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Primary decomposition of general graded structures

Emil Ilić-Georgijević, Mirjana Vuković

Abstract. In this paper we discuss the primary decomposition in the case of general graded modules – moduloids, a generalization of already done work for general graded rings – anneids. These structures, introduced by Marc Krasner are more general than graded structures of Bourbaki since they do not require the associativity nor the commutativity nor the unitarity in the set of grades. After proving the existence and uniqueness of primary decomposition of moduloids, we breafly turn our attention to Krull's Theorem and to the existence of the primary decomposition of Krasner–Vuković paragraded rings.

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1 Introduction

The graded primary decomposition of graded modules graded by torsion free Abelian groups can be found in Bourbaki [1]. The more general case of graduations by finitely generated Abelian groups is covered in papers of M. Perling, S. D. Kummar [16], and S. D. Kummar, S. Behara [15]. However, M. Krasner introduced the theory of general graded structures (groups, rings, modules) where nothing is assumed for the set of grades except its nonemptiness, since, in his definitions, additive graduation and structures of rings and modules will imply operations (generally partial) in the set of grades. It all started when M. Krasner defined the notion of a corpoid while he was studying valued division rings [4]. Corpoid is actually the homogeneous part of a division ring, graded by an arbitrary set, with induced operations among which the induced addition is, naturally, a partial operation, since the sum of two homogeneous elements does not have to be homogeneous. As a generalization of a corpoid, we have the notion of an anneid – the homogeneous part of a graded ring, and finally, the notion of a moduloid over an anneid. The general graded theory continued to develop [2, 3, 5–9, 14]. Particularly, M. Chadeyras considered the existence and uniqueness of the primary decomposition of commutative anneids in [2]. Unlike the abstract case, an irreducible ideal of a Noetherian anneid is not in general a primary ideal, but it is under certain assumptions. In this paper we will extend these results to the case of moduloids. Since proofs of propositions in the case of moduloids are similar to ones in the case of anneids discussed in [2], which were inspired by those given by O. Zariski, P. Samuel in [19], we will give proofs only of some results. Krull's Theorem for moduloids is followed by the brief observation of the existence of primary decomposition of Noetherian paragraded rings, actually of their homogeneous parts. Paragraduations were introduced by M. Krasner and M. Vuković [11–14] and these structures are closed with respect to the direct sum (direct product) in the sense that the homogeneous part of obtained structure is the direct sum (direct product) of homogeneous parts of components (see [14, 18]), which is not necessarily true for their graded counterparts.

2 Preliminaries

According to the definition of Bourbaki–Krasner [10, 14], a graded group G, with the neutral element e, is a group with *graduation* $\gamma : \Delta \to \mathrm{Sg}(G)$, $\gamma(\delta) = G_\delta$ ($\delta \in \Delta$), such that $G = \bigoplus_{\delta\in\Delta} G_\delta$, where Δ is a nonempty set, called the *set of grades*, and $\mathrm{Sg}(G)$ the set of subgroups of G. We also say that G has graduation (Δ, γ). The set $H = \bigcup_{\delta\in\Delta} G_\delta$ is called the *homogeneous part* of G and elements from H are called *homogeneous*. The grade $\xi \in \Delta$ for which $e \neq x \in G_\xi$ is called the *grade* of a homogeneous element x and is denoted by $\delta(x)$. The neutral element e does not have a grade but we may associate a grade to it from $\Delta \setminus \Delta^*$, where $\Delta^* = \{\delta \in \Delta \mid G_\delta \neq \{e\}\}$, and call it the *zero grade* and write $\delta(e) = 0$. If $\Delta = \Delta^* \cup \{0\}$, a graded group $G = \bigoplus_{\delta\in\Delta} G_\delta$ is called *proper*. We will assume graduations throughout the paper to be proper. Multiplicative operation on G induces the partial operation on H. If $x, y \in H$, then xy is defined in H if and only if $xy \in G$ is the element from H, and in that case the result is the same and we write it in the same way. If this situation occurs, we say that the elements x, y are *composable* or *multipliable* (*addible* in the case of an additive operation) and we write $x\#y$ [10, 14]. Elements x, y are composable if and only if they come from the same subgroup $G(a) = \{x \in H \mid ax \in H\}$, $a \in H^* = H \setminus \{e\}$. In [10, 14] it is proved that when H with the operation induced from G is given, then we may reconstruct G up to H-isomorphism, namely $G = \bigoplus_{G(a)\in D^*} G(a)$, where $D^* = \{G(a) \mid a \in H^*\}$. Also, from [10, 14] we know that H is the homogeneous part of some graded group G, with operation induced from that group, if and only if:

i) $(\exists e \in H)(\forall x \in H)\ x\#e \wedge xe = x$;

ii) $(\forall x \in H)\ x\#x$;

iii) $(\forall x, y, z \in H)\ x\#y \wedge y\#z \wedge y \neq e \Rightarrow x\#z$;

iv) for all $a \in H^*$, $H(a) = \{x \in H \mid a\#x\}$ is a group.

The structure that satisfies formentioned axioms is called a *homogroupoid* [10].

If R is a ring whose additive group has a proper graduation (Δ, γ), it is called a *graded ring* if for all $\xi, \eta \in \Delta$ there exists $\zeta \in \Delta$ such that $R_\xi R_\eta \subseteq R_\zeta$ [10]. Clearly, this ζ is unique if $R_\xi R_\eta \neq \{0\}$. According to Krasner's Lemma [3], if (a, a') and (b, b') are two pairs of elements of the same grade and if $ab \neq 0$, $a'b' \neq 0$, then ab and $a'b'$ are of the same grade. Hence, we may define a multiplicative operation on

Δ in the following manner [10,14]: if $\xi, \eta \in \Delta$ we put

$$\xi\eta = \begin{cases} 0, & R_\xi R_\eta = \{0\}; \\ \zeta, & \{0\} \neq R_\xi R_\eta \subseteq R_\zeta. \end{cases}$$

This multiplication does not have to be associative. However, if $R_\xi R_\eta R_\zeta \neq \{0\}$, then $(\xi\eta)\zeta = \xi(\eta\zeta)$. Thus, this definition represents a generalization of the definition of a graded ring given by Bourbaki [1]. If $A = \bigcup_{\delta\in\Delta} R_\delta$ – the homogeneous part of R, then A, with respect to operations induced by R, satisfies the following axioms [10,14]:

i) A is a multiplicative semigroup with a biabsorbing element 0, i.e. with an element 0 for which $a0 = 0a = 0$, for all $a \in A$;

ii) $(\forall a, b, c \in A)\ a\#b \wedge b\#c \wedge b \neq 0 \Rightarrow a\#c$;

iii) $(\forall a \in A)\ a\#0$;

iv) $(\forall a, b \in A)\ a\#b \Rightarrow b\#a$;

v) $(\forall a \in A)\ a\#a$;

vi) $(\forall a \in A \setminus \{0\})$ the set $A(a) = \{x \in A \mid x\#a\}$ is a group with respect to addition;

vii) distributivity of multiplication with respect to addibility and additivity holds.

An *anneid* [10] is a nonempty set A which satisfies the axioms $i) - vii)$. If A is an anneid, then we put $\Delta = \Delta^* \cup \{0\}$, where $\Delta^* = \{A(a) \mid a \in A \setminus \{0\}\}$. Then $\overline{A} = \bigoplus_{A(a)\in\Delta^*} A(a)$ is a graded ring and is called an *associated graded ring* to an anneid A or a *linearization* of A [2,3,10].

Let R be a graded ring with graduation (Δ, γ), $(M, +)$ a commutative graded group with graduation (D, Γ), and let M be an R-module with external multiplication $(a, x) \to ax$ $(a \in R,\ x \in M)$. M is a *graded R-module* if $(\forall \xi \in \Delta)(\forall s \in D)(\exists t \in D)\ R_\xi M_s \subseteq M_t$ [10].

