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On 2-absorbing Primary Subsemimodules over Commutative Semirings

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Abstract. In this paper, we define 2-absorbing primary subsemimodules of a semimodule M over a commutative semiring S with $1 \neq 0$ which is a generalization of primary subsemimodules of semimodules. A proper subsemimodule N of a semimodule M is said to be a 2-absorbing primary subsemimodule of M if $abm \in N$ implies $ab \in \sqrt{(N:M)}$ or $am \in N$ or $bm \in N$ for some $a, b \in S$ and $m \in M$. It is proved that if K is a subtractive subsemimodule of M and $\sqrt{(K:M)}$ is a subtractive ideal of S, then K is a 2-absorbing primary subsemimodule of M if and only if whenever $IJN \subseteq K$ for some ideals I, J of S and a subsemimodule N of M, then $IJ \subseteq \sqrt{(K:M)}$ or $IN \subseteq K$ or $JN \subseteq K$. In this paper, we prove a number of results concerning 2-absorbing primary subsemimodules.

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1 Introduction

The notion of a semiring was first introduced by H. S. Vandiver in 1934 [16]. After that various research have been done in this area and several applications have been found in various branches of mathematics and computer science. The concepts of prime and primary ideals are essential ingredients in ideal theory and algebraic geometry. Prime subsemimodule and primary subsemimodules have been used in soft mathematics and studied by many authors (for example see [3], [4], [7], [8] and [10]) during the last decade. The concept of 2-absorbing subsemimodule which is a generalization of a prime subsemimodule was studied in [13]. In this paper, we introduce the concept of 2-absorbing primary subsemimodule which is a generalization of the primary subsemimodule. Throughout the paper, a semiring S will be considered as commutative with identity $1 \neq 0$ and a left S-semimodule means a unitary semimodule.

A commutative semiring is a commutative semigroup $(S, \cdot)$ and a commutative monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$, both are connected by ring like distributivity. A non-empty subset I of a semiring S is called an ideal of S if whenever $a, b \in I$ and $s \in S$, then $a + b \in I$ and $sa, as \in I$. An ideal I of S is said to be proper if $I \neq S$. A left S-semimodule M is a commutative monoid $(M, +)$ which has a zero element 0_M, together with an operation $S \times M \to M$, denoted by $(a, x) \to ax$ such that for all $a, b \in S$ and

$x, y \in M$,
(i) $a(x+y) = ax + ay$,
(ii) $(a+b)x = ax + bx$,
(iii) $(ab)x = a(bx)$,
(iv) $0_S \cdot x = 0_M = a \cdot 0_M$.

A non-empty subset N of an S-semimodule M is a subsemimodule of M if N is closed under addition and scalar multiplication. A proper subsemimodule N of an S-semimodule M is called subtractive if whenever $a, a+b \in N$, $b \in M$ then $b \in N$. Let N be a subsemimodule of M. Then, an associated ideal of N is defined as $(N :_S M)$ or simply $(N : M)$ denote the ideal $\{s \in S : sM \subseteq N\}$ and $(N : m) = \{a \in S : am \in N \text{and} \, m \in M\}$. Recall ([3], [4], [9], [10], [13], [15]) the following: A non-zero proper ideal I of S is said to be a 2-absorbing ideal of S if whenever $abc \in I$ for any $a, b, c \in S$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A proper ideal I of S is said to be a 2-absorbing primary ideal of S if whenever $a, b, c \in S$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, where $\sqrt{I} = \{s \in S$: there exists $n \in N$ with $s^n \in I\}$ denotes the radical of I. A proper subsemimodule N of an S-semimodule M is said to be a prime subsemimodule if for $a \in S$, $m \in M$, and $am \in N$, either $m \in N$ or $a \in (N :_S M)$. A proper subsemimodule N of M is said to be a 2-absorbing subsemimodule of M if whenever $a, b \in S$, $m \in M$ with $abm \in N$, then $ab \in (N :_S M)$ or $am \in N$ or $bm \in N$. A proper subsemimodule N of M is said to be a primary subsemimodule of M if whenever $a \in S$, $m \in M$ and $am \in N$, then $a \in \sqrt{(N : M)}$ or $m \in N$, where $\sqrt{(N : M)} = \{a \in S : a^n M \subseteq N \, \text{for some} \, n \geq 1\}$. A proper subsemimodule N of an S-semimodule M is said to be a strong subsemimodule if for each $x \in N$ there exists $y \in N$ such that $x + y = 0$.

