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Stability radius bounds in multicriteria Markowitz portfolio problem with venturesome investor criteria

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Abstract. The lower and upper bounds on the stability radius are obtained in multicriteria Boolean Markowitz investment problem with criteria of extreme optimism (MAXMAX) about portfolio return in the case when portfolio and financial market states spaces are endowed with Hölder metric, and criteria space of economical efficiency of investment projects is endowed with Chebyshev metric.

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1 Introduction

In the papers [1,2], vector investment Boolean problems with Savage and Wald's criteria are formulated based on Markowitz portfolio theory [3]. Stability radius bounds are obtained only for particular cases, when three-dimensional problem parameters space is equipped with different combinations of l_1 and l_∞ metric. In the present paper, multicriteria portfolio problem with venturesome investor under the same Markowitz model framework is considered. The investor maximizes various portfolio efficiency types when financial market is in the most favorable state, i.e. with criteria of extreme optimism (MAXMAX). We investigate such a kind of stability of the problem which is a discrete analogue of the property to be semicontinuous from above in Hausdorff's sense of a point-set mapping which transforms any set of parameters of the investment problem into the corresponding Pareto set. As a result of the conducted parametric analysis, power and upper bounds on the stability radius of the problem are obtained in the case when portfolio space and financial market spaces are endowed with Hölder metric $l_p, 1 \leq p \leq \infty$, and criteria space of economical efficiency of investment projects is endowed with Chebyshev metric l_∞.

2 Problem statement and definitions

Based on [2,4], consider multicriteria variant of Markowitz investment management problem [3].

Let m be the number of possible financial market states $(A_1, A_2, \dots, A_m)$, n be the number of alternative investment projects $(B_1, B_2, \dots, B_n)$ and s be the number of types (measures) of the project economical efficiency $(C_1, C_2, \dots, C_s)$. Given the expected evaluation of economical efficiency e_{ijk} for an arbitrary investment project B_j of type C_k in the case when market is in the state A_i. We denote three-dimensional matrix $[e_{ijk}] \in \mathbf{R}^{m\times n\times s}$ by E and its k-th cut by $E_k \in \mathbf{R}^{m\times n}$. Let $x = (x_1, x_2, \dots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio where $\mathbf{E} = \{0,1\}$, $x_j = 1$ if the investor chooses the project B_j and $x_j = 0$ otherwise; $X \subseteq \mathbf{E}^n$ be the set of all possible investment portfolios, i.e. those realization of which does not exceed investor's initial budget and admissible level of risk.

Note that there are several approaches to evaluate efficiency of investment projects (see e.g. the bibliography in [2]).

In the portfolio space X we introduce vector objective function

$$f(x,E) = (f_1(x,E_1), f_2(x,E_2), \dots, f_s(x,E_s)),$$

components of which are well known in the decision making theory criteria of extreme optimism (MAXMAX)

$$f_k(x,E_k) = \max_{1\le i\le m} e_{ik}x = \max_{1\le i\le m} \sum_{j=1}^{n} e_{ijk}x_j \to \max_{x\in X}, \quad k \in N_s = \{1,2,\dots,s\},$$

where $e_{ik} = (e_{i1k}, e_{i2k}, \dots, e_{ink})$ is the i-th row of the cut E_k. Using this criteria venturesome investor optimizes the efficiency $e_{ik}x$ of the portfolio x under the assumption that market is in the most favorable state for him. In other words when a portfolio return is maximal. It is evident that the approach is based on the behavior stereotype of reckless optimism ("make or mar", "who does not risk cannot win" etc.). It is worth to notice that such situations in economics when we have to behave this way are common. Such dealing is inherent in not only optimist but investors with his (her) back to the wall.

Under a multicriteria Boolean investment problem $Z^s(E), s \in \mathbf{N}$ we understand the problem of searching the Pareto set $P^s(E)$, i.e. the set of Pareto optimal investment portfolio

$$P^s(E) = \{x \in X \ : \ X(x,E) = \varnothing\},$$

where

$$X(x,E) = \{x' \in X \ : \ f(x,E) \le f(x',E) \ \& \ f(x,E) \ne f(x',E)\}.$$

It is obvious that $P^s(E) \ne \varnothing$ for any matrix $E \in \mathbf{R}^{m\times n\times s}$.

