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On the number of ring topologies on countable rings

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Abstract. For any countable ring R and any non-discrete metrizable ring topology τ_0, the lattice of all ring topologies admits:
– Continuum of non-discrete metrizable ring topologies stronger than the given topology τ_0 and such that $\sup\{\tau_1, \tau_2\}$ is the discrete topology for any different topologies;
– Continuum of non-discrete metrizable ring topologies stronger than τ_0 and such that any two of these topologies are comparable;
– Two to the power of continuum of ring topologies stronger than τ_0, each of them being a coatom in the lattice of all ring topologies.

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1 Introduction

The study of possibility to set a non-discrete Hausdorff topology on infinite algebraic systems in which existing operations are continuous was begun in [1]. In this article for any countable group a method of constructing such group topologies was given.

For countable rings the problem of the possibility to set non-discrete Hausdorff ring topologies was studied in [2, 3]. In these articles for any countable ring a method of obtaining any ring metrizable topology was given and it was proved that any countable ring admits such topology.

The present article is a continuation of research in this direction. The main result of this paper is Theorem 3.1, in which for any countable ring R and any non-discrete metrizable ring topology τ_0, the number of topologies which have some properties in the lattice of all ring topologies is specified.

For countable groups similar result was obtained in [4, 5].

2 Notations and preliminaries

To present the main results we remind the following well-known result (see, for example, [2], Proposition 1.2.2 and Theorem 1.2.5).

Theorem 2.1. *A set Ω of subsets of a ring R is a basis of filter of neighborhoods of zero for some Hausdorff ring topology on the ring R if and only if the following conditions are satisfied:*

1) $\bigcap_{V\in\Omega} V = \{0\}$;

2) *For any subsets* V_1 *and* $V_2 \in \Omega$ *there exists a subset* $V_3 \in \Omega$ *such that* $V_3 \subseteq V_1 \cap V_2$;

3) *For any subset* $V_1 \in \Omega$ *there exists a subset* $V_2 \in \Omega$ *such that* $V_2 + V_2 \subseteq V_1$;

4) *For any subset* $V_1 \in \Omega$ *there exists a subset* $V_2 \in \Omega$ *such that* $-V_2 \subseteq V_1$;

5) *For any subset* $V_1 \in \Omega$ *and any element* $r \in R$ *there exists a subset* $V_2 \in \Omega$ *such that* $R \cdot V_2 \subseteq V_1$ *and* $V_2 \cdot r \subseteq V_1$;

6) *For any subset* $V_1 \in \Omega$ *there exists a subset* $V_2 \in \Omega$ *such that* $V_2 \cdot V_2 \subseteq V_1$.

Definition 2.2. A subset V of the ring R is called *symmetric* if $-V = V$.

Notation 2.3. If $V_1, V_2, \dots$ and $S_1, S_2, \dots$ are non-empty symmetric subsets of a ring R, then for any natural number k we define by induction the subset $F_k\big(S_1, \dots, S_k; V_1, \dots, V_k\big)$ of the ring R:

We take $F_1(S_1; V_1) = V_1 + V_1 + V_1 \cdot V_1 + S_1 \cdot V_1 + V_1 \cdot S_1$ and

$$F_{k+1}(S_1, S_2, \dots, S_{k+1}; V_1, V_2, \dots, V_k) = F_1(S_1; V_1 \cup F_k\big(S_2, \dots, S_{k+1}; V_2, \dots, V_{k+1}\big)).$$

Proposition 2.4. *If* $V_1, V_2, \dots$ *and* $S_1, S_2, \dots$ *are some sequences of non-empty finite symmetric subsets of a ring* R *and* $0 \in V_i$ *for any natural numbers* i*, then for any natural number* k *the following statements are true:*

2.4.1. $V_1 + V_1 \subseteq F_1(S_1; V_1)$, $V_1 \cdot V_1 \subseteq F_1(S_1; V_1)$, $S_1 \cdot V_1 \subseteq F_1(S_1; V_1)$ *and* $V_1 \cdot S_1 \subseteq F_1(S_1; V_1)$;

2.4.2. *For any natural number* k *the set* $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *is a finite symmetric set;*

2.4.3. $F_k(S_1, \dots, S_k; \{0\}, \dots, \{0\}) = \{0\}$ *for any natural number* k*;*

2.4.4. *If* k *is a natural number and* $U_i \subseteq V_i \subseteq R$ *and* $T_i \subseteq S_i \subseteq R$ *for any natural number* $1 \le i \le k$*, then*

$$F_k(T_1, \dots, T_k; U_1, \dots, U_k) \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k);$$