2 2-absorbing primary subsemimodules

In this section, we introduce the concept of 2-absorbing primary subsemimodule of a semimodule M over a commutative semiring S and investigate some properties and results.

Definition 1. Let M be a semimodule over a commutative semiring S and N be a proper subsemimodules of M. Then N is said to be a 2-absorbing primary subsemimodule of M if whenever $abm \in N$ where $a, b \in S$ and $m \in M$, then $ab \in \sqrt{(N : M)}$ or $am \in N$ or $bm \in N$.

It is easy to see that every 2-absorbing subsemimodule of a semimodule M over a commutative semiring S is a 2-absorbing primary subsemimodule of M but converse need not be true. For instance, consider a Z_0^+-semimodule $M = Z_{16} = \{0, 1, 2, ..., 15\}$. Take a subsemimodule $N = \{0, 8\}$, generated by 8. Then $(N : M) = \{a \in S : aM \subseteq N\} = \{0, 8, 16, ...\}$ and $\sqrt{(N : M)} = \{a \in S : a^n \in (N : M)\} = \{0, 2, 4, 8....\}$. Now, $2.2.2 \in N$ but $2.2 \notin N$ and $2.2 \notin (N : M)$. Therefore, N is not a 2-absorbing subsemimodule of M but it is a 2-absorbing primary subsemimodule of

M, as $2.2 \in \sqrt{(N:M)}$. Also, every primary subsemimodule of M is a 2-absorbing primary subsemimodule but converse need not be true. For example, let S be $Z^* = Z_0^+$ and let $M = Z^* \times Z^*$ be a semimodule over S. If we take the subsemimodule $N = \{0\} \times 4Z^*$ of M, then $(N : M)$=$\{0\}$ and $\sqrt{(N:M)} = \{0\}$. Here, N is a 2-absorbing primary subsemimodule of M but N is not a primary subsemimodule of M. Because $2 \cdot (0, 2) \in N$ but $2 \notin \sqrt{(N:M)}$ and $(0, 2) \notin N$.

Result 1. *Let M be a semimodule and N be a proper subtractive subsemimodule of M and let $m \in M$. Then the following holds:*

(i) $(N : M)$ *is a subtractive ideal of S.*

(ii) $(0 : M)$ *and* $(N : m)$ *are subtractive ideals of S, where* $(0 : M)$=$\{a \in S : aM \subseteq \{0\}\}$.

Proof. Proof is straightforward. □

Theorem 1. *Let N be a subtractive 2-absorbing primary subsemimodule of a semimodule M. Then, $(N : M)$ is a 2-absorbing primary ideal of S.*

Proof. Let $abc \in (N : M)$ for some $a, b, c \in S$. Let $ab \notin (N : M)$ and $bc \notin \sqrt{(N:M)}$. This implies $ab \notin (N : M)$ and $bc \notin (N : M)$. Therefore, there exists $x, y \in M$ such that $abx \notin N$ and $bcy \notin N$ but $ac(bx + by) \in N$. Since N is a 2-absorbing primary subsemimodule of M, we have either $ac \in \sqrt{(N:M)}$ or $a(bx + by) \in N$ or $c(bx + by) \in N$. If $ac \in \sqrt{(N:M)}$, then there is nothing to prove. If $a(bx + by) \in N$, then $aby \notin N$ (as N is a subtractive). Consider $abcy \in N$. Since N is a 2-absorbing primary subsemimodule and $aby \notin N$, $bcy \notin N$, we have $ac \in \sqrt{(N:M)}$. Similarly, if $c(bx + by) \in N$, then we have $cbx \notin N$. Consider $abcx \in N$. Since N is a 2-absorbing primary subsemimodule and $abx \notin N$, $bcx \notin N$, we have $ac \in \sqrt{(N:M)}$. This implies that $(N : M)$ is a 2-absorbing primary ideal of S. □

