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On the distinction between one-dimensional Euclidean and hyperbolic spaces

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Abstract. The difference between Euclidean and hyperbolic spaces is clear starting with dimension two. However, the difference between elliptic space and both Euclidean and hyperbolic ones can be described also for dimension one. Does it mean that there is no difference between one-dimensional Euclidean and hyperbolic lines, or it is necessary to better define the difference between them? This paper proposes one possible way to draw clear distinction between one-dimensional Euclidean and hyperbolic lines.

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1 Introduction

The difference between hyperbolic (named also Lobachevsky in Russian literature) and Euclidean spaces is space curvature — zero in Euclidean case and negative in hyperbolic case. The space curvature is intrinsic space property starting with dimension two [1], it cannot be used to distinguish one-dimensional Euclidean and hyperbolic spaces.

Another approach is to distinguish geometries. Hyperbolic geometry differs from Euclidean one by Parallel axiom [2]:
Euclidean Parallel axiom: On the plane with given line l, through a point $P \notin l$ *exactly one* line a goes so that $a \cap l = \varnothing$.
Hyperbolic Parallel axiom: On the plane with given line l, through a point $P \notin l$ *at least two* lines a, b go so that $a \cap l = \varnothing$, $b \cap l = \varnothing$.

These axioms, as well as all their equivalents, assume the existence of two parallel lines, triangles or other figures, that are essentially two-dimensional.

On the other hand, there exists a clear distinction between one-dimensional elliptic and both Euclidean and hyperbolic spaces. Define points of some line *separable* if among any three different points A, B, C one (let it be B) divides the line into two half–lines, and remaining two points A, C lie on different half–lines. In this case we can speak that B lies *between* A and C. Otherwise, we call points *non-separable*.

The elliptic points are non-separable, because no point devides elliptic line into two half–lines and among any three points no one lies between two others [3]. Euclidean and hyperbolic points are separable. In order to make the difference between them, we refine the point separability property.

2 Uniform model of elliptic, Euclidean and hyperbolic lines

Before we can speak about the tuning of points separability property, we need one universal model for all three one-dimensional spaces constructed in spirit of [7].

Definition 1. Define a *characteristic* to be a number $k \in \{1, 0, -1\}$. The characteristic is *elliptic* if $k = 1$, *linear* or *parabolic* if $k = 0$ and *hyperbolic* if $k = -1$.

Definition 2. For $x, y \in \mathbb{R}^2$ define $x \odot y = x_0 y_0 + k x_1 y_1$. Define the *metaplane* $\mathbb{M}^2 = \{\mathbb{R}^2, \odot\}$.

Definition 3. Define the *line* $\mathbb{B}^1 = \{x \in \mathbb{M}^2 \,|\, x \odot x = 1,\ -x \equiv x\}$ (Figure 1).

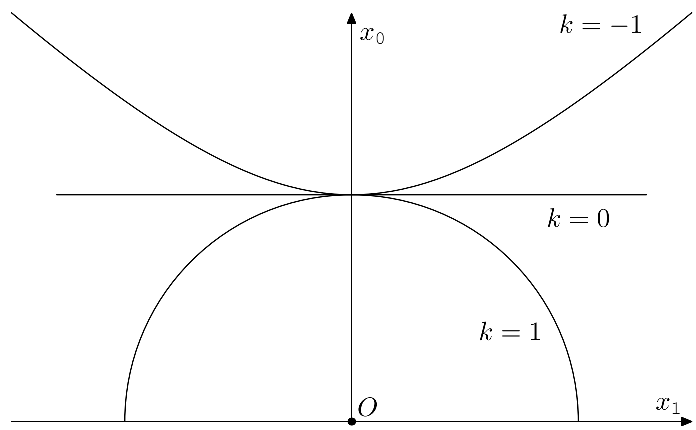


Figure 1. One-dimensional models of elliptic, Euclidean and hyperbolic spaces.