2.4.5. *If* k *and* p *are natural numbers and* $V_{k+j} = \{0\}$ *for any natural number* $1 \le j \le p$*, then*

$$F_k(S_1, \dots, S_k; V_1, \dots, V_k) = F_{k+p}(S_1, \dots, S_{k+p}; V_1, \dots, V_{k+p});$$

2.4.6. *For any natural number* $k \ge 2$ *the following equality is true*

$$F_k(S_1, \dots, S_k; V_1, \dots, V_k) =$$

$$F_k(S_1, \dots, S_k; V_1 \cup F_{k-1}\ (S_2, \dots, S_k; V_2, \dots, V_k), \dots, V_{k-1} \cup F_1(S_k; V_k), V_k);$$

2.4.7. $V_t \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *for any natural numbers* $1 \le t \le k$;

2.4.8. $F_{k+1}(S_s, \dots, S_{k+s}; V_s, \dots, V_{k+s}) \subseteq F_{k+s-t+1}(S_t, \dots, S_{k+s}; V_t, \dots, V_{k+s})$ *for any natural numbers* k, s, t *and* $t \le s$.

Proof. For proof of Statements 2.4.1 – 2.4.7 see in [2] the proof of Proposition 5.3.2.

We prove Statement 2.4.8 by induction on the number $s-t$.

If $s-t=0$, then $t=s$, and then

$$F_{k+1}(S_s,\ldots,S_{k+s};V_s,\ldots,V_{k+s}) = F_{k+(s-t)+1}(S_t,\ldots,S_{k+s};V_t,\ldots,V_{k+s}).$$

Assume that the required inclusion is proved for the number $s-t=n$ and any natural numbers k,s and let $s-t=n+1$. Then, from the induction assumption and Statement 4.7 it follows

$$F_{k+1}(S_s,\ldots,S_{k+s};V_s,\ldots,V_{k+s}) \subseteq V_s \cup F_{k+n+1}(S_s,\ldots,S_{n+s};V_2,\ldots,V_{k+n+s}) \subseteq$$

$$F_1(S_s;V_s \cup F_{k+n}(S_{s+1},\ldots,S_{k+n+s};V_{s+1},\ldots,V_{k+n+s})) =$$

$$F_{k+n+1}(S_s,\ldots,S_{k+n+s};V_s,\ldots,V_{k+n+s}) = F_{k+s-t+1}(S_s,\ldots,S_{k+s-t};V_s,\ldots,V_{k+s-t})$$

for any natural numbers s,k.

Thus Statement 2.4.8 is proved, and hence, Proposition 2.4 is proved. □

Definition 2.5. If R is a ring and x is some variable, then we denote by $R[x]$ the free ring generated by the set $R\bigcup\{x\}$.

We call elements of the ring $R[x]$ the generalized polynomials over the ring R in variable x.

Definition 2.6. For any element $a \in R$, we consider:

– The mapping $\varphi_a : R\bigcup\{x\} \to R$ such that $\varphi_a(x) = a$ and $\varphi_a(b) = b$ for any $b \in R$;

– The ring homomorphism $\tilde{\varphi}_a : R[x] \to R$, which is an extension of the mapping $\varphi_a : R\bigcup\{x\} \to R$;

– If $f(x) \in R[x]$, then we denote by $f(a)$ the element $\tilde{\varphi}_a(f(x))$ of the ring R;

– We call the element $f(0)$ of the ring R the *free term of generalized polynomial* $f(x) \in R[x]$;

– An element b of the ring R is called *a root of a generalized polynomial* $f(x) \in R[x]$ if $f(b) = 0$.

Notation 2.7. If $R = \{0, \pm r_1, \pm r_2, \ldots\}$ is a countable ring, then for any natural number k let $S_k = \{\pm r_1, \pm r_2, \ldots, \pm r_k\}$.

Theorem 2.8. *If τ is a non-discrete Hausdorff ring topology on a countable ring $R = \{0, \pm r_{1,}, \pm r_2, \ldots\}$ and $f(x)$ is a generalized polynomial over the ring R with nonzero free term, then there exists a neighborhood W of zero such that each element $r \in W$ is not a root of this polynomial.*

Proof. Since (R, τ_0) is a Hausdorff space, then there exists a countable basis $\{V_1, V_2, \ldots\}$ of the filter neighborhoods of zero such that $-V_k = V_k$ and $V_k \cap S_k = \emptyset$ (definition of the set S_k see above) and

$$F_1(S_{k+1};V_{k+1}) = V_{k+1} + V_{k+1} + V_{k+1}\cdot V_{k+1} + S_{k+1}\cdot V_{k+1} + V_{k+1}\cdot S_{k+1} \subseteq V_k$$

for any natural number k.