Theorem 2. *Let N be a 2-absorbing primary subsemimodule of an S-semimodule M. Then $\sqrt{(N:M)}$ is a 2-absorbing ideal of S.*

Proof. Let N be a 2-absorbing primary subsemimodule of an S-semimodule M. Then by Theorem 1, we have $(N : M)$ is a 2-absorbing primary ideal of S. By [Theorem 2, [15]], we have $\sqrt{(N:M)}$ is a 2-absorbing ideal of S. □

Theorem 3. *Let N be a 2-absorbing primary subsemimodule of a semimodule M such that $\sqrt{(N:M)} = P$ for some prime ideal P of S. Then for some $m \in M \setminus N$, $\sqrt{(N:m)}$ is a prime ideal of S.*

Proof. Let N be a 2-absorbing primary subsemimodule of M. Then by Theorem 2, $\sqrt{(N:M)}$ is a 2-absorbing ideal of S. Let $a, b \in S$ be such that $ab \in \sqrt{(N:m)}$, where $m \in M \setminus N$. Therefore, $(ab)^n \in (N : m)$, that is, $a^n b^n m \in N$ for some positive integer n. This gives, either $a^n m \in N$ or $b^n m \in N$ or $a^n b^n \in \sqrt{(N:M)}$ since N

is a 2-absorbing primary subsemimodule of M. If $a^n m \in N$ or $b^n m \in N$, that is, $a^n \in (N : m)$ or $b^n \in (N : m)$, then $\sqrt{(N : m)}$ is prime. If $a^n b^n \in \sqrt{(N : M)}$, we have $(a^n b^n)^m \in (N : M)$ for some positive integer m. Thus, $ab \in \sqrt{(N : M)} = P$. Therefore, either $a \in P$ or $b \in P$ since P is prime. Hence $a \in \sqrt{(N : M)} \subseteq \sqrt{(N : m)}$ or $b \in \sqrt{(N : M)} \subseteq \sqrt{(N : m)}$. Consequently, $\sqrt{(N : m)}$ is a prime ideal of S. □

Theorem 4. *Let $f : M \mapsto M'$ be a homomorphism of a S-semimodules M and M'. If N is a 2-absorbing primary subsemimodule of M', then $f^{-1}(N)$ is also a 2-absorbing primary subsemimodule of M.*

Proof. Let $abm \in f^{-1}(N)$ for some $a, b \in S$ and $m \in M$. Then $f(abm) \in N$, that is, $abf(m) \in N$. Since N is a 2-absorbing primary subsemimodule of M', therefore $ab \in \sqrt{(N : M')}$ or $af(m) \in N$ or $bf(m) \in N$. Hence, $ab \in f^{-1}(\sqrt{(N : M')})$ or $am \in f^{-1}(N)$ or $bm \in f^{-1}(N)$. Since $f^{-1}(\sqrt{(N : M')}) \subseteq \sqrt{f^{-1}(N : M')}$, we have $f^{-1}(N)$ is a 2-absorbing primary subsemimodule of M. □

Theorem 5. *Let M be an S-semimodule, N be a 2-absorbing primary subsemimodule of M and K be a subsemimodule of M such that $K \nsubseteq N$. Then $N \cap K$ is a 2-absorbing primary subsemimodule of K.*

