BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(77), 2015, Pages 20–47
ISSN 1024–7696

$l_p(R)$-equivalence of topological spaces and topological modules

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Abstract. Let R be a topological ring and E be a unitary topological R-module. Denote by $C_p(X, E)$ the class of all continuous mappings of X into E in the topology of pointwise convergence. The spaces X and Y are called $l_p(E)$-equivalent if the topological R-modules $C_p(X, E)$ and $C_p(Y, E)$ are topological isomorphisms. Some conditions under which the topological property $\mathcal{P}$ is preserved by the $l_p(E)$-equivalence (Theorems 8 – 11) are given.

Mathematics subject classification: Primary 54C35, 54C10, 54C60; Secondary 13F99, 54C40, 54H13.
Keywords and phrases: Function space, topology of pointwise convergence, support, linear homeomorphism, perfect properties, open finite-to-one properties.

1 Preliminaries

Let G be an Abelian group under addition operation and R be a ring. We call G a left R-module or, simply, an R-module if on it the operation of multiplication between an element of R and an element of G is defined, say $ra \in G$, where $r \in R$ and $a \in G$, with the following properties: $r(a + b) = ra + rb$, $(r + s)a = ra + sa$, $r(sa) = (rs)a$, for any $r, s \in R$ and $a, b \in G$. In other words, an R-module is a vector space where the base field is replaced by a base ring R. Usually the operation of multiplication $(r, a) \mapsto ra$ is called scalar multiplication. Obviously, any ring R is a module over itself. An R-module is unitary [11] if R possesses a multiplicative identity 1 and $1x = x$ for every $x \in G$.

An additive topological group is an additive group G with a topology such that the addition operation $(x, y) \to x + y$ and inverse operation $x \to -x$ are continuous mappings [11]. A topological ring is a ring R with a topology making R into an additive Abelian topological group such that the multiplication is a continuous mapping [11].

Let R be a topological ring. A topological R-module is an R-module E together with a topology such that E is an additive Abelian topological group and scalar multiplication is a continuous mapping.

Let E and F be R-modules. The mapping $\varphi : E \to F$ is a linear mapping if it satisfies the conditions:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$, for any $x, y \in E$;

(ii) $\varphi(\alpha x) = \alpha\varphi(x)$, for any $x \in E$ and $\alpha \in R$.

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Throughout this paper, by a "space" we will mean a "completely regular space", by a "topological ring" we will mean "topological unitary ring" and by a "topological module" we will mean a "topological unitary module".

Fix a space X, a topological ring R and a topological R-module E. By $C(X,E)$ we will denote the family of all E-valued continuous functions with the domain X and by $C_p(X,E)$ we will denote the space $C(X,E)$ endowed with the topology of pointwise convergence. Recall that the family of sets of the form $W(x_1,x_2,...,x_n,U_1,U_2,...,U_n) = \{f : C(X,E) : f(x_i) \in U_i \textit{ for any } i \leq n\}$, where $x_1,x_2,...,x_n \in X$, $U_1,U_2,...,U_n$ are open sets of E and $n \in \mathbb{N}$, is an open base of the space $C_p(X,E)$.

Let E be a topological R-module. The spaces X and Y are called $l_p(E)$-equivalent if the spaces $C_p(X,E)$ and $C_p(Y,E)$ are linearly homeomorphic and we denote $X \overset{E}{\sim} Y$.

Recall that an embedding of X into Y is a mapping $e : X \to Y$ such that e is a homeomorphism of X onto $e(X) \subseteq Y$.

Proposition 1. *Fix a topological R-module E. Then $C_p(X,E)$ is a topological R-module and E is embedded in a natural way in $C_p(X,E)$ as a closed submodule of $C_p(X,E)$.*

Proof. $C_p(X,E)$ is a group under operation of pointwise addition and respectively is unitary module over the ring R. We put $a_X(x) = a$ for any $a \in E$ and $x \in X$, i.e. a_X is a constant function.

Let $e : E \to C_p(X,E)$, where $e(a) = a_X(x)$ for every $a \in E$. The mapping e is injective, since, if $a, b \in E$, with $a \neq b$, then $a_X(x) = a \neq b = b_X(x)$ for every $x \in X$.

The sets $W(x_1,x_2,...,x_n,U_1,U_2,...,U_n) = \{f \in C_p(X,E) : f(x_i) \in U_i\}$, where $n \in \mathbb{N}$, $x_1,x_2,...,x_n \in X$ and $U_1,U_2,...,U_n$ are open subsets of E, form an open base of the space $C_p(X,E)$.

If $x \in X$ and U is open in E, then $e(U) = \{a_X : a \in U\} = e(E) \cap W(x_1,x_2,...,x_n,U,U,...,U)$ and $e^{-1}(W(x_1,x_2,...,x_n,U_1,U_2,...,U_n)) = \{a \in E : a_X \in U_1 \cap U_2 \cap ... \cap U_n\} = U_1 \cap U_2 \cap ... \cap U_n$. Hence, e is an embedding of E in $C_p(X,E)$.

Fix $g \in C_p(X,E)$ such that $g \notin e(E)$. There exist two distinct points $x_1, x_2 \in X$ such that $g(x_1) \neq g(x_2)$. Now we fix two open sets U_1 and U_2 from R such that $g(x_1) \in U_1$, $g(x_2) \in U_2$ and $U_1 \cap U_2 = \emptyset$. Then $g \in W(x_1,x_2,U_1,U_2)$ and $W(x_1,x_2,U_1,U_2) \cap e(R) = \emptyset$. Hence the set $e(E)$ is closed. □

A space X is zero-dimensional if $indX = 0$ (small inductive dimension is zero), i.e. X has a base of clopen (open and closed) subsets.

Proposition 2. *If E is a zero-dimensional topological R-module, then $C_p(X,E)$ is a zero-dimensional topological R-module.*

Proof. Since $C_p(X,E)$ is a subspace of E^X, the proof is complete. □

2 The evaluation mapping

Let X be a space, R be a topological ring and E be a non-trivial topological R-module. Fix $x \in X$. Then the mapping $\xi_x : C_p(X,E) \to E$ defined by $\xi_x(f) = f(x)$ is called the evaluation mapping at x.

Proposition 3. *The evaluation mapping* $\xi_x : C_p(X,E) \to E$ *is continuous and linear for every point* $x \in X$.

Proof. Fix a point $x \in X$. For every open set U of E we have $\xi_x^{-1}(U) = \{f \in C_p(X,E) : f(x) \in U\} = W(x,U)$. But $W(x,U)$ is an element of the subbase of the topology on $C_p(X,E)$, i.e. $\xi_x^{-1}(U)$ is open in $C_p(X,E)$ for every open $U \subseteq E$.

Obviously $\xi_x(f+\alpha g) = (f+\alpha g)(x) = f(x)+\alpha g(x) = \xi_x(f)+\alpha\xi_x(g)$. Hence ξ_x is a linear continuous mapping. □

We now define the canonical evaluation mapping $e_X : X \to C_p(C_p(X,E),E)$, where $e_X(x) = \xi_x$ for any $x \in X$.

Proposition 4. *The canonical evaluation mapping* $e_X : X \to C_p(C_p(X,E),E)$ *is continuous. Moreover, the set* $e_X(X)$ *is closed in the space* $C_p(C_p(X,E),E)$.

Proof. Let $U = W(f_1, f_2, ..., f_k, U_1, U_2, ..., U_k) \cap e_X(X)$ be a standard open set in $C_p(C_p(X,E),E) \cap e_X(X)$. Without loss of generality, we can assume that $U \subseteq e_X(X)$, i.e. $U = \{\xi_x \in e_X(X) : \xi_x(f_i) \in U_i, x \in X, i = \overline{1,k}\} = \{\xi_x \in e_X(X) : f_i(x) \in U_i, x \in X, i = \overline{1,k}\}$. On the other hand $e_X^{-1}(U) = \{x \in X : f_i(x) \in U_i, i = \overline{1,k}\} = \cap\{f_i^{-1}(U_i) : i = \overline{1,k}\}$. Since for every $i = \overline{1,k}$ and $f_i \in C(X,E)$, the set $e_X^{-1}(U)$ is a finite intersection of open sets, therefore it is open.

Fix $\varphi \in C_p(C_p(X,E),E) \setminus e_X(X)$. There exist $f \in C_p(X,E)$ and $b \in X$ such that $\varphi(f) \neq f(b) = \xi_b(f)$. Fix in E two open sets V and W such that $\varphi(f) \in V$, $f(b) \in W$ and $V \cap W = \emptyset$. The set $U = \{\psi C_p(C_p(X,E),E) : \psi(f) \in V\}$ is open in $C_p(C_p(X,E),E)$ and $U \cap e_X(X) = \emptyset$. The proof is complete. □

Let X and Y be spaces, Φ a family of functions $f : X \to Y$. We say that Φ *separates points* of X (or simply is *separating* [1]) if for any $x, y \in X$, $x \neq y$, there exists $f \in \Phi$ such that $f(x) \neq f(y)$. We also say that Φ *separates points from closed sets* (or is *regular* [1]) if for any closed subset F of X and any point $x \in X \setminus F$ there exists $f \in \Phi$ such that $f(x) \notin cl_Y f(F)$.

Proposition 5. *If* $C_p(X,E)$ *is a separating and regular family, then the canonical evaluation mapping* $e_X : X \to C_p(C_p(X,E),E)$ *is a homeomorphism from* X *to the subspace* $e_X(X)$ *of* $C_p(C_p(X,E),E)$.

Proof. Since canonical evaluation mapping is continuous, it is clear that it is surjective and we have only to prove that e_X is injective and the inverse function is continuous.

First, we show that e_X is injective. Let $x, y \in X$, $x \neq y$. By assumption, $C_p(X,E)$ is a separating collection, i. e., we can find a function $f \in C_p(X,E)$ such that $f(x) \neq f(y)$, hence $\xi_x \neq \xi_y$.

Now, we prove that e_X^{-1} is continuous. Let F be a closed subset of X. By assumption $C_p(X,E)$ is a regular collection, i.e. for any $x \notin F$ we can find $f \in C_p(X,E)$ such that $f(x) \notin cl_E f(F)$. Therefore $f(x)$ has a neighbourhood $U_{f(x)}$ for which $U_{f(x)} \cap f(F) = \emptyset$. Then $W(f, U_{f(x)})$ is a neighbourhood of ξ_x that is not contained in $e_X(F)$, i.e. $e_X(F)$ is closed. Hence e_X^{-1} is continuous. □

A space X is called:

(i) *R-completely regular* if for any closed subset F of X and any point $a \in X \setminus F$ there exists $f \in C(X,R)$ such that $f(a) \notin cl_R f(F)$;

(ii) *R-Tychonoff* if for any closed subset F of X, any point $a \in X \setminus F$ there exists $g \in C(X,R)$ such that $g(a) = 0$ and $F \subseteq g^{-1}(1)$.

The space R is R-completely regular. The Cartesian product of R-completely regular spaces is R-completely regular and the Cartesian product of R-Tychonoff spaces is an R-Tychonoff space. A subspace of an R-Tychonoff (R-completely regular) space is an R-Tychonoff (R-completely regular) space.

Remark 1. Let X be an R-Tychonoff space. Then:

(i) X is a Tychonoff space.

(ii) If E is a topological R-module, then for each closed set F of X, any point $a \in X \setminus F$ and any point $b \in E$, there exists $f \in C(X,E)$ such that $f(a) = 0$ and $f(F) = b$.

(iii) X is R-completely regular.

Remark 2. Let E be a topological R-module and X be an R-completely regular space. Then:

(i) X is a Tychonoff space.

(ii) For any closed subset F of X and any point $a \in X \setminus F$ there exists $f \in C(X,E)$ such that $f(a) \notin cl_E f(F)$.

(iii) $C(X,E)$ is a separating and regular family of continuous mappings.

Proposition 6. *A space X is R-completely regular if and only if the family $C(X,E)$ is separating and regular for any non-trivial topological R-module E.*

Proof. It is obvious. □

Proposition 7. *If $indX = 0$, then the space X is R-Tychonoff.*

Proof. If C is a clopen subset, then χ_C is continuous, where $\chi_C(C) = 1$ and $\chi_C(X \setminus C) = 0$. Fix a point $x \in X$ and closed subset F of X such that $x \in X \setminus F$. Since $indX = 0$ we can find a clopen subset C such that $C \subseteq X \setminus F$. Then $X \setminus C$ is also clopen and $F \subseteq cl_X F \subseteq X \setminus C$. The characteristic function $g = \chi_{X \setminus C}$ is continuous, $g(x) = 0$ and $F \subseteq g^{-1}(1)$. Hence X is R-Tychonoff. □

Let R be a topological ring. A topological R-module E is called:

(i) *simple* if it does not contain a non-trivial submodule over R;

(ii) *locally simple* if E is not trivial and there exists an open subset U of E such that $0 \in U$ and U does not contain non-trivial R-submodules of E;

(iii) *R-closed* if there exists a continuous mapping $\varphi_E : E \longrightarrow R$ onto R such that $\varphi_E(x+y) = \varphi_E(x) + \varphi_E(y)$ and $\varphi_E(tx) = t\varphi_E(x)$ for any $t \in R$ and $x, y \in E$.

