increasing in the third argument, i. e.

$$W(x, t, z) \ \uparrow \text{w.r.t.} z. \tag{2.4}$$

Indeed, since $\rho(z) = \left(\frac{s^2}{z} + 1\right)\sqrt{z}e^{-\frac{s^2}{z}} \uparrow w.r.t.\ z, z \ge 0$, then from the representation of W it follows that $W \uparrow$ w.r.t. z.

Below we prove by induction that $F_n(x)$ is monotone decreasing in n

$$1) \quad F^{(n)} \downarrow \text{w.r.t. } n \quad \text{and} \quad 2) \quad F^{(n)}(x) \ge g(x). \tag{2.5}$$

Let $n = 0$. We have

$$\begin{gathered} F_1(x) = g(x) + \int\limits_0^\infty W(x, t, F_0(t))dt = \\ = J_1(x) + c_0 - J_2(x) = F_0(x) + J_1(x) - J_2(x), \end{gathered} \tag{2.6}$$

where

$$J_1(x) = \frac{2\varepsilon c}{3\sqrt{\pi}} \int_0^\infty e^{-xs} e^{-\frac{1}{s^2}} \left(\frac{1}{s^2} + 1\right) \frac{ds}{s^2}, \tag{2.7}$$

$$J_2(x) = \frac{2\varepsilon}{3\sqrt{\pi}} \sqrt{c_0} \int_0^\infty e^{-xs} e^{-\frac{1}{s^2 c_0}} \left(\frac{1}{s^2 c_0} + 1\right) \frac{ds}{s^2}. \tag{2.8}$$

We must prove that $J_2(x) \geq J_1(x)$ for each $x \in \mathbb{R}^+$. It is sufficient to prove that for each $x \in \mathbb{R}^+$ the inequality holds

$$c e^{-\frac{1}{s^2}} \left(\frac{1}{s^2} + 1\right) \leq \sqrt{\frac{1}{c_0}} e^{-\frac{1}{s^2 c_0}} \left(c_0 + \frac{1}{s^2}\right). \tag{2.9}$$

Let us consider the following function

$$\varphi(s^2) = c\sqrt{c_0} e^{\frac{1}{s^2}\left(\frac{1}{c_0} - 1\right)} \left(\frac{1}{s^2} + 1\right), \ s^2 \in \mathbb{R}^+. \tag{2.10}$$

Note that $s_0^2 = c_0 - 1$ is the unique maximum point for φ. Therefore

$$\varphi(s^2) \leq \varphi(s_0^2) = c\sqrt{c_0} \left(\frac{1}{c_0 - 1} + 1\right) e^{-\frac{1}{c_0}}. \tag{2.11}$$

Using the well-known inequality

$$e^{-x} \leq \frac{1}{1 + x}, \quad x \geq 0, \tag{2.12}$$

from (2.11) we obtain

$$\varphi(s^2) \leq \frac{c c_0^2 \sqrt{c_0}}{(c_0^2 - 1)}. \tag{2.13}$$

First we prove that

$$\frac{c\sqrt{c_0} c_0}{(c_0^2 - 1)} \leq 1. \tag{2.14}$$

Since $c_0 = t_0^2 > 1$ (because $t_0^4 = c t_0^3 + 1 > 1 \Rightarrow t_0^2 > 1$), then inequality (2.14) is equivalent to the following inequality:

$$c\sqrt{c_0} c_0 \leq (c_0^2 - 1). \tag{2.15}$$

As $\xi(t) \uparrow$ in t on $[t_0, +\infty)$, then $\xi(\sqrt{c_0}) \geq \xi(t_0) = 0$ or $\xi(\sqrt{c_0}) = c_0^2 - c\sqrt{c_0} c_0 - 1 \geq 0$, i. e. (2.14) is proved. Taking into consideration (2.14), from (2.13), we obtain

$$\varphi(s^2) = \frac{c c_0^2 \sqrt{c_0}}{(c_0^2 - 1)} \leq c_0 \leq c_0 + \frac{1}{s^2}. \tag{2.16}$$

From (2.16) follows (2.9). Therefore we have $J_2(x) \geq J_1(x)$. Considering the last inequality and relation (2.6) we come to the inequality $F_1(x) \leq F_0(x)$. We assume that $F_n(x) \leq F_{n-1}(x)$ for some $n \in \mathbb{N}$. Since $W(x,t,z)$ monotonically increases in the third argument z then from (2.2) it follows that

$$F_{n+1}(x) \leq F_n(x). \tag{2.17}$$

Now we prove that the sequence of functions $\{F_n(x)\}_{n=0}^{\infty}$ is bounded by $g(x)$.

First, we show that $t_0^2 > \frac{c}{2}$. Assume the contrary: $t_0^2 \leq \frac{c}{2}$. Since $t_0 > c$ then we have $c < \sqrt{\frac{c}{2}}$ or

$$c < \frac{1}{2}. \tag{2.18}$$

On the other hand,

$$0 = t_0^4 - ct_0^3 - 1 < t_0^4 - 1.$$

Hence, we obtain $t_0^2 > 1$. But since $t_0^2 < \frac{c}{2}$ then we obtain inequality $c > 2$.