Let M be a commutative additive homogroupoid, A an anneid, and let the external multiplication $(a, x) \to ax$ $(a \in A, x \in M, ax \in M)$ have the following properties: $i)$ $a\#b \Rightarrow ax\#bx$ and $(a + b)x = ax + bx$; $ii)$ $x\#y \Rightarrow ax\#ay$ and $a(x + y) = ax + ay$; $iii)$ $a(bx) = (ab)x$. Then M is called an *A-moduloid* [10]. If A is unitary, then M is unitary if $1x = x$, for all $x \in M$. We will assume all moduloids unitary. A graded $\overline{A}$-module $\overline{M} = \bigoplus M(x)$, where $M(x)$ runs through the set of addibility groups of a homogroupoid M, is called an *associated graded module* to a moduloid M. A nonempty subset N of an A-moduloid M is called a *submoduloid* if $x - y \in N$ for all addible $x, y \in N$, and $ax \in N$ for all $a \in A$ and $x \in N$.

3 Primary decomposition of submoduloids

Like in the abstract case, we begin with the definition of a primary submoduloid of a moduloid. The notion of a primary decomposition of a submoduloid is clear enough. Also, the notion of a Noetherian moduloid is analogous to the ungraded case.

Definition 1. A submoduloid N of a moduloid M over an anneid A is called *primary* if $N \neq M$ and whenever $a \in A$, $x \in M$ and $ax \in N$ implies $x \in N$ or $a^n M \subseteq N$, for some $n \in \mathbb{N}$.

The following result is straightforward.

Lemma 1. *If N is a primary submoduloid of an A-moduloid M, then $\sqrt{N : M} = \{a \in A \mid (\exists n \in \mathbb{N})\ a^n M \subseteq N\}$ is a prime ideal of A. If $P = \sqrt{N : M}$, then we say that N is P-primary.*

Before we proceed, we need to make few observations which are analogous to what Krasner [10] and Chadeyras [2] did in the case of anneids. First, let X be a subset of a moduloid M over an anneid A. Denote by X_+ an additive homogroupoid generated by X and if X and Y are subsets of M, let $X + Y$ be the set $\{x + y \mid x \in X \wedge y \in Y \wedge x\#y\}$. If $AX \subseteq X$ and $AY \subseteq Y$, then it is easy to prove that $A(X + Y) \subseteq X + Y$. Also, if X and Y are additive subhomogroupoids of M, then $X + Y$ is also an additive subhomogroupoid and so, if X and Y are submoduloids of M, $X + Y$ is also a submoduloid of a moduloid M. If $AX \subseteq X$, then $AX_+ \subseteq X_+$ and X_+ is a submoduloid of a moduloid M. We are particularly interested in the case $X = \{m\}$. We denote $(m) = (Am)_+$. Let M and M' be two A-moduloids, where A is an anneid. The mapping $f : M \to M'$ is called a *quasihomomorphism* if $x\#y \Rightarrow f(x)\#f(y)$ and in that case $f(x+y) = f(x)+f(y)$, and also $f(ax) = af(x)$, where $x, y \in M$ and $a \in A$ (see [2, 10, 14] for more details). In [2] M. Chadeyras observed *agglutinations* $M^{(f)} = \bar{f}^{-1}(M')$, where $\bar{f} : \overline{M} \to \overline{M'}$, $f = \bar{f}|_M$, is a *quasihomogeneous homomorphism* of graded modules, that is, it is a homomorphism of modules and $f(M) \subseteq M'$ [2]. In particular, for $a \in A$, the mapping $f_a : M \to M$, $f(x) = ax$ $(x \in M)$ is a quasihomomorphism and let $M^{(a)} = \overline{f_a}^{-1}(M)$. M. Krasner in [10] proved that if M is Noetherian with every element from Δ being semiregular, that is, if M is *strong Noetherian*, then the chain $M^{(a)} \subseteq M^{(a^2)} \subseteq \dots$ is stationary, i.e. there exists n such that $M^{(a^n)} = M^{(a^{n+1})}$; the smallest such n is called an *exponent of semiregularity*. An element $\delta \in \Delta$ is called *semiregular* if the sequence $(\epsilon^{(a^n)})$ is finite, where $\epsilon^{(a)}$ is an equivalence of grades defined by $d_1 \sim d_2 \Leftrightarrow \delta(a)d_1 = \delta(a)d_2$, $d_1, d_2 \in D$, where D is the set of grades of $\overline{M}$.

Lemma 2. *Each submoduloid of a Noetherian moduloid is the intersection of finitely many irreducible submoduloids.*

Proof. This follows from Zorn's Lemma. □

The following lemma is crucial in our discussion, since an irreducible submoduloid is not in general a primary submoduloid. The assumption of strongness imposed on a Noetherian moduloid removes this issue.