Example 1. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. The rings $\mathbb{R}$ and $\mathbb{C}$ are simple rings.

Example 2. If R is the field of real numbers or of complex numbers, then any locally convex linear space over R is an R-closed module.

Example 3. Let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{R}$ and $\mathbb{C}$ are locally simple, not simple and not $\mathbb{Q}$-closed $\mathbb{Q}$-modules.

Example 4. If R is a locally simple ring, then R^n is a locally simple R-closed R-module for any natural number $n \geq 1$.

Example 5. Let $\mathbb{Z}$ be the ring of integers. Relative to discrete topology $\mathbb{Z}$ is locally simple non-simple ring. If $p \geq 2$ and $\mathbb{Z}_p = p \cdot \mathbb{Z}$, then $\mathbb{Z}_p$ is an ideal and $\{n\mathbb{Z}_p : n \in \mathbb{N}\}$ is a base at 0 of the ring topology. In that topology $\mathbb{Z}$ is not a locally simple ring.

Let R be a ring and E be an R-module. For any $a \in E$ we put $E_a = Ra = \{ta : t \in R\}$.

Lemma 1. *Let R be a ring and E be an R-module. Then Ra is an R-module for any $a \in E$.*

Proof. Fix $a \in E$, $a \neq 0$. By definition, $Ra = \{xa : x \in R\}$. In the first we will prove that Ra is an Abelian group.

Let $x, y \in Ra$. Then there exist $x_1, y_1 \in R$ such that $x = x_1 a$ and $y = y_1 a$. Then $x + y = x_1 a + y_1 a = (x_1 + y_1)a \in Ra$. Also $0 + x_1 a = x_1 a + 0 = x_1 a$ and $x_1 a + (-x_1)a = (-x_1)a + x_1 a = 0$. Hence Ra is a group under addition operation.

Now we will prove that Ra is an R-module. Let $\alpha, \beta \in R$ and $x, y \in Ra$. By definition, $x = x_1 a$ and $y = y_1 a$ for some $x_1, y_1 \in R$. Then $\alpha(x_1 a + y_1 a) = \alpha x_1 a + \alpha y_1 a \in Ra$, $(\alpha + \beta)x_1 a = \alpha x_1 a + \beta x_1 a$ and $\alpha(\beta x) = \alpha(\beta x_1 a) = (\alpha\beta)x_1 a = (\alpha\beta)x$. □

Remark 3. It is obvious that, by Lemma 1, for any simple ring R and any $a \in R$, $a \neq 0$, we have $Ra = R$, i. e. R is a field. Moreover, for any simple R-module E and any $a \in E$, $a \neq 0$, we have $Ra = E$.

Proposition 8. *Let E be a locally simple R-closed R-module, $a \in E$ and $\varphi_E(a) = 1$. Then $E_a = \{ta : t \in R\}$ is an R-submodule of E with the following properties:*

1. *$v_a = \varphi_E | E_a : E_a \longrightarrow R$ is a topological isomorphism of the R-module E_a onto R-module R.*
2. *The mapping $\psi_a : E \longrightarrow E_a$, where $\psi_a(x) = v_a^{-1}(\varphi_E(x))$ for each $x \in E$, is an open continuous homomorphism of the R-module E onto the R-module E_a.*
3. *The space E is homeomorphic to the space $\varphi_E^{-1}(0) \times E_a$.*
4. *The set E_a is closed in E.*

Proof. The set $\varphi_E^{-1}(1)$ is non-empty. Fix $a \in \varphi_E^{-1}(1)$. If $t \in R$, then $\varphi_E(ta) = t\varphi_E(a) = t \cdot 1 = t$. Thus $\varphi_E(E_a) = R$ and v_a is a one-to-one continuous homomorphism of E_a onto R. Since $(t,x) \mapsto tx$ is a continuous mapping of $R \times E$ onto E, the mapping v_a is a homeomorphism, i. e. $v_a^{-1} : R \longrightarrow E_a$ is continuous. Assertion 1 is proved. Assertion 2 follows from the Assertion 1.

The mapping $\psi : \varphi_E^{-1}(0) \times E \longrightarrow E$, where $\psi(x,y) = x + y$ for any $x \in \varphi_E^{-1}(0)$ and $y \in E_a$, is a homeomorphism. □

Let X be a space, R be a topological ring and E be a topological R-module. We will consider two subsets of $C_p(C_p(X,E),E)$:

(i) $L_p(X,E) = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n : \alpha_i \in R, x_i \in e_X(X), n \in \mathbb{N}\}$.

(ii) $M_p(X,E)$ the subspace of all linear mappings from $C_p(X,E)$ into E.

(iii) If F is a topological R-module, then $\mathfrak{L}_p(F,E)$ is the space of all linear continuous mappings $\varphi : F \to E$ as a subspace of the space $C_p(F,E)$.

Proposition 9. *Let R be a locally simple ring and X be an R-Tychonoff space. Then $M_p(X,R) = L_p(X,R)$.*

Proof. We will show that every continuous linear mapping $\mu \in M_p(X,R)$ can be written as $\mu = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$ for some concrete $n \in \mathbb{N}$, $\alpha_i \in R$, $x_i \in e_X(X)$ and $1 \leq i \leq n$.

Fix $\mu \in M_p(X,R)$. Assume that $\mu \neq 0$. Let $f \in C(X,R)$, $f = 0$. Then $\mu(f) = 0$, since μ is linear. Fix a neighbourhood U of $0 \in R$ which does not contain R-submodules. Since μ is continuous, we can find $n \in \mathbb{N}$, distinct points $x_1, x_2, ..., x_n \in X$, and $V = V_1 = V_2 = ... = V_n \subseteq U$ such that $\mu(W(x_1, x_2, ..., x_n, V_1, V_2, ..., V_n)) \subseteq U$.

Let $g \in C_p(X,R)$ and $g(x_1) = g(x_2) = ... = g(x_n) = 0$. Clearly, $\alpha g \in W(x_1, x_2, ..., x_n, V_1, V_2, ..., V_n)$ for any $\alpha \in R$. Thus $\mu(\alpha g) \in U$ and, since μ is a linear functional, we have $\alpha\mu(g) \in U$ for every $\alpha \in R$. From Lemma 1 it follows that $R\mu(g)$ is an R-submodule of R and $R\mu(g) \subseteq U$, a contradiction. Therefore $R\mu(g) = \{0\}$ and $\mu(g) = 0$.

Fix $g_i \in C(X,R)$ such that $g_i(x_i) = 1$ and $g_i(x_j) = 0$ for $j \neq i$.

We put $\alpha_i = \mu(g_i)$. Consider $\mu' = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$. Then $\mu'(g) = \alpha_1 g(x_1) + \alpha_2 g(x_2) + ... + \alpha_n g(x_n)$ for any $g \in C(X,R)$.

Let $g \in C_p(X,R)$ and $g' = g - g(x_1)g_1 - g(x_2)g_2 - ... - g(x_n)g_n$. Obviously, $g' \in C_p(X,R)$ and $g'(x_i) = 0$ for each $i \leq n$. Hence $\mu(g') = 0$. Since μ is a linear mapping, $0 = \mu(g') = \mu(g) - \mu(\Sigma\{g(x_i)g_i : i \leq n\})$ and $\mu(g) = \mu(\Sigma\{g(x_i)g_i : i \leq n\})$ $= \Sigma\{g(x_i)\mu(g_i) : i \leq n\} = \Sigma\{\alpha_i g(x_i) : i \leq n\}$. Hence $\mu = \mu'$ □

Remark 4. Let E be a topological R-module and $Hom(E)$ be the set of all continuous homomorphisms $\varphi : E \longrightarrow E$. If $\mu \in M_p(X,E)$ and $\varphi \in Hom(E)$, then $\varphi \circ \mu \in M_p(X,E)$. As follows from the next example, there exist a topological ring R, a topological R-module E, a space X, $\mu \in L_p(X,E)$ and $\varphi \in Hom(E)$ such that $\varphi \circ \mu \in M_p(X,E) \setminus L_p(X,E)$.

Example 6. Let R be a topological ring with the identity 1, A be a non-empty set and $E = R^A$. Then E is an R-closed R-module and a topological ring. For any subset B of A consider the point $1_B = (t_\alpha(B) : \alpha \in A) \in E$, where $t_\alpha(B) = 1$ for $\alpha \in B$, and $t_\alpha(B) = 0$ for $\alpha \in A \setminus B$. The point 1_B generate the homomorphism $\varphi_B \in Hom(E)$, where $\varphi_B(x) = x \cdot 1_B$ for any $x \in E$. If $E_B = \{x \cdot 1_B : x \in E\} = \varphi_B(E)$, then E_B is a a subring of E and R-submodule of E. The homomorphism φ_B is a retraction of E onto E_B.

Let X be a non-empty space. We consider two cases.

Case 1. $|X| + |R| + \aleph_0 \leq |A|$.

Fix $a \in X$. We put $\psi_B(g) = \varphi_B(g(a))$ for any $g \in C_p(X, E)$. We identify $a = e_X(a)$. Hence $\psi_B = \varphi_B \circ a$. We have $\psi_B(C(X, E)) = E_B$. Therefore $|M_p(X, E)| \geq |\{\psi_B : B \in A\}| = 2^{|A|} > |A|$, where $|Y|$ is the cardinality of the set Y. Obviously, $|L_p(X, E)| \leq |X| + |R| + \aleph_0$. Hence, since $|X| + |R| + \aleph_0 \leq |A|$, we have $|L_p(X, E)| < |M_p(X, E)|$ and $M_p(X, E) \setminus L_p(X, E) \neq \emptyset$.

Case 2. *R is a simple ring and* $2 \leq |A|$.

In this case R is a field and E is a locally simple R-module provided the set A is finite. Fix a non-empty proper subset subset B of A and $\psi \in \mathfrak{L}_p(C_p(X, E), E_B)$, where $\psi \neq 0$. We affirm that $\psi \in M_p(X, E) \setminus L_p(X, E)$. Suppose that we can find $n \in \mathbb{N}$, distinct points $x_1, x_2, ..., x_n \in X$ and $\alpha_1, \alpha_2, .., \alpha_n \in R \setminus \{0\}$ such that $\psi(g) = \alpha_1 g(x_1) + \alpha_2 g(x_2) + ... + \alpha_n g(x_n)$ for any $g \in C(X, R)$. Let $C = A \setminus B$. Fix a function $h \in C(X, E)$ for which $h(x_1) = 1_C$ and $h(x_i) = 0$ for each $i \geq n$. Then $\psi(h) = \alpha_1 h(x_1) \in E_C \setminus E_B$, a contradiction with the condition $\psi(B) \subseteq E_B$. In particular, we have $\varphi_B \in M_p(X, E) \setminus L_p(X, E)$.

Proposition 10. *Let R be a ring, E be a topological R-module and X be a space. Then for any $g \in C(X, E)$ there exists a unique linear mapping $\overline{g} \in \mathfrak{L}_p((L_p(X, E), E)$ such that $g = \overline{g} \circ e_X$, where $e_X : X \to L_p(X, E)$ is the evaluation mapping.*

Proof. Let $E_f = E$ for any $f \in C_p(X, E)$. By definition, $e_X(X) \subseteq L_p(X, E) \subseteq E^{C(X,E)} = \Pi\{E_f : f \in C(X, E)\}$. We consider the projection $\pi_f : E^{C(X,E)} \longrightarrow E_f = E$. Let $\overline{f} = \pi_f | L_p(X, E) : L_p(X, E) \longrightarrow E$. Then $\overline{f}$ and π_f are continuous linear mappings. If $x \in X$, then $\overline{f}(e_X(x))) = f(x)$. Hence $f = \overline{f}|_X$ and for any $f \in C(X, E)$ there exists a linear mapping $\overline{f} \in \mathfrak{L}_p(L_p(X, E), E)$ such that $f = \overline{f} \circ e_X$. Since the subspace $e_X(X)$ generates the linear space $L_p(X, E)$, the linear mapping $\overline{f}$ is unique. □

Theorem 1. *Let R be a ring, E be a topological R-module and X be a space. Consider the space $e_X(X)$, where $e_X : X \to L_p(X, E)$ is the evaluation mapping. Then the linear spaces $C_p(X, E)$, $C_p(e_X(X), E)$ and $\mathfrak{L}_p(L_p(X, E), E)$ are linearly homeomorphic.*

Proof. Let $E_f = E$ for any $f \in C_p(X, E)$. By definition, $e_X(X) \subseteq L_p(X, E) \subseteq E^{C_p(X,E)} = \Pi\{E_f : f \in C(X, E)\}$. We consider the projection $\pi_f : E^{C(X,E)} \longrightarrow E_f = E$. Let $\overline{f} = \pi_f | L_p(X, E) : L_p(X, E) \longrightarrow E$. Then $\overline{f}$ and π_f are continuous linear mappings.