For any natural number d in the real space $\mathbf{R}^d$ we define Hölder metric $l_p, p \in [1,\infty]$, i.e. under the norm of vector $y = (y_1, y_2, \dots, y_d)^T \in \mathbf{R}^d$ we understand the number

$$\|y\|_p = \begin{cases} (\sum\limits_{i=1}^{d} |y_i|^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\limits_{1\le i\le d} |y_i| & \text{if } p = \infty. \end{cases}$$

It is well known that for any vectors $a, b \in \mathbf{R}^n$ the Hölder inequality holds

$$|a^T b| \le \|a\|_p \|b\|_q, \tag{1}$$

and for numbers p and q the following relation is true

$$1/p + 1/q = 1.$$

Here $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$. Thereby suppose $1/p = 0$ for $p = \infty$. Further we assume that the domain of variation of p and q is the segment $[1, \infty]$.

In the portfolio space $\mathbf{R}^n$ and financial market states space $\mathbf{R}^m$ define an arbitrary Hölder metric $l_p,\ p \in [1, \infty]$, and in the criteria space of measures of project economical efficiency $\mathbf{R}^s$ define Chebyshev metric l_∞, since under the norm of a matrix $E \in \mathbf{R}^{m\times n\times s}$ we understand the number

$$\|E\|_{pp\infty} = \|(\|E_1\|_{pp}, \|E_2\|_{pp}, \dots, \|E_s\|_{pp})\|_\infty,$$

where

$$\|E_k\|_{pp} = \|(\|e_{1k}\|_p, \|e_{2k}\|_p, \dots, \|e_{mk}\|_p)\|_p, \quad k \in N_s.$$

Obviously,

$$\|e_{ik}\|_p \le \|E_k\|_{pp} \le \|E\|_{pp\infty}, \quad i \in N_m,\ k \in N_s. \tag{2}$$

Therefore using Hölder inequality (1), it is easy to see that for any portfolios $x, x' \in X$ and matrix $E \in \mathbf{R}^{m\times n\times s}$ the following inequalities are valid

$$e_{ik}x - e_{i'k}x' \ge -\|E\|_{pp\infty}\|x + x'\|_1^{1/q}, \quad i, i' \in N_m, \quad k \in N_s. \tag{3}$$

Indeed

$$e_{ik}x - e_{i'k}x' \ge -(\|e_{ik}\|_p\|x\|_q + \|e_{i'k}\|_p\|x'\|_q) \ge$$

$$-\|(\|e_{ik}\|_p, \|e_{i'k}\|_p)\|_p\|(\|x\|_q, \|x'\|_q)\|_q \ge -\|E\|_{pp\infty}\|x + x'\|_1^{1/q}.$$

Moreover, it is easy to see that for vector $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ with conditions $|a_i| = \alpha,\ i \in N_n$, for any number $p \in [1, \infty]$ the following equality holds

$$\|a\|_p = \alpha n^{1/p}. \tag{4}$$

According to [1, 2, 5], with respect to the metrics defined, under the stability radius of the problem $Z^s(E), s \in \mathbf{N}$, we understand the number

$$\rho = \rho(m, n, s, p) = \begin{cases} \sup \Xi_p & \text{if } \Xi_p \ne \varnothing, \\ 0 & \text{if } \Xi_p = \varnothing, \end{cases}$$

where

$$\Xi_p = \{\varepsilon > 0 \ :\ \forall E' \in \Omega_p(\varepsilon) \ \ (P^s(E + E') \subseteq P^s(E))\},$$

$$\Omega_p(\varepsilon) = \{E' \in \mathbf{R}^{m\times n\times s} \ :\ \|E'\|_{pp\infty} < \varepsilon\}$$

be the set of perturbed matrices, $P^s(E+E')$ be the Pareto set of perturbed problem $Z^s(E + E')$. Thus the stability radius of problem $Z^s(E)$ is the limit level of perturbations of matrix E elements in normed vector space $\mathbf{R}^{m\times n\times s}$ which do not lead to appearance of new Pareto optimal portfolios. It is obvious that for $P^s(E) = X$ the stability radius of the problem is supposed to be infinite. The problem for which $P^s(E) \ne X$ is called nontrivial.

3 Bounds on the stability radius of the problem

For the nontrivial problem $Z^s(E)$ put

$$\varphi = \varphi(m,n,s,p) = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x',x)}{\|x'+x\|_1^{1/q}},$$

$$\psi = \psi(m,n,s) = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x',x)}{\|x'-x\|_1},$$

where

$$\gamma(x',x) = \min\{f_k(x',E_k) - f_k(x,E_k) : k \in N_s\},$$

$$P(x,E) = X(x,E) \cap P^s(E).$$

It is easy to see that $\varphi,\ \psi \geq 0$.