Lemma 3. *Let N be a submoduloid of a strong Noetherian A-moduloid M, where A is an anneid. If N is irreducible, then it is primary.*

Proof. Suppose N is not primary. Then there exist $m \in M \setminus N$ and $a \in A$ such that $am \in N$ and $a^n M \nsubseteq N$, for every $n \in \mathbb{N}$. Then we have an ascending chain of submoduloids $(N : \{a^n\}) = \{x \in M \mid a^n x \in N\}$. Since M is Noetherian, there exists $s \in \mathbb{N}$ such that $(N : \{a^s\}) = (N : \{a^{s+1}\})$. Also, since M is strong Noetherian, by M. Krasner [10] there exists $r \in \mathbb{N}$ such that $M^{(a^r)} = M^{(a^{r+1})}$. Let $n = r + s$. Submoduloids $N_1 = N + (m)$ and $N_2 = N + (a^n)$ strictly contain N. Let $0 \neq x \in N_1 \cap N_2$. Then there exist $\alpha, \beta \in N$, $\xi \in M^{(a^n)}$ and $\eta \in A$ such that $x = \alpha + \xi a^n = \beta + \eta m$. All elements are mutually addible. Now, $ax = a\beta + \eta am$ and since $am \in N$, we have $ax \in N$ and so $\xi a^{n+1} \in N$. Let $\zeta = \xi a^r$. Then $\zeta \in M$ and $\zeta a^{s+1} = \xi a^{n+1} \in N$ and from $(N : \{a^s\}) = (N : \{a^{s+1}\})$ we have that $\xi a^n = \zeta a^s \in N$. Hence, $x \in N$. ☐

The notion of a *reduced primary decomposition* is defined as in the case of abstract modules [19].

Corollary 1. *Each submoduloid of a strong Noetherian moduloid has a reduced primary decomposition.*

Corollary 2. *A Noetherian module has a reduced primary decomposition.*

Proof. Let M be a Noetherian R-module. Then R may be viewed as a graded ring via trivial graduation, and as its homogeneous part coincides with R, it may be regarded as an anneid. Analogously, M is a moduloid over an anneid R, if observed as a graded module with trivial graduation. Hence, M is a strong Noetherian R-moduloid with the exponent of semiregularity equal to 1 and it admits a primary decomposition. ☐

Definition 2. Let M and M' be two unitary moduloids over an anneid A, $f : M \to M'$ a quasihomomorphism, and N, N' submoduloids of M, M', respectively. Then $(N')^c := f^{-1}(N')$ is a submoduloid of M called *a contraction* of N' and $N^e := \langle f(N) \rangle$ is a submoduloid of M' called *an extension* of N.

It is easy to prove the following

Lemma 4. *Let M and M' be two unitary moduloids over an anneid A and $f : M \to M'$ a quasihomomorphism. If N' is a P'-primary submoduloid of M', then $(P')^c$ is a prime ideal and $(N')^c$ is $(P')^c$-primary.*

Corollary 3. *Let M be an A-moduloid. If $\overline{M}$ is Noetherian, then M admits a reduced primary decomposition.*

Proof. Let N be a submoduloid of M and let $\overline{N}$ be its linearization, that is a homogeneous submodule of a graded module $\overline{M}$. Then, since $\overline{M}$ is Noetherian, $\overline{N}$ has a primary decomposition. The assertion now follows from the previous lemma and the fact that $N = \overline{N} \cap M$. □

Remark 1. It should be noted that if a moduloid admits a primary decomposition, then this does not imply that its linearization has the same property.

Let us now give the uniqueness theorems.