If $g : e_X(X) \to E$ is a continuous mapping, then $g \circ e_X = f$ for a unique $f \in C(X,E)$. Therefore, $g = \pi_f|e_X(X)$ and the correspondence $f \to \pi_f|e_X(X)$ is a linear homeomorphism of $C_p(X,E)$ onto $C_p(e_X(X),E)$.

Hence, without loss of generality, we can assume that $X = e_X(X) \subseteq L_p(X,E)$.

By virtue of Proposition 10, the correspondence $\psi : C_p(X,E) \to \mathfrak{L}_p(L_p(X,E),E)$, where $\psi(f) = \overline{f}$, is a one-to-one linear mapping of $C(X,E)$ onto $\mathfrak{L}_p(L_p(X,E),E)$.

For each $y \in L_p(X,E)$ there exist the minimal $n = n(y) \in \mathbb{N}$, the unique points $x_1(y),...,x_n(y) \in X$ and the unique points $\alpha_1(y),...,\alpha_n(y) \in R$ such that $y = \alpha_1(y)x_1(y) + ... + \alpha_n(y)x_n(y)$. Hence, the correspondence ψ is continuous and linear.

Since $\psi(f)|X = f$, the mapping ψ^{-1} is continuous. □

Remark 5. We say that $e_X(X)$ is the E-replica of the space X. If X is R-completely regular, then $X = e_X(X)$.

Corollary 1. *Let X, Y be spaces and R be a locally simple R-module. The spaces $C_p(X,R)$ and $C_p(Y,R)$ are linearly homeomorphic if and only if the spaces $L_p(X,R)$ and $L_p(Y,R)$ are linearly homeomorphic.*

For $n \geq 1$, an R-module E and a space X we put $L_{p,n}(X,E) = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n : x_i \in e_X(X), \alpha_i \in R, i \leq n\}$. Obviously, $L_{p,n}(X,E) \subseteq L_{p,n+1}(X,E)$ for each n and $L_p(X,E) = \bigcup\{L_{p,n}(X,E) : n \in \mathbb{N}\}$.

Proposition 11. *The mapping $p_n : R^n \times X^n \longrightarrow L_{p,n}(X,E)$, where $p_n(\alpha_1,\alpha_2,...,\alpha_n,x_1,x_2,...,x_n) = \alpha_1 e_X(x_1) + \alpha_2 e_X(x_2) + ... + \alpha_n e_X(x_n)$, is a continuous mapping of $R^n \times X^n$ onto $L_{p,n}(X,E)$.*

Proof. Follows from Proposition 1. □

Let E be an R-module. We say that the pair $(F(X,E), i_X)$ is an E-free R-module of a space X if it has the following properties:

1. $F(X,E)$ is a submodule of the topological R-module E^τ for some cardinal number τ;

2. $i_X : X \to F(X,E)$ is a continuous mapping and the set $i_X(X)$ algebraically generates the R-module $F(X,E)$;

3. For any continuous mapping $f : X \to E$ there exists a continuous homomorphism $\overline{f} : F(X,E) \longrightarrow E$ such that $f = \overline{f} \circ i_X$.

From the property 2 it follows that the homomorphism $\overline{f}$ is unique and is called the homomorphism generated by the mapping f.

Proposition 12. *For any space X there exists a unique E-free R-module. The pair $(L_p(X,E), e_X)$ is the E-free R-module of the space X.*

Proof. The uniqueness of the E-free R-module of the space X is well known (see [6, 7]). From the method of construction of the object $(L_p(X,E), e_X)$ and from the definition of the E-free R-module it follows that $(L_p(X,E), e_X)$ is the E-free R-module of the space X. □

Proposition 13. *Let X be an R-Tychonoff space and E be an R-module. Then $L_{p,n}(X,E)$ is a closed subset of $L_p(X,E)$ for any $n \in \mathbb{N}$.*

Proof. We follow very closely the proof of Proposition 0.5.16 in [1].

Since e_X is an embedding of X into $L_p(X,E)$, we can assume that $X = e_X(X) \subseteq L_p(X,E)$. In this case a point $x \in X$ as an element of $E^{C(X,E)}$ has the form $x = (f(x) : f \in C(X,E))$.

Fix $y \in L_p(X,E) \setminus L_{p,n}(X,E)$. Then $y = \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_m y_m$, where $m > n$, $y_i \in X$, $\alpha_i \in R$ and $\alpha_i y_i \neq 0$ for any $i \leq m$, $y_i \neq y_j$ provided $i \neq j$. For any $i \leq m$ there exists $h_i \in C(X,E)$ such that $\alpha_i h_i(y_i) \neq 0$. We put $b_i = h_i(y_i)$.

Fix a family of pairwise disjoint open sets $\{V_i : i \leq m\}$ in X and the continuous functions $\{f_i \in C(X,R) i \leq m\}$ such that $y_i \in V_i$, $f_i(y_i) = 1$ and $f_i(X \setminus V_i) = 0$ for any $i \leq m$. Let $g_i(x) = f_i(x)b_i$. By virtue of Proposition 10, each g_i extends to a continuous and linear mapping $\overline{g_i} : L_p(X,E) \longrightarrow E$. The subset $U = \bigcap\{\overline{g_i}^{-1}(E \setminus \{0\})\}$ of $L_p(X,E)$ is open.

We have $\overline{g_i}(\alpha_i y_i) = \alpha_i g(y_i) = \alpha_i b_i \neq 0$. If $j \neq i$, then $\overline{g_i}(\alpha_j y_j) = \alpha_j g(y_j) = \alpha_i 0 = 0$. Hence $\overline{g_i}(y) = \alpha_i b_i \neq 0$ for any $i \leq m$. Therefore $y \in U$.

We will show that $U \cap L_{p,n}(X,E) = \emptyset$. Fix $z \in U$, i.e. for some $k \in \mathbb{N}$, $z_1, z_2, ..., z_k \in X$ and $\beta_1, \beta_2, ..., \beta_k \in R$ we have $z = \beta_1 z_1 + \beta_2 z_2 + ... + \beta_k z_k$, $\beta_i z_i \neq 0$ for any $i \leq k$ and $z_i \neq z_j$ provided $i \neq j$. Then $\overline{g_i}(z) = \beta_1 g_i(z_1) + \beta_2 g_i(xz_2) + ... + \beta_k g_i(xz_k) \neq 0$ for each $i \leq k$. Hence $V_i \cap \{z_1, z_2, ..., z_k\} = V'_i \neq \emptyset$ for each $i \leq k$. As the sets $\{V_i : i \leq m\}$ are pairwise disjoint, it follows that the sets $\{V'_i : i \leq k\}$ are non-empty and pairwise disjoint too. Hence $k \geq m > n$, i.e. $U \cap L_{p,n} = \emptyset$. The proof is complete. □

Let $L^c_{p,n}(X,E) = L_{p,n}(X,E) \setminus L_{p,n-1}(X,E)$ and $H_{p,n}(X,E) = p_n^{-1}(L^c_{p,n}(X,E))$ for any $n \in \mathbb{N}$.

Proposition 14. *Let X be an R-Tychonoff space, R be a simple ring and E be a topological R-module. The following assertions are true:*

1. The mapping $q_n = p_n|_{H_{p,n}(X,E)} : H_{p,n}(X,E) \longrightarrow L^c_{p,n}(X,E)$ is one-to-one.

2. If R is a topological field and the module E is R-closed, then the mapping $q_n = p_n|_{H_{p,n}(X,E)} : H_{p,n}(X,E) \longrightarrow L^c_{p,n}(X,E)$ is a homeomorphism.

Proof. By virtue of Propositions 11 and 1, the mapping q_n is continuous. Since e_X is an embedding of X into $L_p(X,E)$, we can assume that $X = e_X(X) \subseteq L_p(X,E)$.

Since R is a simple ring, R is a field and for any $\lambda \in R \setminus \{0\}$ there exists the inverse element λ^{-1}. The ring R is a topological field provided the mapping $^{-1} : R \setminus \{0\} \to R$ is continuous.

We have $\alpha x \neq 0$ for all $\alpha \in R \setminus \{0\}$ and $x \in E \setminus \{0\}$.

Claim 1. *Let $n \in \mathbb{N}$, $x_1, x_2, ..., x_n \in X$, $\alpha_1, \alpha_2, ..., \alpha_n \in R$ and $x_i \neq x_j$ provided $i \neq j$. If $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0$, then $\alpha_i = 0$ for each $i \leq n$.*

Assume that $\alpha_i \neq 0$ for each $i \leq n$. A point $x \in X$ as an element of $E^{C(X,E)}$ has the form $x = (f(x) : f \in C(X,E))$. Hence, for any $i \leq m$ there exists $h_i \in C(X,E)$ such that $\alpha_i h_i(x_i) \neq 0$. We put $b_i = h_i(x_i)$.

Fix a family of pairwise disjoint open sets $\{V_i : i \leq m\}$ in X and the continuous functions $\{f_i \in C(X,R) i \leq m\}$ such that $x_i \in V_i$, $f_i(x_i) = 1$ and $f_i(X \setminus V_i) = 0$ for any $i \leq m$. Let $g_i(x) = f_i(x)b_i$. Then $0 = 0(g_i) = (\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n)(g_i) = \alpha_1 x_1(g_i) + \alpha_2 x_2(g_i) + ... + \alpha_n x_n(g_i) = \alpha_i b_i \neq 0$, a contradiction.

Claim 2. *Let $n, m \in \mathbb{N}$, $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m \in X$, $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m \in R$, $x_i \neq x_j$ provided $i \neq j$, and $y_l \neq y_k$ provided $l \neq k$. If $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = \beta_1 y_1 + \beta_2 y_2 + ... + \beta_m y_m$, then $m = n$, $\{x_1, x_2, ..., x_n\} = \{y_1, y_2, ..., y_m\}$ and $\alpha_i = \beta_j$ provided $x_i = y_j$.*

Obviously, Claim 2 follows from Claim 1. From Claim 2 it follows that the mapping q_n is one-to-one.

Assume now that R is a topological field and E is an R-closed topological R-module. Fix the continuous homomorphism $\varphi_E : E \longrightarrow R$ of the topological R-module E onto the topological R-module R. Fix $a \in \varphi_E^{-1}(1)$. If $E_a = Ra$, then the mapping $\varphi_E | E_a$ is a homeomorphism of E_a onto R. Hence the mapping φ_E is open and continuous as quotient homomorphism of the topological R-module E onto the topological R-module R.

We can fix the non-empty open subsets $V_1, V_2, ..., V_n$ of R and the non-empty open subsets $W_1, W_2, ..., W_n$ of X such that:

– $U = V_1 \times V_2 \times ... \times V_n \times W_1 \times W_2 \times ... \times W_n \subseteq H_{(p,n)}(X, E)$;

– $W_i \cap W_j = \emptyset$ provided $i \neq j$;

– $0 \notin V_i$ for each $i \leq n$.

We affirm that the set $q_n(U)$ is open in $L^c_{p,n}(X, E)$.

Fix a point $y \in q_n(U)$. By definition, $y = \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n$ and $q_n^{-1}(y)y = (\alpha_1, \alpha_2, ..., \alpha_n, y_1, y_2, ..., y_n)$, where $\alpha_i \in V_i \subseteq R \setminus \{0\}$, $y_i \in W_i \subseteq X$ and $y_j \neq y_i$ for each $i \leq n$ and $j \neq i$. Now we fix $f_i \in C(X, E)$, $i \leq n$, such that $f_i(y_i) = \alpha_i^{-1} a$ and $f_i(X \setminus V_i) = 0$. Then $y(f_i) = \alpha_1 y_1(f_i) + \alpha_2 y_2(f_i) + ... + \alpha_n y_n(f_i) = \alpha_1 f_i(y_1) + \alpha_2 f_i(y_2) + ... + \alpha_n f_i(y_n) = \alpha_i f_i(y_i) = a$.