Proof. Clearly, $N \cap K$ is a proper subsemimodule of K. Let $abx \in N \cap K$ where $a, b \in S$ and $x \in K$. Since $abx \in N$ and N is a 2-absorbing primary subsemimodule of M, therefore either $ax \in N$ or $bx \in N$ or $ab \in \sqrt{(N : M)}$. If $ax \in N$ or $bx \in N$, then $ax \in N \cap K$ or $bx \in N \cap K$. If $ab \in \sqrt{(N : M)}$, then $(ab)^n M \subseteq N$ for some positive integer n. In particular, $(ab)^n K \subseteq N$ which implies $(ab)^n K \subseteq N \cap K$ for some positive integer n. Thus, $ab \in \sqrt{(N \cap K : K)}$. Hence $N \cap K$ is a 2-absorbing primary subsemimodule of K. □

Theorem 6. *Let M and M' be S-semimodules, $f : M \mapsto M'$ be an epimorphism such that $f(0) = 0$ and N be a subtractive strong subsemimodule of M. If N is a 2-absorbing primary subsemimodule of M with $ker f \subseteq N$, then $f(N)$ is a 2-absorbing primary subsemimodule of M'*

Proof. Let N be a 2-absorbing primary subsemimodule of M and $abx \in f(N)$ for some $a, b \in S$ and $x \in M'$. Since $abx \in f(N)$, there exists an element $x' \in N$ such that $abx = f(x')$. Since f is an epimorphism and $x \in M'$, then there exists $y \in M$ such that $f(y) = x$. As $x' \in N$ and N is a strong subsemimodule of M, therefore there exists $x'' \in N$ such that $x' + x'' = 0$, which gives $f(x' + x'') = 0$. Therefore, $abx + f(x'') = 0$ or $f(aby) + f(x'') = 0$ implies $aby + x'' \in ker f \subseteq N$. Thus, we have $aby \in N$, since N is a subtractive subsemimodule of M. Since N is a 2-absorbing primary, we conclude that $ab \in \sqrt{(N : M)}$ or $ay \in N$ or $by \in N$. Thus, $ab \in f(\sqrt{(N : M)}) \subseteq \sqrt{f(N : M)}$ or $f(ay) \in f(N)$ or $f(by) \in f(N)$ and hence $ab \in \sqrt{(f(N) : M')}$ or $ax \in f(N)$ or $bx \in f(N)$. Thus, $f(N)$ is a 2-absorbing primary subsemimodule of M'. □

A subsemimodule N of an S-semimodule M is said to be irreducible if $N = N_1 \cap N_2$, where N_1 and N_2 are subsemimodules of M, then either $N = N_1$ or $N = N_2$. We now give a characterization of 2-absorbing primary subsemimodule of an S-semimodule of M, when N is irreducible.

Theorem 7. *Let N be a proper subtractive subsemimodule of an S-semimodule M. Let $\sqrt{(N : M)}$ be a 2-absorbing ideal of S. If N is an irreducible subsemimodule of M, then N is a 2-absorbing primary subsemimodule of M if and only if $(N : r) = (N : r^2)$ for all $r \in S \setminus \sqrt{(N : M)}$.*