Definition 4. Define generalized by k *cosine*, *sine* and *tangent* functions as:

$$C(t) = \sum_{n=0}^{\infty} (-k)^n \frac{t^{2n}}{(2n)!} = \begin{cases} \cos t, & k = 1, \\ 1, & k = 0, \\ \cosh t, & k = -1; \end{cases}$$

$$S(t) = \sum_{n=0}^{\infty} (-k)^n \frac{t^{2n+1}}{(2n+1)!} = \begin{cases} \sin t, & k = 1, \\ t, & k = 0, \\ \sinh t, & k = -1; \end{cases}$$

$$T(t) = \frac{S(t)}{C(t)} = \begin{cases} \tan t, & k = 1, \\ t, & k = 0, \\ \tanh t, & k = -1. \end{cases}$$

Definition 5. Define the *translation by* φ in $\mathbb{B}^1$ to be the transformation with the matrix

$$\mathfrak{T}(\varphi) = \begin{pmatrix} C(\varphi) & -kS(\varphi) \\ S(\varphi) & C(\varphi) \end{pmatrix}.$$

In these definitions we obtain circle model of one-dimensional elliptic space when $k = 1$. When $k = 0$, the model is identical to one-dimensional Euclidean space identified by the equation $x_0 = 1$ in the metaspace $\mathbb{M}^2$. When $k = -1$ we have hyperbola model of hyperbolic one-dimensional space. This model is equivalent to Beltrami–Klein model of hyperbolic space if instead of coordinates x_0, x_1 use one:

$$x' = \frac{x_1}{x_0},$$

and is equivalent to Poincaré model in a disk if:

$$x' = \frac{x_1}{1 + x_0}.$$

It is important to mention that whatever model or coordonate system is used for one-dimensional space it is always possible to reconstruct its metaplane $\mathbb{M}^2$ by fixing some point O as origin with homogeneous coordonates $(1 : 0)$ and for some line point X coordonates will be $(C(x) : S(x))$, where x is the signed distance $|OX|$.

3 Point unconnectability and angle unmeasurability notions

Because a metaspace $\mathbb{M}^2$ is not Euclidean unless $k = 1$, we need several more important notions. These notions belong to geometry, not to space model constructions. In order to see it, we obtain them from axioms of two-dimensional elliptic, Euclidean and hyperbolic geometries using duality operation. We can generalize Parallel axiom in the following way (Figure 2):

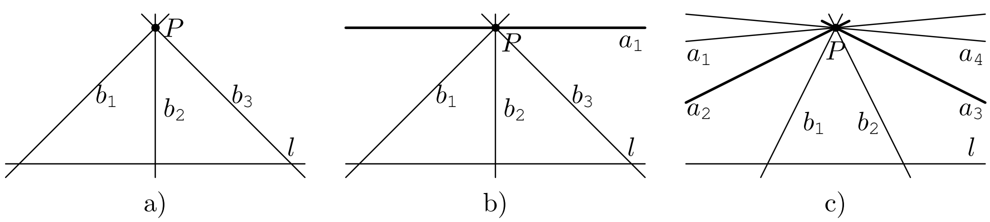


Figure 2. Parallel axiom: a) elliptic, b) Euclidean and c) hyperbolic.

Generalized Parallel axiom: On the plane with given line l, through a point $P \notin l$ 0^k lines $\{a_i\}$ go so that $a_i \cap l = \varnothing$.

Remark. The symbol 0^k is not used in calculus. Its value is:

$$0^k = \begin{cases} 0, & k = 1, \\ 1, & k = 0, \\ \infty, & k = -1. \end{cases}$$

Duality operation on a plane means exchanging the following relations:

$$\begin{aligned} \text{Point } P &\longleftrightarrow \text{Line } p, \\ P \in l &\longleftrightarrow p \ni L, \\ P \notin l &\longleftrightarrow p \not\ni L, \\ l = AB &\longleftrightarrow L = a \cap b, \\ |AB| = \varphi &\longleftrightarrow \measuredangle ab = \varphi, \\ a \parallel b &\longleftrightarrow A, B \text{ have no common line.} \end{aligned}$$

The relation "A, B have no common line" is dual to line parallelism. Such geometries were proposed in [4, 5]. Several of them are described in [6–9].

Definition 6. Two points A, B are *unconnectable* if they have no common line.

In order to see different types of points unconnectability, we need new axiom. Let formulate Connectability axiom, dual to Parallel axiom (Figure 3):
Connectability axiom: On the plane with given point L, in the line $p \not\ni L$ 0^k points $\{A_i\}$ lie so that A_i, L are unconnectable.

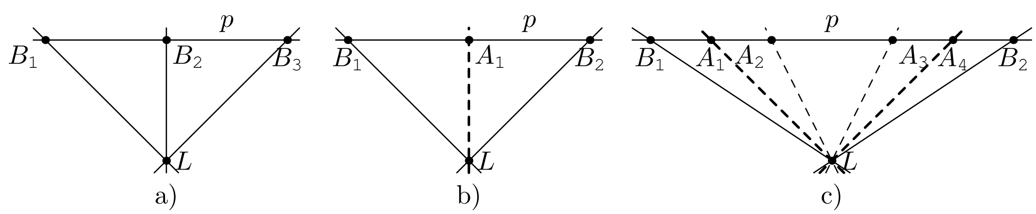


Figure 3. Connectability axiom: a) elliptic, b) parabolic and c) hyperbolic.