Since $\alpha_i \in V_i$, there exist two open subsets $D(1,i)$ and $D(2,i)$ such that $1 \in D(1,i)$, $\alpha_i^{-1} \in D(2,i)$, $0 \notin D(1,i) \cup D(2,i)$ and $D(1,i)D(2,i)^{-1} \subseteq V_i$. By construction, $p_E(f_i(y_i)) \in D(2,i)$. Hence, we can fix $g_i \in C(X, E)$ for which $g_i(y_i) = a$ and $g_i(X \setminus W_i \cap f_i^{-1}(p_E^{-1}(D(2,i))) = 0$. For each $i \leq n$ there exists the unique continuous linear mappings $\overline{f_i}, \overline{g_i} \in \mathcal{L}_p((L_p(X,E), E)$ such that $f_i = \overline{f_i}|X$ and $g_i = \overline{g_i}|X$.

We put $V = \cap \{\overline{f_i}^{-1}(p_E^{-1}(D(1,i))) \cap \overline{g_i}^{-1}(E \setminus \{0\}) : i \leq n\}$. By construction, $\overline{f_i}(y) = a$ and $\overline{g_i}(y) \neq 0$. Hence, $y \in V$. As in the proof of Proposition 12 we establish that $V \cap L_{p,n-1}(X,E) = \emptyset$. Hence, $V \cap L_{p,n}(X,E) \subseteq L^c_{p,n}(X,E)$.

Fix some $z \in V \cap L_{p,n}(X,E)$. Then $z = \beta_1 z_1 + \beta_2 z_2 + ... + \beta_n z_n$, where $\alpha_i \in R \setminus \{0\}$ and $z_j \neq z_i$ for each $i \leq n$ and $j \neq i$. Since $\overline{f_i}(z) \neq 0$ for all $i \leq n$, we can assume that $\overline{f_i}(z_i) \neq 0$ and $z_i \in V_i$.

By construction, $\overline{g_i}(z) = \beta_i g_i(z_i)$ and $\overline{g_i}(z) \neq 0$. Since $p_E(\overline{f_i}(z_i)) \in D(2,i)$, $p_E(\overline{f_i}(z)) \in D(1,i)$ and $\overline{f}_i(z) = \beta_i f(z_i)$, we have $p_E(f_i(z_i)) \in D(2,i)$ and $\beta_i p_E(f_i(z_i)) \in D(1,i)$. Therefore, $\beta_i p_E(f_i(z_i)) \cdot p_E(f_i(z_i))^{-1} \in D(1,i) \cdot D(2,i)^{-1} \subseteq V_i$ and $\beta_i \in V_i$. Hence, $z \in q_n(U)$ and $V \subseteq q_n(U)$, i.e. $q_n(U)$ is an open subset of $L^c_{p,n}(X,E)$ and the mapping p_n^{-1} is continuous. $\square$

Example 7. Let $E = \mathbb{R}$ be the field of reals with the topology generated by the Euclidian distance. Denote by R the topological space $\mathbb{R} \times \mathbb{R}$ with the following operations:

– the additive operation $(\alpha, \beta) + (\delta, \mu) = (\alpha + \delta, \beta + \mu)$;

– the inverse operation $-(\alpha, \beta) = (-\alpha, -\beta)$;

– the multiplicative operation $(\alpha, \beta) \cdot (\delta, \mu) = (\alpha \cdot \delta, \alpha\mu + \beta\delta)$.

Then R is a topological commutative ring with the unity $(1, 0)$. The ring R is a locally simple R-module. The multiplicative operation $\cdot : R \times E \longrightarrow E$ is defined as follows $(\alpha, \beta) \cdot t = \alpha t$. In this case E is a simple $\mathbb{R}$-module and a simple R-module. Obviously, we have the same subspaces $L_{p,n}(X, E)$ when E is considered an $\mathbb{R}$-module or an R-module.

Since $\mathbb{R}$ is a topological field and E is an $\mathbb{R}$-closed topological $\mathbb{R}$-module, the mapping $q_n = p_n|_{H_{p,n}(X,E)} : H_{p,n}(X, E) \longrightarrow L^c_{p,n}(X, E)$ is a homeomorphism if E is considered as an $\mathbb{R}$-module. Hence $indL^c_{p,n}(X, E) = n$ provided $indX = 0$.

Now we consider E as an R-module. In this case $indH_{p,n}(X, E) \geq indR^n = 2n$. Hence the mapping q_n is not one-to-one if E is considered as an R-module. Moreover, the fibers $q_n^{-1}(y)$ have the dimension equal to n and are homeomorphic to the space $\mathbb{R}^n$. Hence the assumption that R is a simple ring in the conditions of Proposition 14 is essential.

Remark 6. For any topological ring ring R and a space X the mapping $q_n = p_n|_{H_{p,n}(X,R)} : H_{p,n}(X, R) \longrightarrow L^c_{p,n}(X, R)$ is one-to-one.

Lemma 2. *Let X be an R-Tychonoff space, Z be a closed subspace of X, E be a topological R-module and $g : X \longrightarrow E$ be a continuous mapping. For any finite subset F of $X \setminus Z$ and any function $f : F \longrightarrow E$ there exists a continuous function $\varphi : X \longrightarrow E$ such that $f = \varphi|_F$ and $\varphi|_Z = g|_Z$.*

Proof. Fix a family $\{U_x : x \in F\}$ of open subsets of X such that $x \in U_x \subseteq X \setminus Z$ for each $x \in F$ and $U_x \cap U_y = \emptyset$ for each distinct points $x, y \in F$. For each $x \in F$ fix a continuous function $f_x : X \longrightarrow R$ such that $f_x(x) = 1$ and $f_x(X \setminus U_x) = 0$. We put $\varphi_x(y) = f_x(y) \cdot f(x)$ for each $x \in F$ and $y \in X$. Let $\varphi_F(y) = \sum\{\varphi_x(y) : x \in F\}$. By construction, the function φ_F is continuous, $\varphi_F|_F = f$ and $\varphi_F(Z) = 0$. Let $g_x(y) = 1 - f_x(y)$ for any $x \in F$ and $y \in X$. We put $g_F(y) = \prod\{g_x(y) : x \in F\}$ for each $y \in X$. The function g_F is continuous, $g_F(F) = 0$ and $g_F(Z) = 1$. Let $\varphi_Z(y) = g_F(y) \cdot g(y)$ for each $y \in Y$. By construction, the function φ_Z is continuous, $\varphi_Z(F) = 0$ and $\varphi_Z|_Z = g|_Z$. Obviously, $\varphi = \varphi_F + \varphi_Z$ is the desired function. □

For any subspace Y of a space X we put $C_p(Y|X, E) = \{f|_Y : f \in C(X, E)\}$.

A subspace Y of X is *E-full* if $C(Y|X, E) = C(Y, E)$.

A space X is called *compactly E-full* if $C(Y|X, E) = C(Y, E)$ for any compact subspace Y of X.

Lemma 3. *Let X be a zero-dimensional space and E be a metrizable space. Then X is a compactly E-full space. Moreover, for any compact subset Y of X and any $f \in C(Y, E)$ there exists $g \in C(X, E)$ such that $g(X) \subseteq f(Y)$ and $f = g|_Y$, i.e. X is compactly E-full.*

Proof. Fix a metric d on E. Let Y be a non-empty compact subspace of the space X. Fix $f \in C(Y, E)$. For each point $y \in Y$ and each $n \in \mathbb{N}$ we fix a clopen subset $U_n y$ of X such that $y \in U_n y$ and $d(f(y), f(z)) < 2^{-n-1}$ for each $z \in U_n y \cap Y$.

There exists a sequence $\{Y_n : n \in \mathbb{N}\}$ of finite subsets of Y and a sequence $\{\gamma_n = \{V_n y : y \in Y_n\} : n \in \mathbb{N}\}$ of families of clopen subsets of the space X such that:

– $Y_n \subseteq Y_{n+1}$ for each $n \in \mathbb{N}$;

– $y \in V_{n+1}y \subseteq V_n y \subseteq U_n y$ for any $n \in \mathbb{N}$ and any $y \in Y_n$;

– $Y \subseteq \cup\{V_n y : y \in Y_n\} = \cup\gamma_n$ for each $n \in \mathbb{N}$;

– for each $n \in \mathbb{N}$ and each $y \in Y_{n+1}$ there exists a unique $z(y) \in Y_n$ such that $V_{n+1}y \subseteq V_n z(y)$;

– if $y_1, y_2 \in Y_n$, $y_1 \neq y_2$ and $n \in \mathbb{N}$, then $V_n y_1 \cap V_n y_2 = \emptyset$.

We put $V_n = \cup\{V_n y : y \in Y_n\}$. Fix $a \in f(Y)$. We will construct a sequence $\{g_n : X \longrightarrow E : n \in \mathbb{N}\}$ of continuous mappings with the next properties:

– $d(g_n(y), f(y)) < 2^{-n}$ for each $n \in \mathbb{N}$ and any $y \in Y$;

– $d(g_n(x), g_{n+k}(x)) < 2^{-n}$ for each $n, k \in \mathbb{N}$ and any $x \in X$;

– $g_n(X) \subseteq f(Y)$ for each $n \in \mathbb{N}$.

We put $f_1(x) = a$ for each $x \in X \setminus V_1$ and $f_1(V_1 y) = f(y)$ for each $y \in Y_1$.

Assume that $n \geq 1$ and the function g_n is constructed. We put $g_{n+1}|_{(X \setminus V_{n+1})} = g_n|_{(X \setminus V_{n+1})}$ and $g_{n+1}(V_{n+1}y) = f(y)$ for each $y \in Y_{n+1}$. The sequence $\{g_n : n \in \mathbb{N}\}$ is constructed. Since $\{g_n : n \in \mathbb{N}\}$ is a fundamental sequence and $(f(Y), d)$ is a compact metric space, there exists the continuous limit $g = \lim g_n$. By construction, we have $f = g|_Y$. The proof is complete. □

3 About theorem of Nagata

Let R be a simple topological ring. We consider only R-Tychonoff spaces. Fix $n \in \mathbb{N}$. A functional $\mu : C(X, R) \longrightarrow R^n$ is called *multiplicative* if it is linear and $\mu(fg) = \mu(f)\mu(g)$ for any $f, g \in C(X, R^n)$.

Denote by $I_{(p,n)}(X, R) = \{\mu \in L_p(X, R, R^n) : \mu \neq 0, \mu \text{ is multiplicative}\}$.

Theorem 2. *The spaces X^n and $I_{(p,n)}(X, R)$ are homeomorphic.*

Proof. Let $1 = (1, 1, ..., 1)$ be the unity of the ring R^n. For each $i \leq n$ we put $R_i = \{(x_1, x_2, ..., x_n) \in R^n : x_j = 0 \text{ for any } j \neq i\}$. Then R_i is a subring of R^n with the unity $1_i = (0, 0, ..., 1, 0, ..., 0) \in R_i$. The ring R and R_i are topologically isomorphic. The mapping $p_i : R^n \longrightarrow R_i$, where $p_i(x_1, x_2, ..., x_n) = 1_i \cdot (x_1, x_2, ..., x_n)$ for each $(x_1, x_2, ..., x_n) \in R^n$ is open, continuous, linear and multiplicative. If $\mu \in I_{(p,n)}(X, R)$, then we put $\pi_i(\mu) = \mu_i \circ p_i$. Then $\pi_i(\mu) \in I_{(p,1)}(X, R)$ and $\mu(f) = (\pi_1(\mu)(f), \pi_2(\mu)(f), ..., \pi_n(\mu)(f))$ for all $f \in C(X, R)$. Hence $I_{(p,n)}(X, R) = I_{(p,1)}(X, R)^n$.

Now is sufficient to prove that $I_p(X, R) = I_{(p,1)}(X, R)$ and X are homeomorphic.

Obviously, $\xi_x \in I_p(X, R)$ for any $x \in X$. Assume that $\mu \in I_p(X, R)$. Then there exists $n \geq 1$, $x_1, x_2, ..., x_n \in X$ and $\alpha_1, \alpha_2, ..., \alpha_n \in R \setminus \{0\}$ such that $\mu = \alpha_1\xi_{x_1} + \alpha_2\xi_{x_2} + ... + \alpha_n\xi_{x_n}$. Since R is a simple ring, we have $\alpha_i\beta_i = 1$ for some

$\beta_i \in R$. For each $i \leq n$ fix $f_i \in C(X,R)$ such that $f_i(x_i) = \beta_i$ and $f_i(x_j) = 0$ for each $j \neq i$. By construction, $\mu(f_i) = \alpha_i f(x_i) = \alpha_i \beta_i = 1$ for each $i \leq n$. Assume that $n \geq 2$. Then $f_i \cdot f_2 = 0$ and $0 = \mu(0) = \mu(f_1 \cdot f_2) = \mu(f_1)\mu(f_2) = 1 \cdot 1 = 1$, a contradiction. Hence $n = 1$ and $\mu = \alpha_1 \xi_{x_1}$.