Theorem 1. *Let M be an A-moduloid and N a submoduloid with a reduced primary decomposition $N = \bigcap_i N_i$ and let $P_i = \sqrt{(N_i : M)}$. Then P_i's are prime ideals P in A for which there exists $x \in M$, $x \notin N$, such that $(N : \{x\})$ is a P-primary ideal.*

Proof. The proof is similar to the classical case [19]. Namely, if $x \in \bigcap_{j \neq i} N_j$, $x \in N_i$, then $(N : \{x\})$ contains the annihilator of M/N_i, and hence it can be proved that $(N : \{x\})$ is P_i-primary. The converse is easy as well. □

Theorem 2. *If N is a submoduloid of an A-moduloid M which has a reduced primary decomposition $N = \bigcap_i N_i$, N_i a P_i-primary, then the minimal elements of the family of all prime ideals P_i are also the minimal elements of the family of all prime ideals P which contain the annihilator of M/N.*

Proof. Let $Q_i = \mathrm{ann}(M/N_i)$. Then Q_i is a P_i-primary ideal and $\mathrm{ann}(M/N) = \bigcap_i Q_i$. Since $\bigcap_i Q_i$ represents the primary decomposition of $\mathrm{ann}(M/N)$, the assertion follows from the known result for anneids [2] which claims that a prime ideal of an anneid A contains an ideal I, which has a reduced primary decomposition $\bigcap_i Q_i$, Q_i a P_i-primary, if and only if it contains one of the P_i's. □

Theorem 3. *Let N be a submoduloid of an A-moduloid M which has a reduced primary decomposition $N = \bigcap_i N_i$, N_i a P_i-primary. The set N_i' of all elements $x \in M$ for which there exists $a \notin P_i$ such that $ax \in N$ is a submoduloid of M which is contained in N_i, and, if P_i is a minimal element of the family $\{P_i\}$, then $N_i' = N_i$.*

Proof. Let $x_1, x_2 \in N_i'$ and $x_1 \# x_2$. Then there exist $a_1, a_2 \notin P_i$ such that $a_1x_1, a_2x_2 \in N$, and hence, $a_1a_2(x_1 - x_2) \in N$ while $a_1a_2 \notin P_i$, which proves that N is a submoduloid of M. The inclusion $N_i' \subseteq N_i$ is clear enough. If P_i is a minimal element of the family $\{P_i\}$, then for all $j \neq i$, we have $P_j \nsubseteq P_i$. Let $a_j \in P_j \setminus P_i$, $n(j) \in \mathbb{N}$ be such that $a_j^{n(j)} M \subseteq N_j$ and $a = \prod_{j \neq i} a_j^{n(j)}$. Then $a \notin P_i$ and, if $x \in N_i$, then $ax \in N$, which means that $x \in N_i'$. □

4 Krull's Theorem

The proof of the following result runs exactly as in the abstract case [19].

Lemma 5. *If A is a strong Noetherian anneid and M a Noetherian A-moduloid, then if Q is an ideal of A and N a submoduloid of M, then there exist an integer s and a submoduloid N' of M such that $QN = N \cap N'$ and $N' \supseteq Q^s M$.*

Lemma 6. *Let Q be an ideal of a unitary anneid A and let N be a submoduloid of M. If $N = QN$ and if N is finitely generated, then for all $0 \neq x \in N$, $x \in Qx$.*

Proof. Let $\{x_1 = x, x_2, \ldots, x_n\}$ be the generators of N. From $N = QN$ we have that each x_i can be written as a linear combination of $x_1, \ldots, x_n$ over Q. Thus, we have n equations

$$-\mu_1^i x_1 - \cdots - (\mu_i^i - 1)x_i - \cdots - \mu_j^i x_j - \cdots = 0 \qquad i = 1, \ldots, n,$$

where $\mu_j^i \in \overline{Q}$. If we do the calculations in $\overline{M}$, we will obtain a determinant of coefficients equal to $1 - \mu$, where $\mu \in \overline{Q}$, such that $(1 - \mu)x_1 = (1 - \mu)x = 0$. By regrouping the addible elements of M in the development of $(1 - \mu)x$, we get $(1-\alpha)x + \beta_1 x + \cdots + \beta_s x = 0$, $\alpha, \beta_1, \ldots, \beta_s \in \overline{Q}$. Since $\overline{M}$ is the direct sum of groups of addibility, we have that $(1 - \alpha)x = 0$ which implies that $\alpha = \sum_k a_k \in \overline{Q}$, where x and $a_k x$ are mutually addible. □

Now we may formulate and prove Krull's Theorem for moduloids.