Theorem 1. *For any $m,n,s \in \mathbf{N}$ and $p \in [1,\infty]$ for the stability radius $\rho(m,n,s,p)$ of the multicriteria nontrivial investment problem $Z^s(E)$ the following bounds are valid*

$$\varphi(m,n,s,p) \leq \rho(m,n,s,p) \leq (mn)^{1/p}\psi(m,n,s). \tag{5}$$

Proof. Let us first show that the inequality $\rho \geq \varphi$ is valid. For $\varphi = 0$ it is evident. Let $\varphi > 0$ and the perturbed matrix $E' \in \mathbf{R}^{m\times n\times s}$ with cuts E'_k, $k \in N_s$, belongs to the set $\Omega_p(\varphi)$, i.e. $\|E'\|_{pp\infty} < \varphi$. According to the definition of number φ for any portfolio $x \notin P^s(E)$ there exists portfolio $x^0 \in P(x,E)$ such that

$$\gamma(x^0,x) \geq \varphi\|x^0+x\|_1^{1/q},$$

i.e. the inequalities

$$f_k(x^0,E_k) - f_k(x,E_k) \geq \varphi\|x^0+x\|_1^{1/q}, \quad k \in N_s$$

hold.

Therefore, taking into account inequality (3), for any index $k \in N_s$ we obtain

$$f_k(x^0,E_k+E'_k) - f_k(x,E_k+E'_k) = \max_{1\leq i\leq m}(e_{ik}+e'_{ik})x^0 - \max_{1\leq i\leq m}(e_{ik}+e'_{ik})x =$$

$$= \min_{1\leq i\leq m}\max_{1\leq i'\leq m}(e_{i'k}x^0 - e_{ik}x + e'_{i'k}x^0 - e'_{ik}x) \geq$$

$$\geq \min_{1\leq i\leq m}\max_{1\leq i'\leq m}(e_{i'k}x^0 - e_{ik}x) - \|E'\|_{pp\infty}\|x^0+x\|_1^{1/q} =$$

$$= f_k(x^0,E_k) - f_k(x,E_k) - \|E'\|_{pp\infty}\|x^0+x\|_1^{1/q} \geq (\varphi - \|E'\|_{pp\infty})\|x^0+x\|_1^{1/q} > 0,$$

where e'_{ik} is the i-th row of the cut E'_k. Thus, any portfolio x, which is not in $P^s(E)$, is not a Pareto optimal portfolio on the perturbed problem $Z^s(E+E')$. Therefore we conclude that for any perturbed matrix $E' \in \Omega_p(\varphi)$ the inclusion $P^s(E+E') \subseteq P^s(E)$ is valid. Hence the inequality $\rho \geq \varphi$ is true.

Further we prove the inequality $\rho \leq (mn)^{1/p}\psi$.

According to the definition of the number ψ there exists portfolio $x^0 \notin P^s(E)$ such that for any portfolio $x \in P(x^0, E)$ there exists $l = l(x) \in N_s$, for which

$$f_l(x, E_l) - f_l(x^0, E_l) \leq \psi \|x - x^0\|_1. \tag{6}$$

Assuming $\varepsilon > (mn)^{1/p}\psi$, we define the k-th cut E_k^0, $k \in N_s$ elements e_{ijk}^0 of the perturbed matrix E^0 by the rule

$$e_{ijk}^0 = \begin{cases} \delta & \text{if} \quad i \in N_s,\ x_j^0 = 1 \\ -\delta & \text{otherwise,} \end{cases}$$

where $\varepsilon/(mn)^{1/p} > \delta > \psi$. Then according to (4) we have

$$\|e_{ik}^0\|_p = n^{1/p}\delta, \quad \|E_k^0\|_{pp} = (mn)^{1/p}\delta, \quad i \in N_m, \quad k \in N_s,$$

$$\|E^0\|_{pp\infty} = (mn)^{1/p}\delta.$$

This means that $E^0 \in \Omega_p(\varepsilon)$. Moreover, all rows e_{ik}^0, $i \in N_m$, of the cut E_k^0, $k \in N_s$, are the same and consist of the components δ and $-\delta$. Therefore, assuming $A = e_{ik}^0$, $i \in N_m$, $k \in N_s$, we have

$$A(x - x^0) = -\delta \|x - x^0\|_1. \tag{7}$$

Hence, taking into account (6), we conclude that for any portfolio $x \in P(x^0, E)$ there exists $l \in N_s$, satisfying the relations

$$f_l(x, E_l + E_l^0) - f_l(x^0, E_l + E_l^0) = \max_{1 \leq i \leq m} (e_{il} + e_{il}^0)x - \max_{1 \leq i \leq m} (e_{il} + e_{il}^0)x^0 =$$