Assume that $\alpha_1 \neq 1$. Then $\beta_1 \neq 1$ and $\beta_1 = 1 \cdot \beta_1 = \alpha_1 \beta_1 \beta_1 = \alpha_1 (f_1 \cdot f_1)(x_1) = \mu(f_1 \cdot f_1) = \mu(f_1) \cdot \mu(f_1) = 1 \cdot 1 = 1$, a contradiction. Hence $\beta_1 = \alpha_1 = 1$ and $\mu = \xi_{x_1}$. The proof is complete. □

Corollary 2. *If the rings $C_p(X,R)$ and $C_p(Y,R)$ are topologically isomorphic, then the spaces X and Y are homeomorphic.*

The Corollary 2 for the ring $\mathbb{R}$ of reals was proved by Nagata (see [1], Theorem 0.6.1).

4 Algebraical classes of spaces

Fix a topological ring R. Assume that R is an R-Tychonoff space. A class $\mathcal{P}$ of topological spaces is called *an algebraical R-class* of spaces if:

(i) any space $X \in \mathcal{P}$ is R-Tychonoff and $Y \in \mathcal{P}$ for any closed subspace Y of X;

(ii) if $f : X \longrightarrow Y$ is a continuous mapping of X onto Y, $X \in \mathcal{P}$ and Y is an R-Tychonoff space, then $Y \in \mathcal{P}$;

(iii) if $\{X_n \in \mathcal{P} : n \in \mathbb{N}\}$ is a sequence of closed subspaces of an R-Tychonoff space X and $X = \cup\{X_n : n \in \mathbb{N}\}$, then $X \in \mathcal{P}$;

(iv) if $X, Y \in \mathcal{P}$, then $X \times Y \in \mathcal{P}$;

(v) $R \in \mathcal{P}$.

Lemma 4. *Let $\mathcal{P}$ be an algebraical R-class of spaces, $\{X_n : n \in \mathbb{N}\}$ be a sequence of subspaces of an R-Tychonoff space X, $X = \cup\{X_n : n \in \mathbb{N}\}$ and $X_n \in \mathcal{P}$ for any $n \in \mathbb{N}$. Then $X \in \mathcal{P}$.*

Proof. Let $Y_n = X_n \times \{n\}$ and Y is the discrete sum of the spaces $\{Y_n : n \in \mathbb{N}\}$. Obviously Y is an R-Tychonoff space, $Y_n \in \mathcal{P}$ and Y_n is closed in Y for any $n \in \mathbb{N}$. Hence $Y \in \mathcal{P}$ and X is a continuous image of Y. □

Theorem 3. *Let $\mathcal{P}$ be an algebraical R-class of spaces, R be a topological ring and E be a topological R-module. Assume that E is an R-Tychonoff space. For an R-Tychonoff space X the following assertions are equivalent:*

(i) $X \in \mathcal{P}$.

(ii) $L_p(X,E) \in \mathcal{P}$.

Proof. Assume that $X \in \mathcal{P}$. We consider the mapping $\varphi_n : X^n \times R^n \longrightarrow L_p(X,E)$ where $\varphi_n((x_1, x_2, ..., x_n), (\alpha_1, \alpha_2, ..., \alpha_n)) = \alpha_1 \xi_{x_1} + \alpha_2 \xi_{x_2} + ... + \alpha_n \xi_{x_n}$. The mapping φ_n is continuous and we put $L_{p,n}(X,E) = \varphi_n(X^n \times R^n)$.

Since $L_p(X,E) \subseteq E^{C(X,E)}$, the space $L_p(X,E)$ is R-Tychonoff. Hence $X^n \times R^n$, $L_{p,n}(X,E) \in \mathcal{P}$ for each n. From Proposition 13 we have $L_p(X,E) = \cup\{L_{p,n}(X,E) : n \in \mathbb{N}\}$. Thus $L_p(X,E) \in \mathcal{P}$.

Assume now that $L_p(X,E) \in \mathcal{P}$. Since $X = L_{p,1}(X,E)$, by virtue of Proposition 13, X is a closed subspace of the space $L_p(X,E)$. Then $X \in \mathcal{P}$. □

Corollary 3. *Let $\mathcal{P}$ be an algebraical R-class of spaces, R be a topological ring and E be a topological R-module. Assume that E is an R-Tychonoff space. If $C_p(X,E)$ and $C_p(Y,E)$ are topologically homeomorphic and $X \in \mathcal{P}$, then $Y \in \mathcal{P}$.*

Remark 7. For the ring $\mathbb{R}$ of reals and $E = \mathbb{R}$ the above assertion is proved in [1], Proposition 0.5.13.

5 The support mapping

Fix a topological ring R and a non-trivial locally simple topological R-module E.

Consider a space X and a functional $\mu \in M_p(X,E)$. We put $\mathcal{S}(\mu) = \{B \subseteq X :$ if $B \subseteq f^{-1}(0)$, then $\mu(f) = 0\})$. Obviously, $X \in \mathcal{S}(\mu)$. Thus the set $\mathcal{S}(\mu)$ is non-empty.

The set $supp_X(\mu)$ is the family of all points $x \in X$ such that for each neighbourhood U of x in X there exists $f \in C_p(X,E)$ such that $f(X \setminus U) = 0$ and $\mu(f) \neq 0$ (see [2,8] for $E = R = \mathbb{R}$, and [3,10] for $R = \mathbb{R}$).

If $f \in C_p(X,E)$ and U is an open neighbourhood of 0 in E, then we put $A(f,L,U) = \{g \in C_p(X,E) : f(x) - g(x) \in U$ for any $x \in L\}$. The family $\{A(f,L,U) : f \in C_p(X,E), L$ *is a finite subset of* X, U *is an open neighbourhood of 0 in* $E\}$ is an open base of the space $C_p(X,E)$.

Theorem 4. *Let X be an R-Tychonoff space, E be a non-trivial locally simple topological E-module, $\mu \in M_p(X,E)$ and $\mu \neq 0$. Then:*

1. *There exists a finite set $K \in \mathcal{S}(\mu)$ such that $supp_X(\mu) \subseteq K$.*
2. *$supp_X(\mu) \in \mathcal{S}(\mu)$ and $supp_X(\mu)$ is a finite subset of X.*

Proof. Fix an open subset V_0 of E such that $0 \in V_0$ and V_0 does not contain non-trivial R-submodules of E.

There exists a finite subset K of X such that $\mu(f) \in V_0$ for each $f \in A(0,K,V_0)$. Let $f \in C_p(X,E)$ and $f(K) = 0$. Then $\alpha f \in A(0,K,V_0)$ for each $\alpha \in R$. Hence $\mu(\alpha f) \in W_0$ for each $\alpha \in R$. Thus $E \cdot \mu(f) \subseteq V_0$ and $E \cdot \mu(f)$ is the trivial R-submodule. Thus $\mu(f) = 0$ and $K \in \mathcal{S}(\mu)$. In this case $supp_X(\mu) \subseteq K$. Hence $supp_X(\mu)$ is a finite set and K is a finite set from $\mathcal{S}(\mu)$.

Let $L \in \mathcal{S}(\mu)$ be a finite set and $x_0 \in L \setminus supp(\mu)$. Then $L_1 = L \setminus \{x_0\} \in \mathcal{S}(\mu)$. Really, since $x_0 \notin supp_X(\mu)$, there exists an open subset H of X such that $x_0 \in H$ and $\mu(f) = 0$ provided $f(X \setminus H) = 0$. Fix $h \in C(X,R)$ such that $h(x_0) = 1$ and $h(X \setminus H) = 0$. Let $f \in C_p(X,E)$ and $f(L_1) = 0$. We put $f_1(x) = h(x)f(x)$ for any $x \in X$ and $f_2 = f - f_1$. Since $f_1(X \setminus H) = 0$, we have $\mu(f_1) = 0$. By construction, $f_2(L) = 0$ and $\mu(f_2) = 0$. Hence $f = f_1 + f_2$ and $\mu(f) = \mu(f_1 + f_2) = \mu(f_1) + \mu(f_2) = 0$. Hence $L_1 \in \mathcal{S}(\mu)$. Since $K \setminus supp_X(\mu)$ is a finite set, we have $supp_X(\mu) \in \mathcal{S}(\mu)$. □

Proposition 15. *If $x \in X$ and $\xi_x(f) = f(x)$ for each $f \in C_p(X, E)$, then $\xi_x \in L_p(X, E)$.*

Proof. Obviously, $\xi_x = \alpha x(f)$, where $\alpha \in R$ and $\alpha = 1$. Thus $\xi_x \in L_p(X, E)$. □

Remark 8. Let R be a locally simple ring, X be an R-Tychonoff space, $\mu \in M_p(X, R)$ and $supp(\mu) = \{x_1, x_2, ..., x_n\}$. Then, by virtue of Proposition 9, there exists $\alpha_1, \alpha_2, ..., \alpha_n \in R \setminus \{0\}$ such that $\mu = \Sigma\{\alpha_1 \xi_{x_i} : i \leq n\}$.

As was mentioned in Remark 4 and Example 6, as a rule $M_p(X, E) \neq L_p(X, E)$. The following result specifies the form of linear functionals for locally simple R-modules.

Theorem 5. *Let X be an R-Tychonoff space, E be a non-trivial topological E-module, $\mu \in M_p(X, E)$, $\mu \neq 0$, $supp_X(\mu) \in \mathcal{S}(\mu)$ and $supp_X(\mu)$ is a finite subset of X. Then $\mu = \varphi \circ \eta$ for some $\varphi \in Hom(E)$ and $\eta \in L_p(X, E)$.*

Proof. Assume that $n \geq 1$ and $supp_X(\mu) = \{b_1, b_2, ..., b_n\}$, where $b_i \neq b_j$ for $i \neq j$. We put $\eta_i = b_i \in L_p(X, E)$ and $\eta = \eta_1 + \eta_2 + ... + \eta_n$. Obviously, $\eta \in L_p(X, E)$ and $supp_X(\eta) = \{b_1, b_2, ..., b_n\} = supp_X(\mu)$.

We can fix the non-empty open subsets $V_1, V_2, ..., V_n$ of X and the functions $f_1, f_2, ..., f_n \in C(X, R)$ such that:

– $V_i \cap V_j = \emptyset$ provided $i \neq j$;

– $b_i \in V_i$, $f_i(b_i) = 1$ and $f_i(X \setminus V_i) = 0$ for each $i \leq n$.

Fix $i \leq n$. For each $y \in E$ we put $\varphi_i(y) = \mu(f_i \cdot y)$, where $(f_i \cdot y)(x) = f_i(x)y$ for each $x \in X$. If $g \in C(X, E)$, then $\mu_i(g) = \mu(f_i \cdot g)$.

Claim 1. *$\varphi_i \in Hom(E)$ for each $i \leq n$.*

The mapping $\psi_i : E \longrightarrow C_p(X, E)$, where $\psi_i(y) = f_i \cdot y$ for each $y \in E$, is continuous and linear. The equality $\varphi_i = \mu \circ \psi_i$ completes the proof of the Claim 1.

Claim 2. *$\mu_i \in M_p(X, E)$ and $supp_X(\mu_i) = \{b_i\}$ for each $i \leq n$.*

It is clear that $\mu_i \in M_p(X, E)$. If $g \in C(X, E)$, then $g(b_i) = (f_i \cdot g)(b_i)$ and $(f_i \cdot g)(b_j) = 0$ provided $i \neq j$. Hence $\mu_i(g) = \mu_i(f_i \cdot g) = \mu(f_i \cdot g)$. Claim 2 is proved.

Claim 3. *$\mu = \mu_1 + \mu_2 + ... + \mu_n$.*

Let $g \in C(X, E)$ and $h = f_1 g + f_2 g + ... + f_n g$. Then $g|supp_X(\mu) = h|supp_X(\mu)$. Hence $\mu(g) = \mu(h) = \mu(f_1 g + f_2 g + ... + f_n g) = \mu(f_1 g) + \mu(f_2 g) + ... + \mu(f_n g) = \mu_1(g) + \mu_2(g) + ... + \mu_n(g)$. Claim 3 is proved.