Proof. Let N be a 2-absorbing primary subsemimodule of M and let $r \in S \setminus \sqrt{(N : M)}$. We will show that $(N : r) = (N : r^2)$. Clearly, $(N : r) \subseteq (N : r^2)$. Let $x \in (N : r^2)$, so $r^2x \in N$. Therefore, either $rx \in N$ or $r^2 \in \sqrt{(N : M)}$, since N is a 2-absorbing primary subsemimodule. If $rx \in N$, then $x \in (N : r)$. Otherwise, $r^2 \in \sqrt{(N : M)}$ gives $r \in \sqrt{(N : M)}$, a contradiction. Thus $(N : r) = (N : r^2)$.
Conversely, let $r_1r_2x \in N$ for some $r_1, r_2 \in S$ and $x \in M$. Let $r_1r_2 \notin \sqrt{(N : M)}$. Then we will show that $r_1x \in N$ or $r_2x \in N$. We claim that $r_1 \notin \sqrt{(N : M)}$ and $r_2 \notin \sqrt{(N : M)}$ because if $r_1 \in \sqrt{(N : M)}$ and $r_2 \in \sqrt{(N : M)}$, then $r_1r_2 \in (\sqrt{(N : M)})^2 \subseteq \sqrt{(N : M)}$, which is a contradiction. Therefore we may assume that either $(N : r_1) = (N : r_1^2)$ or $(N : r_2) = (N : r_2^2)$. Suppose $(N : r_1) = (N : r_1^2)$. Let $r_1x \notin N$ and $r_2x \notin N$, then $N \subseteq (N + Sr_1x) \cap (N + Sr_2x)$. Let $y \in (N+Sr_1x)\cap(N+Sr_2x)$. Then $y = n_1+s_1r_1x = n_2+s_2r_2x$ where $n_1, n_2 \in N$ and $s_1, s_2 \in S$. Now, $r_1y = r_1n_1 + s_1r_1^2x = r_1n_2 + r_1r_2s_2x$ and $s_2r_1r_2x, rn_1, rn_2 \in N$, so $s_1r_1^2x \in N$, as N is subtractive. This implies $s_1x \in (N : r_1^2)$ but $(N : r_1) = (N : r_1^2)$. Therefore $s_1xr_1 \in N$ and so $y \in N$. Hence $(N + Sr_1x) \cap (N + Sr_2x) \subseteq N$. Consequently, $(N + Sr_1x) \cap (N + Sr_2x) = N$, a contradiction (since N is an irreducible subsemimodule). Hence N is a 2-absorbing primary subsemimodule of M. $\square$

Lemma 1. *Let N be a subtractive 2-absorbing primary subsemimodule of an S-semimodule M. Suppose that $abJ \subseteq N$ for some subsemimodule J of M. If $ab \notin \sqrt{(N : M)}$, then $aJ \subseteq N$ or $bJ \subseteq N$.*

Proof. Let $abJ \subseteq N$ for some $a, b \in S$ and for some subsemimodule J of M. Suppose $aJ \nsubseteq N$ and $bJ \nsubseteq N$, then $aj_1 \notin N$ and $bj_2 \notin N$ for some $j_1, j_2 \in J$. Since $abj_1 \in N$ and $ab \notin \sqrt{(N : M)}$ and $aj_1 \notin N$, we have $bj_1 \in N$. Again, since $abj_2 \in N$ and $ab \notin \sqrt{(N : M)}$ and $bj_2 \notin N$, we have $aj_2 \in N$. Now, $ab(j_1 + j_2) \in N$ and $ab \notin \sqrt{(N : M)}$, we have either $a(j_1 + j_2) \in N$ or $b(j_1 + j_2) \in N$. If $a(j_1 + j_2) \in N$ and $aj_2 \in N$, we get $aj_1 \in N$, a contradiction. Similarly, if $b(j_1 + j_2) \in N$ and $bj_1 \in N$, then $bj_2 \in N$ (since N is subtractive), a contradiction. Thus, $aJ \subseteq N$ or $bJ \subseteq N$. $\square$

We know that, if K is a subtractive subsemimodule, then $(K : M)$ is also subtractive. In the next theorem, we will assume that K and $\sqrt{(K : M)}$ are subtractive subsemimodule of M and subtractive ideal of S respectively.