Remark. As in the case of hyperbolic Parallel axiom (Figure 2, c), the limit case between non-intersected and intresected lines is two parallel lines (bold ones), for hyperbolic Connectability axiom (Figure 3, c), the limit case between connectable and unconnectable points is two unconnectable points (also marked with bold).

Remark. Elliptic variant of Connectability axiom is equivalent to the following statement: "Any two different points can be connected by a line", that holds for elliptic, Euclidean and hyperbolic geometries.

Definition 7. Define some angle to be *measurable* if any point from its interior (including the rays) is either connectable or unconnectable with the vertex. Define an angle to be *unmeasurable* if its interior (including the rays) contains both connectable and unconnectable points with the vertex.

4 Points separability in a line

In order to draw the difference between Euclidean and hyperbolic cases of separable points, give more precise definition [10]. This definition is based only on points connectability notion. Although the Connectability axiom also assumes at least two-dimensional plane, this plane is nothing more than extended space of one-dimensional line — its metaplane. No geometric objects are involved other than objects of an one-dimensional line with its structure.

Definition 8. We call points on a line *non-separable* if all points on this line are connectable with any point on the metaplane. We call points on a line *separable* if for any three points A, B, C on this line and some point D on the metaplane, connectable with A, C and unconnectable with B, the angle $\angle ADC$ is unmeasurable (Figure 4).

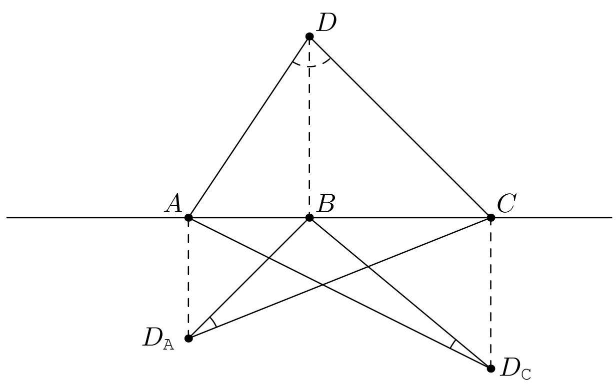


Figure 4. Points separability on a line.

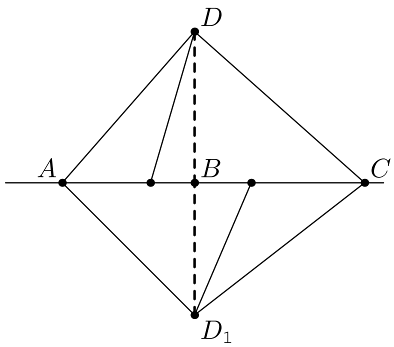
Remark. For separable points A, B, C only a single point (B) has the described property. For other points (A, C) and some unconnectable with them points D_A, D_C, the angles $\angle BD_AC$ and $\angle AD_CB$ are measurable.

If points of some line are separable, then the point B devides the line into two half–lines and points A and C lie on different half–lines defined by B. When points of some line are non-separable, then no point devides the line into half–lines.

Definition 9. In the case of separable points we say that the point B *lies between* points A and C.

Remark. In the case of non-separable points on a line, among any three points no one divides the line into half–lines, and it is impossible to talk about the position of some point between other two.

Definition 10. We call points on some line *weak separable* (Figure 5, left) if any point D of the line metaplane, being unconnectable with point B (that lies between A and C) and connectable with both A, C, is also connectable with all points from some neighborhood of B. We call points on some line *strong separable* (Figure 5, right) if in the same conditions any point D is unconnectable not only with B, but also with all points from some its neighborhood.



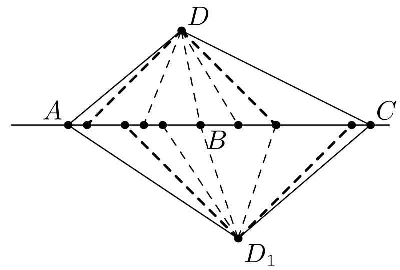


Figure 5. Points separability on a plane: weak (left) and strong (right).

In this definitions, points of elliptic line are non-separable, points of Euclidean line are weak separable, and points of hyperbolic line are strong separable.

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