Claim 4. *$\mu = \varphi \circ \eta$.*

By construction, $supp_X(\mu) = supp_X(\varphi \circ \eta)$. If $g \in C(X, E)$ and $h = f_1 g + f_2 g + ... + f_n g$, then $(\varphi \circ \eta)(g) = (\varphi \circ \eta)(h)$. Since $(\varphi \circ \eta)(f_i g) = ((\varphi_1 + \varphi_2 + ... + \varphi_n) \circ (\eta_1 + \eta_2 + ... + \eta_n))(f_i g)) = \Sigma\{(\varphi_i \circ \eta_j)(f_i g) : i, j \leq n\} = (\varphi_i \circ \eta_i)(f_i g) = \varphi_i(\eta_j(f_i g))$ $= \varphi_i(g(b_i)) = \mu(f_i g)$, we have $(\varphi \circ \eta)(g) = (\varphi \circ \eta)(h) = (\varphi \circ \eta)(f_1 g + f_2 g + ... + f_n g)$ $= \Sigma\{(\varphi \circ \eta)(f_i g) : i \leq n\} = \Sigma\{\mu(f_i g) : i \leq n\} = (\mu(f_1 g + f_2 g + ... + f_n g) = \mu(g)$. Claim is proved. The proof is complete. □

6 Topological properties of the mapping $supp_X$

Fix a topological ring R, a non-trivial locally simple R-module E and an R-Tychonoff space X.

Recall that a set-valued mapping $f : X \to 2^Y$ is lower semicontinuous (l. s. c) if for every open subset U of Y the inverse image of U, $f^{-1}(U) = \{x \in X : f(x) \cap U \neq \emptyset\}$ is open in X.

Proposition 16. *The set-valued mapping $supp_X : M_p(X, E) \to X$ is l. s. c.*

Proof. We follow very closely the proof of [3], Property 4.2, and [8], Lemma 6.8.2(4).

Let U be an open subset of X, and put $V = supp_X^{-1}(U)$, i.e., $V = \{\mu \in M_p(X, E, F) : supp_X(\mu) \cap U \neq \emptyset\}$. Let $\mu \in V$, and take $x \in supp_X(\mu) \cap U$. Fix an open subset W of X such that $x \in W \subseteq cl_X W \subseteq U$. The there exists $f \in C(X, E)$ such that $f(X \setminus W) = \{0\}$ and $\mu(f) \neq 0$.

Let $H = \{\eta \in M_p(X, E, F) : \eta(f) \neq 0\}$. Since the set $\{0\}$ is closed in E, H is the prebasic open set $W(f, E \setminus \{0\}) = \{\eta \in M_p(X, E) : \eta(f) \in E \setminus \{0\}\}$ and $\mu \in W(f, E \setminus \{0\})$.

We claim that $H \subseteq V$. By contradiction, suppose that $\eta \in H \setminus V$, i. e. $\eta(f) \neq 0$ and $supp_X(\eta) \cap U = \emptyset$. Then $X \setminus cl_X W$ is an open neighbourhood of $supp_X(\eta)$ and since $f(X \setminus cl_X W) = \{0\}$, applying Theorem 4, we get that $\eta(f) = 0$. A contradiction, hence V is open in $M_p(X, E)$. □

A subset L of a space X is *bounded* if any continuous real-valued function $f : X \longrightarrow \mathbb{R}$ is bounded on L.

A subset L of a topological R-module E is called:

(i) *precompact* or *totally a-bounded* if for any neighbourhood U of 0 in E there exists a finite subset A of E such that $L \subseteq A + U = U + A$;

(ii) *a-bounded* if for any neighbourhood U of the 0 in E there exists $n \in \mathbb{N}$ such that $L \subseteq nU$.

Any bounded set is precompact. In a topological vector space over field of reals any precompact set is a-bounded.

A topological R-module E is called *locally bounded* if there exists an a-bounded neighbourhood U of 0 in E such that $E = \cup\{nU : n \in \mathbb{N}\}$ and for any $a \in E$, $a \neq 0$, and any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nU$. In this case the set U does not contain R-submodules of E and E is a locally simple R-module.

Example 8. Let E be a normed vector space over reals $\mathbb{R}$. Then E is a locally bounded $\mathbb{R}$-module.

Example 9. Let E be a topological vector space over reals $\mathbb{R}$ and there exists a number $q > 0$ and a functional $||\cdot|| : E \longrightarrow \mathbb{R}$ such that:

1. $0 < q \leq 1$.
2. $||x|| \geq 0$ for any $x \in E$.
3. If $||x|| = 0$, then $x = 0$.
4. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in E$.

5. $||\lambda x|| \leq |\lambda|^q ||x||$ for all $x \in E$ and $\lambda \in \mathbb{R}$.

6. If $x \neq 0$ then $lim_{\lambda \to +\infty} ||\lambda x|| = +\infty$.

The functional $||\cdot||$ is called a *q-norm* if the family $\{V(0,r) = \{x : ||x|| < r\} : r > 0\}$ is a base of E at 0. Any q-normed space is locally bounded.

Theorem 6. *Let E be a non-trivial locally bounded topological R-module, X be an R-Tychonoff space and for any non-bounded subset L of X there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded. Then:*

(i) The set $supp_X(H)$ is bounded in X for any a-bounded subset H of $M_p(X,E)$.

(ii) The set $supp_X(H)$ is bounded in X for any totally a-bounded subset H of $M_p(X,E)$.

(iii) The set $supp_X(H)$ is bounded in X for any bounded subset H of $M_p(X,E)$.

Proof. Fix an a-bounded open neighbourhood W_1 of 0 in E such that $E = \bigcup\{nW_1 : n \in \mathbb{N}\}$ and for any $a \in E$, $a \neq 0$ and any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nW_1$.

Now fix two open neighbourhoods W and W_0 of 0 in E such that $W_0 + W_0 + W_0 \subseteq W = -W \subseteq W_1$ and $W_0 = -W_0$.

By construction, $W_1 \subseteq kW_0$ for some $k \in \mathbb{N}$.

Hence the sets W and W_0 have the following properties:

– W and W_0 are a-bounded subsets of E;

– $E = \bigcup\{nW : n \in \mathbb{N}\} = \bigcup\{nW_0 : n \in \mathbb{N}\}$;

– if L is a bounded or a precompact subset of E, then $L \subseteq nW_0$ for some $n \in \mathbb{N}$;

– if $a \in E$, $a \neq 0$, then for any $n \in \mathbb{N}$ there exists $t \in R$ such that $ta \notin nW$.

Suppose that the set H is a-bounded or precompact in $L_p(X,E)$ and the set $supp_X(H)$ is not bounded in X. Fix $f \in C(X,E)$ such that the set $f(supp_X(H))$ is not a-bounded in E.

By induction, we shall construct a sequence $\{\mu_n : n \in \mathbb{N}\} \subseteq H$, a sequence $\{U_k : k \in \mathbb{N}\}$ of open subsets of X, a sequence $\{x_n \in supp_X(\mu_n) : n \in \mathbb{N}\}$ and a sequence $\{h_k \in C(X,E) : n \in \mathbb{N}\}$ with properties:

1. $x_i \in U_i$, $h_i(X \setminus U_i) = 0$ for any $i \in \mathbb{N}$;
2. $\{U_n : n \in \mathbb{N}\}$ is a discrete family of subsets of X;
3. $\mu_n(h_n) \notin nW$;
4. $supp_X\{\mu_1, \mu_2, ..., \mu_n\} \cap cl_X U_{n+1} = \emptyset$;
5. $f(U_n) \subseteq f(x_n) + W_0$ and $f(x_{n+1}) \notin \bigcup\{f(x_i) + W : i \leq n\}$ for each $n \in \mathbb{N}$.

Fix $\mu_1 \in H$ and $x_1 \in supp_X(\mu_1)$. There exists an open subset U_1 of X and $g_1 \in C(X,E)$ such that $f(U_1) \subseteq W_0 + f(x_1)$, $g_1(X \setminus U_1) = 0$ and $\mu_1(g_1) \neq 0$. There exists $\alpha_1 \in R$ such that $\alpha_1\mu_1(g) \notin W$. We put $h_1 = \alpha_1 g_1$.

Assume that $n \geq 1$ and the objects $\{h_i, x_i, U_i, \mu_i : i \leq n\}$ are constructed. We put $M_n = \bigcup\{supp_X(\mu_i) : i \leq n\}$. The set M_n is finite. Hence the set $f(supp_X(H)) \setminus f(M_n)$ is not a-bounded in E. For some $m_n \in \mathbb{N}$ we have $f(M_n) \subseteq m_n W_0$.

Fix $\mu_{n+1} \in H$ and $x_{n+1} \in supp_X(H)$ such that $f(x_{n+1}) \in E \setminus m_n W$. There exists an open subset U_{n+1} of X and $g_{n+1} \in C(X,E)$ such that $x_{n+1} \in U_{n+1}$, $f(U_{n+1}) \subseteq f(x_{n+1}) + W_0$, $g_{n+1}(X \setminus U_{n+1}) = 0$, $cl_X U_{n+1} \cap M_n = \emptyset$ and $M_{n+1}(g_{n+1}) \neq 0$. There exists $\alpha_{n+1} \in R$ such that $\alpha_{n+1}\mu_{n+1}(g_{n+1}) \notin (n+1)W$. We put $h_{n+1} =$

$\alpha_{n+1}g_{n+1}$. That completes the inductive construction. The objects $\{x_m, \mu_n, h_n, U_n\}$ are constructed for all $n \in \mathbb{N}$. Let $h = \Sigma\{h_n : n \in \mathbb{N}$. Since $\{U_n : n \in \mathbb{N}\}$ is a discrete family and $h_n(X \setminus U_n) = 0$ for any $n \in \mathbb{N}$, we have $h \in C(X, E)$. By construction $\mu_n(h) = \mu_n(h_n) \notin nW_0$ for any n. Then $\{\mu_n(h) : n \in \mathbb{N}\}$ is not a-bounded subset of E. Since the set H is a-bounded, the set $\mu(h) : \mu \in H\}$ is a-bounded too, a contradiction. The proof is complete. □

Remark 9. If R is the fields of real or complex numbers and E is a locally bounded R-module, then:

– E is a metrizable linear space;
– E is a locally simple R-module;
– any precompact set is a-bounded in E.

Remark 10. Any normed space is a locally bounded $\mathbb{R}$-module. If E is a non-trivial normed space, then for any non-bounded subset L of the space X there exists $f \in C(X, E)$ such that the set $f(L)$ is not bounded in E. For a normed space E Theorem 6 was proved by V. Valov in [10].

A space X is *μ-complete* if any closed bounded subset of X is compact.

A space X is *Dieudonné complete* if the maximal uniformity on X is complete. Any Dieudonné complete space is μ-complete.

Denote by PX the space X with the G_δ-topology generated by the G_δ-subsets of X. The set $\delta\text{-}cl_X H = cl_{PX} H$ is called the G_δ-closure of the set H in X. If $\delta\text{-}cl_X H = H$, then we say the set H is G_δ-closed.

If the space X is μ-complete, then any G_δ-closed subspace of X is μ-complete.

A tightness of a space X is the minimal cardinal number τ for which for any subset $L \subseteq X$ and any point $x \in cl_X L$ there exists a subset $L_1 \subseteq L$ such that $|L_1| \leq \tau$ and $x \in cl_X L_1$.

We denote by $t(X)$ and $l(X)$ the tightness and the Lindelöf numbers respectively of a space X.

The following assertion for $E = \mathbb{R}$ was proved by A. V. Arhangel'skii and E. G. Pytkeev (see [1], Theorem II.1.1).

Proposition 17. *Assume that E is a metrizable space and $l(X^n) \leq \tau$ for any $n \in \mathbb{N}$. Then $t(C_p(X, E)) \leq \tau$.*

Proof. The proof is as in [1]. We show only the scheme of the proof.

Fix a metric d on E. Let $A \subseteq C_p(X, E)$ and $f \in clA$. Let $\varepsilon_n = 2^{-n}$. For any $x = (x_1, x_2, ..., x_n) \in X^n$ there exists $g_x \in A$ such that $d(g_x(x_i), f(x_i)) < \varepsilon_n$, $i \leq n$. Since g_x and f are continuous, there exists $O_x = \Pi\{Ox_i : i \leq n\}$ such that $d(g_x(y), f(y)) < \varepsilon_n$ for all $y \in O_x$. The $\{O_x : x \in X^n\}$ is a cover of X^n. Fix $B_n \subseteq X^n$, $|B_n| \leq \tau$ and $\bigcup\{O_x : x \in B_n\} = X^n$. Let $A_n = \{f_x : x \in B_n\} \subseteq A$. Then $f \in cl(\bigcup\{A_n : n \in \mathbb{N}\})$. □

Proposition 18. *Let X and E be spaces and $t(X) \leq \aleph_0$. Then $C_p(X, E)$ is a G_δ-closed subspace of the space E^X. Moreover, if E is μ-complete, then the space $C_p(X, E)$ is μ-complete too.*

Proof. Since the product of μ-complete spaces is μ-complete, the space E^X is μ-complete provided the space E is μ-complete.