Theorem 8. *Let K be a subtractive subsemimodule of M and $\sqrt{(K:M)}$ be a subtractive ideal of S. If K is a 2-absorbing primary subsemimodule of M, then whenever $IJN \subseteq K$ for some ideals I, J of S and a subsemimodule N of M, then $IJ \subseteq \sqrt{(K:M)}$ or $IN \subseteq K$ or $JN \subseteq K$.*

Proof. Let K be a 2-absorbing primary subsemimodule of M and let $IJN \subseteq K$ for some ideals I, J of S and a subsemimodule N of M, such that $IJ \nsubseteq \sqrt{(K:M)}$. We show that $IN \subseteq K$ or $JN \subseteq K$. If possible, suppose that $IN \nsubseteq K$ and $JN \nsubseteq K$. Then there exist $a_1 \in I$ and $b_1 \in J$ such that $a_1N \nsubseteq K$ and $b_1N \nsubseteq K$. Since $a_1b_1N \subseteq K$ and $a_1N \nsubseteq K$ and $b_1N \nsubseteq K$, we have $a_1b_1 \in \sqrt{(K:M)}$ by Lemma 1. Next, we have $IJ \nsubseteq \sqrt{(K:M)}$, therefore for some $a \in I$ and $b \in J$, $ab \notin \sqrt{(K:M)}$. Since $abN \subseteq K$ and $ab \notin \sqrt{(K:M)}$, we have $aN \subseteq K$ or $bN \subseteq K$ by Lemma 1. Here three cases arise.

Case I: $aN \subseteq K$ but $bN \nsubseteq K$. Since $a_1bN \subseteq K$ and $bN \nsubseteq K$ and $a_1N \nsubseteq K$, by Lemma 1 we have $a_1b \in \sqrt{(K:M)}$. Now, $aN \subseteq K$ but $a_1N \nsubseteq K$, therefore $(a+a_1)N \nsubseteq K$. Since $(a+a_1)bN \subseteq K$ and $bN \nsubseteq K$ and $(a+a_1)N \nsubseteq K$ implies $(a+a_1)b \in \sqrt{(K:M)}$ by Lemma 1. Since $(a+a_1)b \in \sqrt{(K:M)}$ and $a_1b \in \sqrt{(K:M)}$, we have $ab \in \sqrt{(K:M)}$, as $\sqrt{(K:M)}$ is subtractive, a contradiction.

Case II: When $bN \subseteq K$ but $aN \nsubseteq K$. Since $ab_1N \subseteq K$ and $aN \nsubseteq K$ and $b_1N \nsubseteq K$, then by Lemma 1, $ab_1 \in \sqrt{(K:M)}$. Since $bN \subseteq K$ and $b_1N \nsubseteq K$, we have $(b+b_1)N \nsubseteq K$. Since $a(b+b_1)N \subseteq K$ and $aN \nsubseteq K$ and $(b+b_1)N \nsubseteq K$, we have $a(b+b_1) \in \sqrt{(K:M)}$ by Lemma 1. Since $a(b+b_1) \in \sqrt{(K:M)}$ and $ab_1 \in \sqrt{(K:M)}$, we have $ab \in \sqrt{(K:M)}$ (since $\sqrt{(K:M)}$ is subtractive), a contradiction.

Case III: When $aN \subseteq K$ and $bN \subseteq K$. Since $bN \subseteq K$ and $b_1N \nsubseteq K$ it implies $(b+b_1)N \nsubseteq K$. Since $a_1(b+b_1)N \subseteq K$ and $(b+b_1)N \nsubseteq K$ and $a_1N \nsubseteq K$, we conclude that $a_1(b+b_1) \in \sqrt{(K:M)}$, by Lemma 1. Since $a_1b_1 \in \sqrt{(K:M)}$ and $a_1(b+b_1) \in \sqrt{(K:M)}$, we have $a_1b \in \sqrt{(K:M)}$, as $\sqrt{(K:M)}$ is subtractive. Again, $aN \subseteq K$ and $a_1N \nsubseteq K$ implies $(a+a_1)N \nsubseteq K$. Since $(a+a_1)b_1N \subseteq K$ and $(a+a_1)N \nsubseteq K$ and $b_1N \nsubseteq K$, then we have $(a+a_1)b_1 \in \sqrt{(K:M)}$ by Lemma 1. Since $a_1b_1 \in \sqrt{(K:M)}$ and $(a+a_1)b_1 \in \sqrt{(K:M)}$, then $ab_1 \in \sqrt{(K:M)}$. Since $(a+a_1)(b+b_1)N \subseteq K$ and $(a+a_1)N \nsubseteq K$ and $(b+b_1)N \nsubseteq K$, then by Lemma 1 $(a+a_1)(b+b_1) \in \sqrt{(K:M)}$. Since $ab_1, a_1b, a_1b_1 \in \sqrt{(K:M)}$, we have $ab \in \sqrt{(K:M)}$ (since $\sqrt{(K:M)}$ is subtractive), a contradiction. Hence $IN \subseteq K$ or $JN \subseteq K$.