Using induction on k it is easy to prove that $F_k(S_{i+1},\dots,S_{i+k};V_{i+1},\dots,V_{i+k})\subseteq V_i$ for any natural numbers i,k.

Since $f(0)\neq 0$, then $f(0)\notin V_{t_0}$ for some natural number t_0.

If S is the set of all nonzero elements of the ring R which are included in the expression of the polynomial $f(x)-f(0)$, then the finiteness of the set S implies that $S\subseteq S_{i_0}$ for some natural number i_0, and hence, $S\subseteq S_i$ for all natural numbers $i\geq i_0$. Besides, if n is a natural number such that the ring operations include not more than n times in the expression of the polynomial $f(x)-f(0)$ and $n\geq i_0$, then from the definition of sets $F_k(S_{i+1},\dots,S_{i+k};V_{i+1},\dots,V_{i+k})$ it follows that

$$f(x)-f(0)\in F_n\big(S_{t_0+1},\dots,S_{t_0+n};\{0\},\dots,\{0\},\{x,0,-x\}\big).$$

Then

$$f(r)-f(0)\in F_n\big(S_{t_0+1},\dots,S_{t_0+n};\{0\},\dots,\{0\},\{r,0,-r\}\big)\subseteq F_n\big(S_{t_0+1},\dots,S_{t_0+n};V_{t_{t_0}+1},\dots,V_{t_0+n}\big)\subseteq V_{t_0}$$

for any $r\in V_{t_0+n}$. And since $F(0)\notin V_{t_0}$, then $f(r)\neq 0$ for any $r\in V_{t_0+n}$.

The theorem is proved. □

Since the intersection of a finite number of neighborhoods of zero is a neighborhood of zero, then from Theorem 8 follows

Corollary 2.9. *If τ is a non-discrete Hausdorff ring topology of a countable ring R, then for any finite set of generalized polynomials over the ring R with nonzero free terms there exists a neighborhood W of zero such that each element $r\in W$ is not a root of each of these polynomials.*

Proposition 2.10. *The following statements are true:*

2.10.1. *There exists a set $\tilde{\mathbb{N}}$ of subsets of the set $\mathbb{N}$ of natural numbers such that the cardinality of the set $\tilde{\mathbb{N}}$ is continuum and $A\cap B$ is a finite set for any different sets A and B* (see. [6], the proof of example 3.6.18);

2.10.2. *If $(\beta\mathbb{N},\tau)$ is the Stone–Čech compactification of the set $\mathbb{N}$ of natural numbers with the discrete topology, then $\mathbb{N}$ is a dense subset of the Hausdorff space $(\beta\mathbb{N},\tau)$ and the cardinality of the set $\beta\mathbb{N}$ is two to the power of continuum* (see [6], Corollary 3.6.12).

3 Basic results

Theorem 3.1. *If $R=\{0,\pm r_1,,\pm r_2,\dots\}$ is a countable ring and τ_0 is a non-discrete Hausdorff ring topology such that the topological ring (R,τ_0) has a countable basis of the filter of neighborhoods of zero, then the following statements are true:*

3.1.1. *For any infinite set A of natural numbers there is a metrizable ring topology $\tau(A)$ such that $\tau_0\leq\tau(A)$;*

3.1.2. $\sup\{\tau(A),\tau(B)\}$ *is the discrete topology for any infinite sets A and B of natural numbers such that $A\cap B$ is a finite set;*

3.1.3. *There are continuum of ring topologies stronger than τ_0 and such that any two of them are comparable to each other;*

3.1.4. *There exist two to the power of continuum of ring topologies such that* $\sup\{\tau_1, \tau_2\}$ *is the discrete topology for any two different topologies τ_1 and τ_2;*

3.1.5. *There exist two to the power of continuum of coatoms in the lattice of ring topologies of the ring R.*

Proof. Since (R, τ_0) is a Hausdorff space, then there exists a countable basis $\{V_1, V_2, \ldots\}$ of the filter of neighborhoods of zero such that $-V_k = V_k$ and $V_k \cap S_k = \emptyset$ and

$$F_1(S_{k+1}; V_{k+1}) = V_{k+1} + V_{k+1} + V_{k+1} \cdot V_{k+1} + S_{k+1} \cdot V_{k+1} + V_{k+1} \cdot S_{k+1} \subseteq V_k$$

for any natural number k.

