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# A note on weak structures due to Császár

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**Abstract.** Weak structures has been introduced by Á. Császár and it has been shown that every generalized topology and every minimal structure is a weak structure. Recently E. Ekici introduced and studied the structure $r(w)$ in a weak structure $w$ on $X$. In general the structure $r(w)$ need not be a topology on $X$. In this paper we have shown that under some conditions $r(w)$ is a topology on $X$. Further, comparision of two weak structures has been studied.

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In [2], Császár introduced and studied generalized stuctures and in [1, 3] introduced generalized operators. Recently in [4], Császár introduced a new notion called weak structures. Let $X$ be a non-empty set and $\mathcal{P}$ be its power set. A structure on $X$ is a subset of $\mathcal{P}$ and an operation on $X$ is a function from $\mathcal{P}$ to $\mathcal{P}$. A structure $w$ on $X$ is called a weak structure on $X$ if and only if $\emptyset \in w$ [4]. Weak structures are briefly notrd as WS. If $w$ is a WS on $X$, then every member of $w$ is known as $w$-open and complement of a $w$-open set is known as $w$-closed. Let $w$ be a WS on $X$ and $A \subset X$ then the union of all $w$-open subsets of $A$ is denoted as $i_wA$ and the intersection of all $w$-closed sets containing $A$ is denoted as $c_wA$. Further with the help of $i_w$ and $c_w$, several other structures such as $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ have been introduced and studied in [4]. E. Ekici in [5], studied properties of the structures $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ and introduced $r(w)$ and $rc(w)$. It is also shown that if $w$ is a WS on $X$ then each of the structures $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ is a generalized topology. So it is natural to ask which structure under which condition becomes topology. In this paper, we have shown that under some conditions $r(w)$ is a topology.

**Definition 1.** [5] Let $w$ be a WS on X and $A \subset X$. Then
(i) $A \in r(w)$ if $A = i_w(c_w(A))$,
(ii) $A \in rc(w)$ if $A = c_w(i_w(A))$.

**Lemma 1.** *Let $w$ be a WS on $X$, then $\emptyset \in r(w)$ if any one of the followings holds:*
*(i) there exist $U, V \in w$ such that $(X - U) \cap (X - V) = \emptyset$.*
*(ii) $\bigcap_{X-U\in w} U = \emptyset$.*
*(iii) for $\bigcap_{X-U\in w} U = V \neq \emptyset$ there does not exist any $W \in w$ such that $W \subset V$.*

*Proof.* (i) Let $w$ be a WS on $X$ and $U, V \in w$ be such that $(X - U) \cap (X - V) = \emptyset$. Since $(X-U)$ and $(X-V)$ are two disjoint $w$-closed sets, $c_w(\emptyset) = \emptyset$. So $i_w(c_w(\emptyset)) = \emptyset$. Hence $\emptyset \in r(w)$.

(ii) If $\bigcap_{X-U\in w} U = \emptyset$, then $c_w(\emptyset) = \emptyset$. Thus $i_w(c_w(\emptyset)) = \emptyset$. Hence $\emptyset \in r(w)$.

(iii) If $\bigcap_{X-U\in w} U = V \neq \emptyset$, then $c_w(\emptyset) = V$. Since there does not exist any $W \in w$ such that $W \subset V$, $i_w(V) = \emptyset$. Thus $i_w(c_w(\emptyset)) = i_w(V) = \emptyset = \emptyset$. Hence $\emptyset \in r(w)$. □

**Lemma 2.** *Let $w$ be a WS on $X$, then $X \in r(w)$ if either $X \in w$ or $\bigcup_{U\in w} U = X$.*

**Lemma 3.** *If $w$ is a WS on $X$ and $U \in w$ is such that for every $V \in w$, $V \subset (X-U)$, then $X \in rc(w)$.*

**Lemma 4.** *Let $w$ be a WS on $X$ in which every pair of members of $w$ is disjoint and $\bigcup_{U\in w} U = X$. Then every member of $w$ belongs to $r(w)$.*