$$= \min_{1 \leq i \leq m} \max_{1 \leq i' \leq m} (e_{i'l}x - e_{il}x^0 + e_{i'l}^0 x - e_{il}^0 x^0) = f_l(x, E_l) - f_l(x^0, E_l) + A(x - x^0) \leq$$

$$\leq (\psi - \delta)\|x - x^0\|_1 < 0.$$

Thus the formula

$$\forall x \in P(x^0, E) \quad (x \notin X(x^0, E + E^0)) \tag{8}$$

is valid. If $X(x^0, E + E^0) = \emptyset$ then $x^0 \in P^s(E + E^0)$. Recall that $x^0 \notin P^s(E)$.

Now suppose that $X(x^0, E + E^0) \neq \emptyset$.

Then due to the external stability of the set $P^s(E + E^0)$ (see e.g. [6, 7]) there exists portfolio $x^* \in P(x^0, E + E^0)$. We show that $x^* \notin P^s(E)$.

We assume the contrary: $x^* \in P^s(E)$. According to (8) the inclusion

$$x^* \in P^s(E) \setminus P(x^0, E)$$

holds. Therefore only two following cases are possible.

Case 1. $f(x^*, E) = f(x^0, E)$. Then for any $k \in N_s$ from equality (7) it follows

$$f_k(x^*, E_k + E_k^0) - f_k(x^0, E_k + E_k^0) = f_k(x^*, E_k) - f_k(x^0, E_k) +$$

$$+A(x^* - x^0) = -\delta\|x^* - x^0\|_1 < 0.$$

Case 2. There exists $q \in N_s$ such that $f_q(x^*, E_q) < f_q(x^0, E_q)$. Then again using (7) we obtain

$$f_q(x^*, E_q + E_q^0) - f_q(x^0, E_q + E_q^0) = f_q(x^*, E_q) - f_q(x^0, E_q) + A(x^* - x^0) < 0.$$

Consequently both cases contradict the inclusion $x^* \in P(x^0, E + E^0)$. Therefore it is proved that $x^* \notin P^s(E)$. Recall that $x^* \in P^s(E + E^0)$.

Thus for any number $\varepsilon > (mn)^{1/p}\psi$ it is guaranteed that there exists a perturbing matrix $E^0 \in \Omega_p(\varepsilon)$ such that there exists portfolio (x^0 or x^*) which is not Pareto optimal portfolio for $Z^s(E)$ but becomes Pareto optimal in the perturbed problem $Z^s(E + E^0)$. Hence the formula

$$\forall \varepsilon > (mn)^{1/p}\psi \quad \exists E^0 \in \Omega_p(\varepsilon) \quad (P^s(E + E^0) \nsubseteq P^s(E))$$

is valid.

Consequently, $\rho \leq (mn)^{1/p}\psi$. □

The well known result follows from Theorem 1.

Corollary 1 [8]. $\varphi(m, n, s, \infty) \leq \rho(m, n, s, \infty) \leq \psi(m, n, s)$.

The following evident statement confirms attainability on these bounds

Corollary 2. *If for any pair $x \notin P^s(E)$ and $x' \in P(x, E)$ the equality*

$$\{j \in N_n \ : \ x_j = x'_j = 1\} = \varnothing$$

holds then the formula

$$\rho(m, n, s, \infty) = \varphi(m, n, s, \infty) = \psi(m, n, s)$$

is valid.

Attainability of the upper bound in (5) for $m = 1$ and $p = \infty$ follows from the following known theorem.

Theorem 2 [9]. $\rho(1, n, s, \infty) = \psi(1, n, s), \quad n, s \in \mathbf{N}$.

Remark 1. From theorem 1 it follows that the upper bound on the stability radius of the problem $\rho(m, n, s, p)$ decreases mn times with number p increasing from 1 to ∞. That is the upper bound decreases from $mn\psi(m, n, s)$ to $\psi(m, n, s)$. At the same time the lower bound also decreases from

$$\varphi(m, n, s, 1) = \min_{x \notin P^s(E)} \ \max_{x' \in P(x,E)} \gamma(x', x)$$

to

$$\varphi(m, n, s, \infty) = \min_{x \notin P^s(E)} \ \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{\|x' + x\|_1}.$$

As follows from Corollary 2, when its conditions hold the lower values of the lower and upper bounds on the stability radius are identical:

$$\varphi(m, n, s, \infty) = \psi(m, n, s).$$

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