Assume that $g \in E^X \setminus C(X, E)$. Then there exists a point $x_0 \in X$ and an open subset U of E such that $g(x_0) \in U$ and $x_0 \in cl_X\{x \in X : g(x) \notin U\}$. Since $t(X) \leq \aleph_0$ there exists a countable subset $L \subseteq \{x \in X : g(x) \notin U\}$ such that $x_0 \in cl_X L$. Fix an open subset V of E such that $g(x_0) \in V \subseteq cl_E V \subseteq U$.

For each $y \in L$ we put $H_y = \{f \in E^X : f(x_0) \in V, f(y) = E \setminus cl_E V\}$. The set H_y is open in E^X and $g \in H_y$. Let $H = \cap\{H_y : y \in L\}$. Then H is a G_δ-subset of E^X.

Assume that $f \in C(X, E)$. If $f(x_0) \notin V$, then $f \notin H_y$ for each $y \in L$. Suppose that $f(x_0) \in V$. There exists an open subset W of X and a point $y \in L$ such that $x_0 \in W$, $y \in L \cap W$ and $f(W) \subseteq V_0$. Then $f \notin H_y$. Hence $H \cap C(X, E) = \emptyset$ and $g \notin \delta\text{-}cl_{E^X} C(X, E)$. Therefore $C(X, E)$ is a G_δ-closed subset of E^X. Any G_δ-closed subset of a μ-complete space is μ-complete. □

Proposition 19. *Let F and E be topological R-modules and $\mathfrak{L}_p(F, E)$ be the space of all linear continuous mappings of F into E. Then $\mathfrak{L}_p(F, E)$ is a closed subspace of the space $C_p(F, E)$.*

Proof. Fix $g \in C_p(F, E) \setminus \mathfrak{L}_p(F, E)$. Then we have one of the following two cases.

Case 1. *There exist $a, b \in F$ such that $g(a + b) \neq g(a) + g(b)$.*

In this case there exist four open subsets V_1, V_2, V and W of E such that $g(a) \in V_1$, $g(b) \in V_2$, $g(a + b) \in W$, $V_1 + V_2 \subseteq V$ and $V \cap W = \emptyset$. The set $H = \{f \in C_p(F, E) : f(a + b) \in W, f(a) \in V_1, f(b) \in V_2\}$ is open in $C_p(F, E)$ and $H \cap \mathfrak{L}_p(F, E) = \emptyset$.

Case 2. *There exist $a \in F$ and $\lambda \in R$ such that $g(\alpha a) \neq \alpha g(a)$.*

In this case there exist three open subsets V_1, V and W of E such that $g(a) \in V_1$, $g(\alpha a) \in W$, $\alpha V_1 \subseteq V$ and $V \cap W = \emptyset$. The set $H = \{f \in C_p(F, E) : f(\alpha a) \in W, f(a) \in V_1\}$ is open in $C_p(F, E)$ and $H \cap \mathfrak{L}_p(F, E) = \emptyset$. The proof is complete. □

Corollary 4. *Let E and F be topological R-modules and $t(F) \leq \aleph_0$. Then $\mathfrak{L}_p(F, E)$ is a G_δ-closed subset of E^F. In particular, if E is μ-complete, then the space $\mathfrak{L}_p(F, E)$ is μ-complete too.*

For any subspace Y of a space X we view $C_p(Y|X, E) = \{f|_Y : f \in C(X, E)\}$ as a subspace of the space $C_p(Y, E)$.

Proposition 20 ([1], Proposition 0.4.1 for $E = \mathbb{R}$). *Let Y be a subspace of the space X, E be a non-trivial topological R-module, X be an R-Tychonoff space and $p_Y(f) = f|_Y$ for each $f \in C_p(X, E)$. Then the mapping $p_Y : C_p(X, E) \longrightarrow C_p(Y|X, E)$ has the following properties:*

(i) p_Y is a continuous mapping.

(ii) If the set Y is closed in X, then the mapping p_Y is open.

(iii) If Y is dense in X, then p_Y is a one-to-one correspondence.

(iv) The subspace $C_p(Y|X, E)$ is dense in $C_p(Y, E)$.

Proof. Let $x_1, x_2, ..., x_n$ be a finite subset of X. The we can assume that there exists $k \leq n$ such that $x_1, x_2, ..., x_{k-1} \in Y$ and $x_k, ..., x_n \in X \setminus Y$. Let $f \in C(X, E)$ and $U_1, U_2, ..., U_n$ be open subsets of E such that $f(x_i) \in U_i$ for each $i \leq n$. We put $W(f, x_1, x_2, ..., x_n, U_1, ..., U_n) = \{g \in C(X, E) : g(x_i) \in U_i \text{ for each } i \leq n\}$ and $W_Y(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1}) = \{g|_Y : g \in W(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1})\}$. We have $p_Y(W(f, x_1, x_2, ..., x_n, U_1, ..., U_n)) \subseteq W_Y(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1})$. Thus the mapping p_Y is continuous. If Y is closed in X, then from Lemma 2 it follows that $p_Y(W(f, x_1, x_2, ..., x_n, U_1, ..., U_n)) = W_Y(f, x_1, ..., x_{k-1}, U_1, ..., U_{k-1})$. Hence the mapping p_Y is open.

Assertion (iii) is obvious. Assertion (iv) follows from Lemma 2. The proof is complete. □

Theorem 7. *Let E be a locally bounded metrizable R-module, X be an R-Tychonoff compactly E-full space and for any non-bounded subset L of X there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in E. Then the space X is μ-complete if and only if the space $M_p(X, E)$ is μ-complete.*

Proof. By virtue of Proposition 5, we can assume that $X = e_X(X)$ is a subspace of the space $M_p(X, E)$. From Proposition 4 it follows that the subspace X is closed in $M_p(X, E)$.

Let $M_p(X, E)$ be a μ-complete space. Since X is a closed subspace of $M_p(X, E)$, the space X is μ-complete too.

Assume that X is a μ-complete space. Let Φ be a closed bounded subset of $M_p(X, E)$. Then the closure Y of the set $\cup\{supp_X(\mu) : \mu \in \Phi\}$ is a compact subset of X.

The restriction mapping $p_Y : C_p(X, E) \longrightarrow C_p(Y, E)$ is an open continuous linear mapping of the R-module $C_p(X, E)$ onto the R-module $C_p(Y, E)$.

Claim 1. *The dual mapping $\varphi : E^{C(Y,E)} \longrightarrow E^{C_p(X,E)}$ is a linear embedding and the set $\varphi(E^{C(Y,E)})$ is closed in $E^{C(X,E)}$.*

The proof of this fact is similar with the prof of Proposition 0.4.6 from [1].

By construction, we have $\Phi \subseteq \varphi(M_p(Y, E)) \subseteq M_p(X, E)$.

Claim 2. *$\varphi(M_p(Y, E))$ is a closed subset of the subspaces $M_p(X, E)$ and $C_p(C_p(X, E), E)$ of the space $E^{C(X,E)}$.*

Follows from Claim 1 and Proposition 19.

Claim 3. *$\varphi(C_p(C_p(Y, E), E)) \subseteq C_p(C_p(X, E), E)$.*

Follows from the continuity of the mapping p_Y.

Claim 4. *The sets $\varphi(M_p(X, E))$ and $\varphi(C_p(C_p(Y, E), E))$ are G_δ-closed in $E^{C(X,E)}$.*

Since Y is compact, from Proposition 17 it follows that $t(C_p(Y, E)) = \aleph_0$. Then, from Proposition 18 it follows that $C_p(C_p(Y, E), E)$ is a G_δ-closed subset of the space $E^{C(Y,E)}$. From Claim 1 it follows that $\varphi(C_p(C_p(Y, E), E))$ is G_δ-closed in $E^{C(X,E)}$. Corollary 4 completes the proof of the claim.

Let G be the G_δ-closure of the set $C_p(C_p(X,E),E))$ in $E^{C(X,E)}$. We have $M_p(X,E) \subseteq G$. Hence Φ is a bounded subset of the space G.

Claim 5. *The sets $\varphi(M_p(X,E))$ and $\varphi(C_p(C_p(Y,E),E))$ are closed in G.*

Follows from Claim 4.

Since E is a metrizable space, E is a μ-complete space. Thus Φ is a closed bounded subset of the μ-complete space G. Therefore the set Φ is compact. The proof is complete. □

7 Relations between spaces generated by a linear homeomorphism

Let R be a topological ring and E be a non-trivial locally bounded topological R-module. Then the R-module E is locally simple.

Fix two non-empty R-Tychonoff spaces X and Y with the properties:

– for any non-bounded subset L of X there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded in E;

– for any non-bounded subset L of Y there exists $f \in C(Y,E)$ such that the set $f(L)$ is not a-bounded in E.

Fix now a continuous linear homeomorphism $u : C_p(X,E) \longrightarrow C_p(Y,E)$. Then the dual mapping $v : M_p(Y,E) \longrightarrow M_p(X,E)$, where $v(\eta) = \eta \circ u$ for each $\eta \in M_p(Y,E)$, is a linear homeomorphism. For each $x \in X$ we put $\varphi(x) = supp_Y(v^{-1}(\xi_x))$ and for any $y \in Y$ put $\psi(y) = supp_X(v(\xi_y))$.

Property 1. *$\varphi : X \to Y$ and $\psi : Y \to X$ are l.s.c. set-valued mappings and $\varphi(x)$, $\psi(y)$ are finite sets for all points $x \in X$, $y \in Y$.*

Proof. Follows from Proposition 16 and Theorem 4. □

Property 2. *Let $y_0 \in Y$, $f \in C(X,E)$ and $f(\psi(y_0)) = 0$. Then $u(f)(y_0) = 0$.*

Proof. For any $\eta \in M_p(Y,E)$ and $g \in C(X,E)$ we have $v(\eta)(g) = \eta(u(g))$. Since $f(supp_X(v(\xi_{y_0}))) = f(\psi(y_0)) = 0$, we have $u(f)(y_0) = \xi_{y_0}(u(f)) = v(\xi_{y_0})(f) = f(supp_X(v(\xi_{y_0}))) = 0$. The proof is complete. □

Corollary 5. *If $f, g \in C(X,E)$ and $f|_{\phi(y)} = g|_{\phi(y)}$, then $u(f)(y) = u(g)(y)$.*

Property 3. *$x \in cl_X\psi(\varphi(x))$ for every point $x \in X$ and $y \in cl_Y\varphi(\psi(y))$ for every point $y \in Y$.*

Proof. Assume that $x_0 \in X$ and $x_0 \notin cl_X\psi(\varphi(x_0)) = F$. Fix $f \in C(X,E)$ such that $f(x_0) = b \neq 0$ and $f(F) = f(\psi(\varphi(x_0))) = 0$. Since $\psi(y) \subseteq F$ and $f(F) = 0$ for any $y \in \varphi(x_0)$ by virtue of Property 2, we have $u(f)(y) = 0$ for each $y \in \varphi(x_0)$. Since $u(f)(y) = 0$ for each $y \in \varphi(x_0)$, by virtue of Property 2, we have $f(x_0) = u^{-1}(u(f))(x_0) = 0$. By construction, we have $f(x_0) \neq 0$, a contradiction. □

Property 4. *$x \in \psi(\varphi(x))$ for every point $x \in X$.*

Proof. For every $x \in X$, $\varphi(x)$ is finite set and $\psi(\varphi(x))$ is compact. Property 3 completes the proof. □

Property 5. *If H is a dense subset of Y, then $\psi(H)$ is a dense subset of X.*

Proof. Assume that $x_0 \notin cl_X\psi(H)$. Then there exists $f \in C(X,E)$ such that $f(x_0) \neq 0$ and $f(\psi(H)) = 0$. Since $f(\psi(H)) = 0$ for any $y \in Y$, by virtue of Property 2, we have $u(f)(y) = 0$ for any $y \in Y$. Thus $u(f) = 0$. Hence $f = 0$, a contradiction. □

Corollary 6. *The space X is separable if and only if the space Y is separable. In general, $d(X) = d(Y)$.*

Property 6. *$\varphi(F)$ is a bounded set of Y for each bounded set F of X.*

Proof. Let F be a bounded subset of X. Then F is a bounded subset of $M_p(X,E)$ and respectively $v^{-1}(F)$ is a bounded subset of $M_p(Y,E)$. By Theorem 6 the set $supp_Y(v^{-1}(F))$ is a bounded subset of Y. The proof is complete. □

Property 7. *Let E be a metrizable space, X and Y be compactly E-full spaces. Then the space X is μ-complete if and only if the space Y is μ-complete.*

Proof. Let X be a μ-complete space. Then $M_p(X,E)$ and $M_p(Y,E)$, by virtue of Theorem 7, are μ-complete spaces. By Theorem 7 the space Y is μ-complete too. The proof is complete. □

As in [3, 4] we say that the pair of set-valued mappings $\theta : X \longrightarrow Y$ and $\pi : Y \longrightarrow X$ is called lower-reflective if it satisfies the following conditions:

1l. θ and π are l.s.c.