□

Definition 2. ([3], Definition 1) A subsemimodule N of an S-semimodule M is called a partitioning subsemimodule (=Q-subsemimodule) if there exists a nonempty subset Q of M such that

(i) $SQ \subseteq Q$, where $SQ = \{rq : r \in S, q \in Q\}$;
(ii) $M = \cup\{q + N : q \in Q\}$;
(iii) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$.

Let M be an S-semimodule, and let N be a Q-subsemimodule of M. Define $M/N_{(Q)} = \{q + N : q \in Q\}$. Then $M/N_{(Q)}$ forms an S-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q_1+N) \oplus (q_2+N) = q_3+N$ where $q_3 \in Q$ is the unique element such that $q_1+q_2+N \subseteq (q_3+N)$ and $r \odot (q_1+N) = q_4+N$, where $r \in S$ and $q_4 \in Q$ is the unique element such that $rq_1 + N \subseteq q_4 + N$. Then, this S-semimodule $M/N_{(Q)}$ is called the quotient semimodule of M by N. By the definition of Q-subsemimodule, there exists a unique $q_0 \in Q$ such that $0_M + N \subseteq q_0 + N$. Then $q_0 + N$ is a zero element of M/N. But, for every $q \in Q$ from (i) one obtains $0_M = 0_s q \in Q$; hence $q_0 = 0_M$.
For deeper understandings of Q-subsemimodules of semimodule, we refer ([3],[4], [8],[14]).

Theorem 9. *Let M be an S-semimodule, N be a Q-subsemimodule of M and P be a subtractive subsemimodule of M such that $N \subseteq P$. Then P is a 2-absorbing primary subsemimodule of M if and only if $P/N_{(Q\cap P)}$ is a 2-absorbing primary subsemimodule of $M/N_{(Q)}$.*

Proof. Let P be a 2-absorbing primary subsemimodule of M. Let $a, b \in S$ and $q + N \in M/N_{(Q)}$ be such that $ab \odot q + N = q_1 + N \in P/N_{(Q\cap P)}$ where $q_1 \in Q \cap P$ is a unique element such that $abq + N \subseteq q_1 + N$. So $abq = q_1 + x_1$, for some $x_1 \in N \subseteq P$. Since P is a 2-absorbing primary subsemimodule of M, either $(ab)^n \in (P : M)$ or $aq \in P$ or $bq \in P$ for some positive integer n. First, let $a^n b^n \in (P : M)$. Consider, $a^n b^n \odot q_2 + N = q_3 + N$ where $q_2 + N \in M/N_{(Q)}$ and $q_3 \in Q$ is a unique element such that $a^n b^n q_2 + N \subseteq q_3 + N$. So, $a^n b^n q_2 = q_3 + x_2$ for some $x_2 \in N \subseteq P$. Since $a^n b^n \in (P : M)$, we have $a^n b^n q_2 \in P$, which gives $q_3 \in P$, as P is subtractive. Thus, we have $q_3 \in Q \cap P$ which gives $a^n b^n \odot q_2 + N = q_3 + N \in P/N_{(Q\cap P)}$ and hence $ab \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)})}$. If $aq \in P$, consider $a \odot q + N = q_4 + N$ where $q_4 \in Q$ is a unique element such that $aq + N \subseteq q_4 + N$. This gives, $aq = q_4 + x_3$ for some $x_3 \in N \subseteq P$. Since P is subtractive, we have $q_4 \in P$ Hence $a \odot (q + N) = q_4 + N \in P/N_{(Q\cap P)}$. Similarly, we can prove that $b \odot (q + N) \in P/N_{(Q\cap P)}$. Consequently, $P/N_{(Q\cap P)}$ is a 2-absorbing primary subsemimodule of $M/N_{(Q)}$.