Then by induction on n it is easy to prove that

$$F_n(S_{i+1}, \ldots, S_{i+n}; V_{i+1} \ldots, V_{i+n}) \subseteq V_i$$

for any natural numbers i and n.

Further the proof of Statement 3.1.1 will be realized in several steps.

Step I. By induction we construct a sequence $k_1, k_2, \ldots$ of natural numbers such that $k_i \geq i$, for any positive integer number i and a sequence $h_1, h_2, \ldots$ of nonzero elements of the ring R such that $\{-h_i, h_i\} \subseteq V_{k_i}$ and

$$F_n(S_1, \ldots, S_k; U_{A,1}, \ldots, U_{A,n}) \bigcap F_n(S_1, \ldots, S_k; U_{B,1}, \ldots, U_{B,n}) = \{0\}$$

for all subsets A and B of the set of all natural numbers such that $A \cap B = \emptyset$, where $U_{C,i} = \{h_i, 0, -h_i\}$ if $i \in C$ and $U_{C,i} = \{0\}$ if $i \notin C$, for any set C of natural numbers.

We take $k_1 = 2$, and as h_1 we take an arbitrary element of the set $V_2 \backslash \{0\}$.

If A and B are some sets of natural numbers such that $A \bigcap B = \emptyset$, then $k_1 \notin A$ or $k_1 \notin B$, and hence, $U_{A,1} = \{0\}$ or $U_{B,1} = \{0\}$. Then $F_1(S_1; U_{A,1}) \cap F_1(S_1; U_{B,1}) = \{0\}$ for any sets A and B of natural number such that $A \cap B = \emptyset$.

Suppose that we defined natural numbers $k_1 < k_2 < \ldots < k_n$ such that $k_i \geq i$ and nonzero elements $h_1, h_2, \ldots, h_n$ of the ring R such that $\{h_i, -h_i\} \subseteq V_{k_i}$ and

$$F_n(S_1, \ldots, S_k; U_{A,1}, \ldots, U_{A,n}) \cap F_n(S_1, \ldots, S_k; U_{B,1}, \ldots, U_{B,n}) = \{0\}$$

for any sets A and B of natural numbers such that $A \cap B = \emptyset$.

For any subsets $A' \subseteq \{1, \ldots, n\}$ and $B' \subseteq \{1, \ldots, n\}$ we consider a finite set

$$\Omega_{(A',B')} = F_{n+1}(S_1, \ldots, S_{n+1}; U_{A',1} \ldots, U_{A',n}, \{x, 0, -x\}) -$$

$$\big(F_n(S_1, \ldots, S_{n+1}; U_{B',1} \ldots, U_{B',n}) \backslash \{0\}\big)$$

of generalized polynomials over the ring R in variable x.

Since, according to Statement 4.5,

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A',1}\dots,U_{A',n},\{0\})=F_n(S_1,\dots,S_{n+1};U_{A',1},\dots,U_{A',n})$$

and, according to inductive assumption,

$$F_n(S_1,\dots,S_n;U_{A',1},\dots,U_{A',n})\cap F_n(S_1,\dots,S_n;U_{B',1},\dots,U_{B',n})=\{0\}$$

then the free term of generalized polynomial from $\Omega_{(A',B')}$ is nonzero.

Since the set $\{1,\dots,n\}$ has a finite number of subsets, then the set $\Phi_n = \bigcup\limits_{A',B'\subseteq\{1,\dots,n\},A'\bigcap B'=\emptyset}\Omega_{(A',B')}$ is a finite set of generalized polynomials with nonzero free term.

Then, by Corollary 2.9, there exists a neighborhood W of zero in the topological ring (R,τ_0) such that any element $r\in W$ is not a root of any polynomial of the set $\Phi_{n+1}(x)$.

Then there exists a natural number k_{n+1} such that $k_{n+1}>n+1$ and $V_{k_{n+1}}\subseteq W$. We take h_{n+1}, any element of the set $V_{k_{n+1}}\setminus\{0\}$.

We prove that

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n+1})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n+1})=\{0\}$$

for all sets A and B of natural numbers such that $A\cap B=\emptyset$.