2l. $\theta(x)$ and $\pi(x)$ are finite sets for all points $x \in X$ and $y \in Y$.

3l. $x \in \pi(\theta(x))$ and $y \in \theta(\pi(y))$ for all points $x \in X$ and $y \in Y$.

Also, as in [3, 4] we say that the pair of set-valued mappings $\theta : X \longrightarrow Y$ and $\pi : Y \longrightarrow X$ is called upper-reflective if it satisfies the following conditions:

1u. $\theta(F)$ is a bounded subset of Y for each bounded subset F of X.

2u. $\pi(\Phi)$ is a bounded subset of X for each bounded subset Φ of Y.

3u. $x \in cl_X\pi(\theta(x))$ and $y \in cl_Y\theta(\pi(y))$ for all points $x \in X$ and $y \in Y$.

From the above properties follows

Corollary 7. *The space X is separable if and only if the space Y is separable. In general, $d(X) = d(Y)$.*

General conclusion: The set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ form an equivalence of X and Y in sense of articles [3,4]. Thus the general theorems from [3] can be extended for the mappings in topological R-modules. In the following sections we formulate the general theorems for that case.

8 Application to perfect properties

We say that the property $\mathcal{P}$ is a *perfect property* if for any continuous perfect mapping $f : X \longrightarrow Y$ of X onto Y we have $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$. We say that the property $\mathcal{P}$ is a *strongly perfect property* if it is perfect and any space with property $\mathcal{P}$ is μ-complete.

Example 10. By virtue of Example 6.2 from [3] (see also [4]), the following properties are perfect:

1. To be a compact space.
2. To be a paracompact p-space.
3. To be a paracompact space.
4. To be a metacompact space.
5. To be a k-scattered space.
6. To be a monotonically p-space.
7. To be a monotonically Čech complete space.
8. To be a Čech complete space.
9. To be a Lindelöf space.
10. To be a Lindelöf Σ-space.
11. To be a subparacompact space.
12. To be a locally compact space.

Example 11. The following properties are strongly perfect:

1. To be a compact space.
2. To be a paracompact p-space.
3. To be a paracompact space.
4. To be a μ-complete metacompact space.
5. To be a k-scattered μ-complete space.
6. To be a μ-complete monotonically p-space.
7. To be a μ-complete monotonically Čech complete space.
8. To be a μ-complete Čech complete space.

9. To be a Lindelöf space.

10. To be a Lindelöf Σ-space.

11. To be a μ-complete subparacompact space.

12. To be a μ-complete locally compact space.

A space X is called a *wq-space* if for any point $x \in X$ there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $x \in \cap\{U_n : n \in \mathbb{N}\}$ and each set $\{x_n \in U_n : n \in \mathbb{N}\}$ is bounded in X.

A space X is *pseudocompact* if the set X is bounded in the space X. A pseudocompact space is a μ-complete space if and only if it is compact. Any pseudocompact space is a wq-space.

Theorem 8. *Let R be a topological ring and E be a non-trivial locally bounded topological R-module. Fix two non-empty R-Tychonoff spaces X and Y with the properties:*

– for any non-bounded subset L of X there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded in E;

– for any non-bounded subset L of Y there exists $f \in C(Y,E)$ such that the set $f(L)$ is not a-bounded in E.

Assume that $u : C_p(X,E) \longrightarrow C_p(Y,E)$ is a linear homeomorphism. Then:

1. X is a pseudocompact space if and only if Y is a pseudocompact space.

2. If $\mathcal{P}$ is a perfect property and X, Y are μ-complete wq-spaces, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

Proof. Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in Section 7.

Let X be a pseudocompact space. Then X is a bounded subset of the space X. Hence $Y = \varphi(X)$ is a bounded subset of Y and Y is a pseudocompact space. Assertion 1 is proved.

Assume that $\mathcal{P}$ is a perfect property and X, Y are μ-complete wq-spaces. Suppose that $X \in \mathcal{P}$. By virtue of Theorem 2.5 from [3], there exist a space Z and two perfect single-valued mappings $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ onto X and Y, respectively. Hence, $Y, Z \in \mathcal{P}$. Assertion 2 is proved. The proof is complete. □

Theorem 9. *Let R be a topological ring and E be a non-trivial metrizable locally bounded topological R-module. Fix two non-empty R-Tychonoff compactly E-full spaces X and Y with the properties:*

– for any non-bounded subset L of X there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded in E;

– for any non-bounded subset L of Y there exists $f \in C(Y,E)$ such that the set $f(L)$ is not a-bounded in E.

Assume that $u : C_p(X,E) \longrightarrow C_p(Y,E)$ is a linear homeomorphism. Then:

1. The space X is μ-complete if and only if the space Y is μ-complete.

2. X is a compact space if and only if Y is a compact space.

3. If $\mathcal{P}$ is a strongly perfect property and X, Y are wq-spaces, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

Proof. Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in Section 7. Assertion 1 follows from Property 7.

Assume that $\mathcal{P}$ is a strongly perfect property and X, Y are wq-spaces. Suppose that $X \in \mathcal{P}$. By definition of a strongly perfect property, X is a μ-complete space. From Assertion 1 it follows that Y is a μ-complete space too. By virtue of Theorem 2.5 from [3], there exist a space Z and two perfect single-valued mappings $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ onto X and Y, respectively. Hence, we have $Y, Z \in \mathcal{P}$. Assertion 3 is proved.

Let X be a compact space. By virtue of Theorem 8, Y is a pseudocompact space. Hence X and Y are wq-spaces. Assertion 3 completes proof of Assertion 2. The proof is complete. □

9 Application to open properties

We say that the property $\mathcal{P}$ is an *of-property* (*open-finite property*) if for any continuous open finite-to-one mapping $f : X \longrightarrow Y$ and any subspace Z of X we have $Z \in \mathcal{P}$ if and only if $f(Z) \in \mathcal{P}$.

Example 12. From the results from [3],[4] and [5] the following properties are of-properties:

1. To be hereditarily Lindelöf.
2. To be σ-space.
3. To be hereditarily separable.
4. To be σ-metrizable.
5. To be σ-scattered.
6. To be σ-discrete space.

Example 13. Let τ be an infinite cardinal. Consider the properties:

1. $X \in e(\tau)$ if and only if $e(X) \leq \tau$;
2. $X \in d(\tau)$ if and only if $d(X) \leq \tau$;
3. $X \in hd(\tau)$ if and only if $hd(X) \leq \tau$;
4. $X \in hl(\tau)$ if and only if $hl(X) \leq \tau$.

Then $e(\tau), d(\tau), hd(\tau), hl(\tau)$ are of-properties.

Theorem 10. *Let R be a topological ring and E be a non-trivial locally bounded topological R-module. Fix two non-empty R-Tychonoff spaces X and Y with the properties:*

– for any non-bounded subset L of X there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded in E;

– for any non-bounded subset L of Y there exists $f \in C(Y,E)$ such that the set $f(L)$ is not a-bounded in E.

Assume that $u : C_p(X,E) \longrightarrow C_p(Y,E)$ is a linear homeomorphism. If $\mathcal{P}$ is an of-property, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

Proof. Consider the set-valued mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ constructed in the Section 7. As in [3] (see Theorem 2.1 from [3]) we put $Z = \cup\{\{x\} \times \varphi(x) : x \in X\}$ and $S = \cup\{\psi(y) \times \{y\} : y \in Y\}$ as subspaces of the spaces $X \times Y$, $f(x,y) = x$ and $g(x,y) = y$ for any point $(x,y) \in X \times Y$. Then $f : Z \longrightarrow X$ and $g : S \longrightarrow Y$ are continuous open finite-to-one mappings. If $D = Z \cap S$, then from Property 4 it follows that $f(D) = X$ and $g(D) = Y$. Hence $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$. The proof is complete. □

10 $l_p(E)$-equivalence and metrizability

Theorem 11. *Let R be a topological ring and E be a non-trivial metrizable locally bounded topological R-module. Fix two non-empty R-Tychonoff compactly E-full spaces X and Y with the properties:*

– for any non-bounded subset L of X there exists $f \in C(X,E)$ such that the set $f(L)$ is not a-bounded in E;

– for any non-bounded subset L of Y there exists $f \in C(Y,E)$ such that the set $f(L)$ is not a-bounded in E.

Let X and Y be $l_p(E)$-equivalent spaces. Then:

1. X is a compact metrizable space if and only if Y is a compact metrizable space.

2. If X is a metrizable space, then the space Y is metrizable if and only if Y is a wq-space.

Proof. Any metrizable space is a wq-space.

Let X be a metrizable space and Y be a wq-space. Since X is metrizable, by virtue of Theorem 8, Y is a paracompact p-space. From Theorem 10 it follows that Y is a σ-space. If a paracompact space Y is a σ-space and a p-space, then Y is metrizable [9]. Assertion 2 is proved.

Assertion 1 follows from the Assertion 2 and Theorem 8. The proof is complete. □

11 Final remarks and examples

The requirements on spaces R, E and X in the conditions of Theorems 8, 9 and 10 are essential.

First, the space X must have a sufficient number of continuous mappings into R and E. Moreover, these mappings must determine the topology of the space X and should feel certain properties of subsets relative to their position in the space X. These are explained the requirements:

– the space X is a non-empty R-Tychonoff space;

– for any non-bounded subset L of X there exists $f \in C(X, E)$ such that the set $f(L)$ is not a-bounded in E.

Obviously, the requirement "X is a non-empty R-Tychonoff space" may be changed by the requirement "X is a non-empty E-Tychonoff space". A space X is an E-Tychonoff space if for each closed set F of X, any point $a \in X \setminus F$ and any point $b \in E$, there exists $f \in C(X, E)$ such that $f(a) = 0$ and $f(F) = b$.

Second, the space E should have unbounded sets and some neighborhoods of zero should be able to distinguish boundedness and unboundedness of sets from E.

Example 14. Let $\mathbb{Z}$ be the discrete ring of integers, $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle with the binary operation $(x, y) + (u, v) = (xu - yv, xv + yu)$. Then any zero-dimensional space is $\mathbb{Z}$-Tychonoff and E is a locally simple compact $\mathbb{Z}$-module. Any Tychonoff space is an E-Tychonoff space. For any space X the $\mathbb{Z}$-modules $C_p(X, E)$ and $M_p(X, E)$ are a-bounded. In this case the assertions (i) and (ii) of Theorem 6 are not true.

Example 15. Let R be a finite ring, $0 \neq 1$, and $E = R^A$, where A is an infinite discrete space. Then the ring $\mathbb{R}$ is locally simple and compact. The R-module E is compact and not locally simple. Any zero-dimensional space is R-Tychonoff. The spaces $C_p(X, E)$, $C_p(X \times A, E)$, $C_p(X, E)^A$, $C_p(X, R)^A$, $C_p(X, E)^A$, and $C_p(X \times A, R)$ are linearly homeomorphic. The space X may be compact and the space $X \times A$ is not pseudocompact. If the space A is countable, the module E is metrizable.

Example 16. Let $E = \mathbb{R}^A$, where A is an infinite discrete space. The ring of reals $\mathbb{R}$ is a locally simple and locally compact topological field. The $\mathbb{R}$-module E is not locally simple and not locally a-bounded. Any Tychonoff space is an $\mathbb{R}$-Tychonoff space. The spaces $C_p(X, E)$, $C_p(X, \mathbb{R})$, $C_p(X, E)^A$, $C_p(X, \mathbb{R})^A$, $C_p(X, E)^A$, and $C_p(X \times A, \mathbb{R})$ are linearly homeomorphic. The space X may be compact and the space $X \times A$ is not pseudocompact. If the space A is countable, the module E is metrizable.

Acknowledgements. The authors express theirs gratitude to Professors Alexander V. Arhangel'skii and Vesco Valov for useful conversations.

The article was supported by the project RM 15.817.02.18F.

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Received December 9, 2014

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