Theorem 4 (Krull's Theorem)**.** *Let A be a Noetherian strong anneid with unity, and Q an ideal of A. If M is a Noetherian A-moduloid, then $\bigcap_{n=1}^{\infty} Q^n M = \{0\}$ if and only if $x \notin Qx$, for all $0 \neq x \in M$.*

Proof. Let $x \neq 0$ and $x \in Qx$. Then there exists $\alpha = \sum_k a_k$ $(a_k \in Q)$ such that a_k are mutually addible as well as $a_i x \# a_j x$ $(i \neq j)$ and $x = \alpha x = \sum_k a_k x$. Hence,

$$x = \sum_k a_k (\sum_{k'} a_{k'} x) = \sum_k a_k (\sum_{k'} a_{k'} (\sum_{k''} a_{k''} x)) = \ldots$$

and so $x \in Q^n M$ for all $n \in \mathbb{N}$. Thus, $\bigcap_{n=1}^{\infty} Q^n M \neq \{0\}$. Conversely, let $\bigcap_{n=1}^{\infty} Q^n M = N \neq \{0\}$. Then there exist an integer s and a submoduloid N' such that $QN = N \cap N'$ with $N' \supseteq Q^s M$. So, $QN \supseteq N \cap Q^s M = N$, which means that $N = QN$, and so for all $0 \neq x \in N \subseteq QM$, $x \in Qx$. □

5 Primary decomposition of quasianneids

We start by giving less known notions introduced in [11–14]. The mapping $\pi : \Delta \to \mathrm{Sg}(G)$, $\pi(\delta) = G_\delta$ $(\delta \in \Delta)$, of partially ordered set $(\Delta, <)$, which is from below a complete semi-lattice and from beyond inductively ordered, to the set $\mathrm{Sg}(G)$ of subgroups of the group G, is called a *paragraduation* if it satisfies the following six-axiom system:

i) $\pi(0) = G_0 = \{e\}$, where $0 = \inf \Delta$; $\delta < \delta' \Rightarrow G_\delta \subseteq G_{\delta'}$;

Remark 2. $H = \bigcup_{\delta \in \Delta} G_\delta$ is called *the homogeneous part* of G with respect to π.

Remark 3. If $x \in H$, we say that $\delta(x) = \inf\{\delta \in \Delta \mid x \in G_\delta\}$ is the grade of x. We have $\delta(x) = 0$ if and only if $x = e$. Elements $\delta(x)$, $x \in H$, are called *principal grades* and they form a set which we will denote by Δ_p.

ii) $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta \in \theta} G_\delta = G_{\inf \theta}$;

iii) If $x, y \in H$ and $yx = zxy$, then $z \in H$ and $\delta(z) \leq \inf\{\delta(x), \delta(y)\}$;

iv) Homogeneous part H is a generating set of G;

v) Let $A \subseteq H$ be a subset such that for all $x, y \in A$ there exists an upper bound for $\delta(x), \delta(y)$. Then there exists an upper bound for all $\delta(x)$, $x \in A$;

vi) G is generated by H with the set of H-inner and left commutation relations:

1. $xy = z$ (H-inner relations);
2. $yx = z(x, y)xy$ (left commutation relations).

A group with paragraduation is called a *paragraded group.* A ring R is called *paragraded* if its additive group is paragraded and if for all $\xi, \eta \in \Delta$ there exists $\zeta \in \Delta$ such that $R_\xi R_\eta \subseteq R_\zeta$. If R is a paragraded ring with homogeneous part H, then we may observe restrictions of operations on R to H. Induced addition is partial and we write $x\#y$ if and only if $x + y \in H$. The obtained structure is called a *paraanneid* [14]. If $x \in H$, let $H(x) = \{y \in H \mid x\#y\}$. Paraanneid certainly satisfies the following axioms:

i) There exists an element $0 \in H$ such that $H = H(0)$ and such that for all $x \in H$ we have $0 + x = x$;

ii) If $a \in H$, $x + y$ is always defined on $g(a) = \{x \in H \mid H(x) \supseteq H(a)\}$ and $(g(a), +)$ is an Abelian group;

iii) If $A \subseteq H$ is such that for all $x, y \in A$ we have $x\#y$, then there exists $g \subseteq H$ such that $x + y \in g$ for all $x, y \in g$, $x \in g$ implies $g(x) \subseteq g$ and $A \subseteq g$;

iv) $H^2 \subseteq H$;

v) $x\#x'$ and $y\#y'$ imply $xy\#x'y'$.