Taking into consideration (2.18), from the last inequality we come to contradiction. Therefore,

$$t_0^2 > \frac{c}{2}. \tag{2.19}$$

Now, due to (2.19) from (2.3), we have

$$F_0(x) = t_0^2 > \frac{c}{2} \geq g(x),$$

because

$$g(x) \leq \frac{2}{3\sqrt{\pi}} c \int_0^\infty e^{-\frac{1}{s^2}} \left(\frac{s^2+1}{s^4} \right) ds = \frac{c}{2}.$$

Let $F_n(x) \geq g(x)$ for some $n \in \mathbb{N}$. Then taking into consideration monotonicity and nonnegativity of the function W, we obtain

$$F_{n+1}(x) \geq g(x) + \int_0^\infty W(x,t,g(t))dt \geq g(x). \tag{2.20}$$

Therefore the sequence of functions $\{F_n(x)\}_{n=0}^{\infty}$ has a pointwise limit as $n \to \infty$. In accordance with B.Levi's theorem the function F satisfies equation (1.1) and the double inequalities

$$g(x) \leq F(x) \leq c_0 \equiv t_0^2. \tag{2.21}$$

Thus the following theorem holds

Theorem 1. *Let $0 < c \leq 1$ is a given number. Then nonlinear integral equation (1.1) has a positive measurable and bounded solution $F(x)$. The following inequalities hold*

$$g(x) \leq F(x) \leq c_0 \equiv t_0^2, \tag{2.22}$$

where t_0 is the unique positive root of the following algebraic equation $t^4 - ct^3 - 1 = 0$.

3 Linearization of a Urysohn nonlinear integral equation (1.1). Qualitative difference between solutions in the linear and nonlinear cases

Usually in kinetic theory in linear approximation the function $F(x)$ is represented as:

$$F(x) = 1 + \Delta f(x), \tag{3.1}$$

where $\Delta f(x)$ is the temperature perturbation $(\Delta f(x) \ll 1)$. Taking into account (3.1), expanding the function $W(x, t, F(t))$ by the third argument in a power series about zero and holding the first expansion term, we obtain the following Wiener-Hopf-Hankell type linear integral equation with respect to $\Delta f(x)$:

$$\Delta f(x) = g_1(x) + \int_0^\infty [K(x-t) + (1-\varepsilon)K(x+t)]\Delta f(t)dt. \tag{3.2}$$

Here

$$\begin{gathered} K(x) = \int_0^\infty e^{-|x|s} G(s)ds, \\ G(s) = \frac{2}{3}\frac{1}{\sqrt{\pi}}\frac{1}{s}e^{-\frac{1}{s^2}}\left(\frac{1}{s^4} + \frac{1}{2s^2} + \frac{1}{2}\right), \end{gathered} \tag{3.3}$$

$$\begin{gathered} g_1(x) = \int_0^\infty e^{-xs} G_1(s)ds, \\ G_1(s) = \frac{2\varepsilon}{3\sqrt{\pi}s^4}(c-1)(s^2+1)e^{-\frac{1}{s^2}}. \end{gathered} \tag{3.4}$$

It is easy to check that kernel (3.3) satisfies the conservative condition

$$K \geq 0, \qquad \int_{-\infty}^{+\infty} K(x)dx = 1. \tag{3.5}$$

Due to linearity the solution of equation (3.2) can be written as:

$$\triangle f(x) = -\triangle f_1(x) + \triangle f_2(x), \tag{3.6}$$

where $\triangle f_1(x)$ and $\triangle f_2(x)$ are the solutions of inhomogeneous and homogeneous equations, respectively

$$\Delta f_1(x) = -g_1(x) + \int_0^\infty [K(x-t) + (1-\varepsilon)K(x+t)]\Delta f_1(t)dt \tag{3.7}$$

$(-g_1(x) \geq 0$ because of $c \in (0,1])$,

$$\Delta f_2(x) = \int_0^\infty [K(x-t) + (1-\varepsilon)K(x+t)]\Delta f_2(t)dt. \tag{3.8}$$

There are numerous works devoted to study and solutions of equations (3.7) and (3.8) (see [4, 5] and references therein). Without going into details we note that equation (3.7) has positive bounded solution, which possesses finite limit at infinity (see [4, 6]).

The solution of corresponding homogeneous equation (3.8) has the form (see [5])

$$\triangle f_2(x) = \frac{1}{\sqrt{\nu_2}} x + q(x), \tag{3.9}$$

here $q(x)$ is the well-known Hopf function, and ν_2 is the second moment of the kernel $K(x)$. Thus we have

$$\begin{gathered} \triangle f(x) = \frac{1}{\sqrt{\nu_2}} x + q(x) - \triangle f_1(x) \quad \text{and} \\ \triangle f(x) \sim \frac{1}{\sqrt{\nu_2}} x, \quad \text{as} \quad x \to +\infty. \end{gathered} \tag{3.10}$$

Conclusion. Note that the linear equation (3.2) possesses an alternating solution with the asymptotic $O(x)$ as x tends to $+\infty$, while the solution of initial nonlinear equation (1.1) is a positive bounded function $F(x)$. Moreover, $g(x) \leq F(x) \leq c_0,\ x \in \mathbb{R}^+$. The qualitative difference between the solutions is conditioned by linearization of equation (1.1). In fact the linearization can distort the problem and the corresponding linear equation can not adequately describe the problem from a physical point of view.

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