Conversely, let $P/N_{(Q\cap P)}$ be a 2-absorbing primary subsemimodule of $M/N_{(Q)}$. Let $abx \in P$ for some $a, b \in S$ and $x \in M$. Since, N is a Q-subsemimodule of M and $x \in M$, we have $x \in q + N$ where $q \in Q$. So $abx \in abq + N$. Now, let $ab \odot (q+N) = q_5 + N$ where $q_5 \in Q$ is a unique element such that $abq + N \subseteq q_5 + N$. This gives, $abx = q_5 + x_4$ for some $x_4 \in N \subseteq P$. Therefore, we have $q_5 \in P$, since P is subtractive. Thus, $q_5 \in Q \cap P$ and hence $ab \odot (q + N) = q_5 + N \in P/N_{(Q\cap P)}$. Thus, we have $a^m b^m \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ or $a \odot (q+N) \in P/N_{(Q\cap P)}$

or $b \odot (q+N) \in P/N_{(Q\cap P)}$ for some positive integer m, since $P/N_{(Q\cap P)}$ is a 2-absorbing primary subsemimodule. Let $a^m b^m \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ for some positive integer m. Let $y \in M$. Then there exists a unique element $q_6 \in Q$ such that $y \in q_6 + N \in M/N_{(Q)}$. Now, $a^m b^m \odot (q_6 + N) \in P/N_{(Q\cap P)}$. Therefore, there exists a unique element $q_7 \in Q \cap P$ such that $a^m b^m q_6 + N \subseteq q_7 + N$ which gives $a^m b^m y \in a^m b^m q_6 + N \subseteq q_7 + N$. Thus, $a^m b^m y = q_7 + x_5 \in P$ for some $x_5 \in N \subseteq P$. Hence $a^m b^m y \in P$ and hence $a^m b^m \in (P : M)$. Let $a \odot (q+N) \in P/N_{(Q\cap P)}$. Then, there exists unique $q_8 \in Q \cap P$ such that $aq + N \subseteq q_8 + N$. We have, $x \in q + N$ implies $ax \in aq + N \subseteq q_8 + N$. Therefore, $ax = q_8 + x_6$ for some $x_6 \in N \subseteq P$. Hence $ax \in P$. Similarly, $bx \in P$. □

Theorem 10. *Let M be an S-semimodule, N be a Q-subsemimodule of M and P be a subtractive subsemimodule of M such that $N \subseteq P$. If N and $P/N_{(Q\cap P)}$ are 2-absorbing primary subsemimodules of M and $M/N_{(Q)}$ respectively, then P is a 2-absorbing primary subsemimodule of M.*

Proof. Let N and $P/N_{(Q\cap P)}$ be 2-absorbing primary subsemimodules of M and $M/N_{(Q)}$ respectively. Let $abx \in P$ for some $a, b \in S$ and $x \in M$. If $abx \in N$, then we are done (since N is a 2-absorbing primary subsemimodule of M). So, let $abx \notin N$. Since $x \in M$, there exists a unique element $q_1 \in Q$ such that $x \in q_1 + N$ gives $abx \in ab \odot (q_1 + N)$. This gives, $abx \in abq_1 + N \subseteq q_2 + N$ where q_2 is a unique element of Q. Since $abx \in P$ and $N \subseteq P$, we have $q_2 \in P$. Therefore, $ab \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Thus, either $ab \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)})}$ or $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Now, it is similar to the proof of the converse part of the last theorem. □

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