Structure $(H, +, \cdot)$ which satisfies axioms $i) - v)$ is called a *quasianneid* [14]. A quasianneid however does not have to be a paraanneid; it is under few more assumptions [14]. Let us notice that *iv*) and *v*) imply

vi) If $x\#y$ then $z(x + y) = zx + zy$ and $(x + y)z = xz + yz$.

Let us now suppose that a paraanneid H is commutative, and let us consider the mapping $\varphi_a : x \to ax$ $(x \in H)$, where $a \in H$. It is a quasiendomorphism [14] of H (definition is analogous to the notion of a quasiendomorphism for anneids) and let $H^{[a]} = \varphi_a^{-1}(\hat{H})$, where $\hat{H} = \langle \varphi_a(H) \rangle$. The mapping φ_a defines the equivalence ϵ_a on the set of grades Δ of H in the following manner: $\delta_1 \sim \delta_2 \Leftrightarrow \delta(a)\delta_1 = \delta(a)\delta_2$. Obviously, $(H^{[a]}, +) = (H, +)$ implies discrete equivalence $\theta[x \sim y \Rightarrow x = y]$. Since $\varphi_{ab} = \varphi_a\varphi_b$, we have $H^{[ab]} \supseteq H^{[b]}$, and we write $H^{[ab]} \geq H^{[b]}$. Also, the equivalence ϵ_b is finer than ϵ_{ab} and we write $\epsilon_{ab} \geq \epsilon_b$. Hence $H \leq H^{[a]} \leq H^{[a^2]} \leq \dots$ and

$\theta \leq \epsilon_a \leq \epsilon_{a^2} \leq \dots$. If the sequence $(H^{[a^n]})$ resp. (ϵ_{a^n}) is stationary, then we say that a is a semiregular element resp. semiregular grade [14]. A paraanneid H is called *strong* [14] if every $a \in H$ is semiregular. The notion of a Noetherian paraanneid is clear. A nonempty subset Q of a paraanneid H is called an *ideal* if $x - y \in Q$ for all addible $x, y \in Q$, and if $ax \in Q$ for all $a \in H$ and $x \in Q$. The notion of a *primary ideal* is analogous to the abstract case. Now, as in the case of anneids and moduloids, one can prove the following

Theorem 5. *Each ideal of a Noetherian paraanneid is the intersection of finitely many irreducible ideals. If Q be an irreducible ideal of a strong Noetherian paraanneid H, Q is primary. A strong Noetherian paraanneid has a primary decomposition.*

If a ring R is *extragraded* [13, 14], that is, if $(R, +)$ is an extragraded group, i.e. if instead of the axiom *vi*) we have the following axiom:

vi′) If $\delta_1, \dots, \delta_s \in \Delta_p$ are incomparable in pairs and if $x_i, x'_i \in H$ $(i = 1, \dots, s)$ are such that $x_1 + \dots + x_s = x'_1 + \dots + x'_s$ and $x_i, x'_i \in R_{\delta_i}$ for all $i = 1, \dots, s$, then $\delta(-x_i + x'_i) < \delta_i$,

then the ascending chain condition on R implies the existence of a primary decomposition in the corresponding homogeneous part, which we call an *extraanneid*.

Theorem 6. *If R is an extragraded Noetherian ring, then its extraanneid has a primary decomposition.*

Proof. Let H be a homogeneous part of R. Since R is Noetherian, every ideal, and particularly, every homogeneous ideal Q, has a primary decomposition. Since R is extragraded, $Q \cap H$ is an ideal in an extraanneid H and Q is generated exactly by $Q \cap H$ [14]. Also, if Q is a primary ideal in R, then $Q \cap H$ is a primary ideal in H and the claim follows. □

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Emil Ilić-Georgijević
University of Sarajevo
Faculty of Civil Engineering
Patriotske lige 30, 71000 Sarajevo
E-mail: *emil.ilic.georgijevic@gmail.com*

Mirjana Vuković
Academy of Sciences and Arts of Bosnia and Herzegovina
Bistrik 7, 71000 Sarajevo
E-mail: *mvukovic@anubih.ba*