*Proof.* Let $w$ be a WS on $X$. Let every pair of members of $w$ be disjoint and $\bigcup_{U\in w} U = X$. Then for every $A \in w$, $c_w A = \cap\{B : B \in w, A \subset (X - B)\} = A$. Since $A \in w$, $i_w c_w A = i_w A = A$. Thus $A \in r(w)$. □

**Lemma 5.** *Let $w$ be a WS on $X$ in which every pair of members of $w$ is disjoint and $\bigcup_{U\in w} U = X$. Then arbitrary union of members of $w$ belongs to $r(w)$.*

*Proof.* Let $w$ be a WS on $X$ and let $A_\alpha$ be a collection of members of $w$. Since $\bigcup_{U\in w} U = X$ and every pair of members of $w$, $c_w(\cup A_\alpha) = \cap\{B : (X - B) \in w, \cap A_\alpha \subset B\} = \cup A_\alpha$. So $i_w c_w(\cup A_\alpha) = i_w(\cup A_\alpha) = \cup A_\alpha$. Thus $\cup A_\alpha \in w$. □

**Theorem 1.** *Let $w$ be a WS on $X$ in which every pair of members of $w$ is disjoint and $\bigcup_{U\in w} U = X$. Then $r(w)$ is a topology on $X$.*

*Proof.* Since every pair of members of $w$ is disjoint and $\bigcup_{U\in w} U = X$, either (ii) or (iii) of Lemma 1 holds. Thus $\emptyset \in r(w)$. Since $\bigcup_{U\in w} U = X$, by Lemma 2, $X \in r(w)$. By Lemma 4, every member of $w$ belongs to $r(w)$ and arbitrary union of members of $w$ also belogs to $w$ by Lemma 5. Since the intersection of members of $w$ is empty, finnite intersection of members of $w$ belongs to $r(w)$. Hence $r(w)$ is a topology on $X$. □

*Remark* 1. Let $w$ be a WS on $X$ in which every pair of members of $w$ is disjoint and $\bigcup_{U\in w} U = X$. Then it can also be shown that $rc(w)$ is a topology on $X$.

Let $w$ and $\nu$ be two structures on $X$. The structure $\nu$ is said to be finer than $w$ if for every member of $w$ is a member of $\nu$. The power set $\mathcal{P}$ of $X$ is the finest structure on $X$ and $\{\emptyset\}$ is the weakest structure on $X$. Two structures $w$ and $\nu$ are said to be non-comparable if neither $w$ is finer than $\nu$ nor $\nu$ is finer than $w$.

*Observation* 1. Let $w$ and $\nu$ be two WSs on $X$ and $\nu$ is finer than $w$. Then $r(w)$ and $r(\nu)$ are non-comparable.

*Observation* 2. Let $w$ and $\nu$ be two WSs on $X$. Then
(i) $r(w) \cap r(\nu) \neq r(w \cap \nu)$.
(ii) $r(w) \cup r(\nu) \neq r(w \cup \nu)$.

The above observations are established by the following example.

**Example 1.** Let $X = \{a, b, c\}$, $w = \{\emptyset, \{a\}, \{b\}\}$ and $\nu = \{\emptyset, \{a\}, \{b\}, \{b, c\}$.
(i) Then $\nu$ is finer than $w$ but $r(w) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $r(\nu) = \{\emptyset, \{a\}, \{b, c\}, X\}$ are non-comparable.
(ii) $r(w \cap \nu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $r(w) \cap r(\nu) = \{\emptyset, \{a\}\}$.
(iii) $r(w \cup \nu) = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $r(w) \cup r(\nu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$

**Lemma 6.** *Let $w$ and $\nu$ be two WSs on $X$ and $\nu$ is finer than $w$. Then $r(w) \cap r(\nu) \subset r(w \cap \nu)$ and $r(w \cup \nu) \subset r(w) \cup r(\nu)$ .*

If WSs $w$ and $\nu$ are non-comparable then the above result need not hold can be seen from the following example.

**Example 2.** Let $X = \{a, b, c\}$, $w = \{\emptyset, \{a\}, \{b\}\}$ and $\nu = \{\emptyset, \{a\}, \{c\}$.
$r(w) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $r(\nu) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$.
$r(w \cup \nu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ and
$r(w) \cup r(\nu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$.

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