Assume the contrary, and let

$$0\neq r\in F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n+1})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n+1}).$$

Since $A\cap B=\emptyset$ then from inductive assumption it follows that

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n},\{0\})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n},\{0\})=$$

$$F_n(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n})\cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n})=\{0\},$$

and hence $U_{A,n+1}=\{h_{n+1},0,-h_{n+1}\}$ or $U_{B,n+1}=\{h_{n+1},0,-h_{n+1}\}$ and from the definition of sets $U_{C,i}$ it follows that $U_{A,n+1}=\{0\}$ or $U_{B,n+1}=\{0\}$.

Assume, for definiteness, that $U_{A,n+1}=\{0\}$ and $U_{B,n+1}=\{h_{n+1},0,-h_{n+1}\}$.

Then

$$0\neq r\in F_n(S_1,\dots,S_n;U_{A,1},\dots,U_{A,n})\bigcap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n+1}),$$

and hence, $r=f(h_{n+1})$ for some generalized polynomial

$$f(x)\in F_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n},\{x,0,-x\}).$$

Since $r\notin F_n(S_1,\dots,S_n;U_{B,1}\dots,U_{B,n})$, the free term of the generalized polynomial $f(x)-r$ is nonzero, and the element h_{n+1} is a root of the generalized polynomial $f(x)-r$, and

$$f(x)-r\in F_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n},\{x,0,-x\})-$$

$$(F_n(S_1,\dots,S_{n+1};U_{B,1}\dots,U_{B,n})\setminus\{0\}).$$

As $U_{C,i} = U_{C\bigcap\{1,\dots,n\},i}$ for any natural number $1 \le i \le n$ and any set C of natural numbers, then $f(h) - r \in F_{n+1}(S_1,\dots,S_{n+1};U_{A',1},\dots,U_{A',n},\{h,0,-h\})-(F_n(S_1,\dots,S_{n+1};U_{B',1}\dots,U_{B',n})\setminus\{0\})$ for some subsets $A',B' \subseteq \{1,\dots,n\}$.

We have contradiction with the definition of the element h_{n+1}. Therefore

$$F_{n+1}(S_1,\dots,S_{n+1};U_{A,1}\dots,U_{A,n+1}) \cap F_{n+1}(S_1,\dots,S_{n+1};U_{B,1}\dots,U_{B,n+1}) = \{0\}.$$

So, we defined the sequence $k_1, k_2, \dots$ of natural numbers such that $k_i \ge i$ for any number i and the sequence $h_1, h_2, \dots$ of nonzero elements of the ring R such that $\{-h_i, h_i\} \subseteq V_{k_i}$ for any natural number i and

$$F_n(S_1,\dots,S_k;U_{A,1},\dots,U_{A,n}) \cap F_n\ (S_1,\dots,S_k;U_{B,1},\dots,U_{B,n}) = \{0\}$$

for all sets A and B of natural numbers such that $A \cap B = \emptyset$ and any natural number n.

Step II. For any pair (i,j) of natural numbers we consider the set

$$U_{(i,j),A} = F_j(U_{i+1,A},\dots,U_{i+j,A};S_{i+1},\dots,S_{i+j}),$$

where $U_{i,A} = \{0\}$ if $i \notin A$ and $U_{i,A} = \{0, h_i, -h_i\}$ if $i \in A$.

We show that for the sets $U_{(i,j),A}$ the following inclusions are true:

1. From Statement 2.4.3 it follows that $0 \in U_{(i,j),A}$ for any natural numbers i, j and

$$U_{(i,n),A} = F_n(S_{i+1},\dots,S_{i+n};U_{i+1,A},\dots,U_{i+n,A}) \subseteq$$
$$F_n(S_{i+1},\dots,S_{i+n};V_{i+1},\dots,V_{i+n}) \subseteq V_i$$

for any natural numbers i, n and any set A of natural numbers.

2. From Statements 2.4.4 and 2.4.5 it follows that $U_{(k,j),A} \subseteq U_{(k,n),A}$ for any natural numbers $j \le n$.

3. From Statement 2.4.8 it follows that $U_{(i,j),A} \subseteq U_{(k,j),A}$ for any natural numbers $k \le i$ and j.

4. From Statement 2.4.2 it follows that $U_{(i,j),A}$ is a symmetric set, i.e. $-U_{(I,j),A} = U_{(i,j),A}$ for any natural numbers i, j.

5. $U_{(i+1,j),A} \cdot U_{(i+1,j),A} \subseteq U_{(i,j),A}$ and $U_{(i+1,j),A} + U_{(i+1,j),A} \subseteq U_{(i,j),A}$ for any natural numbers i and $j > 1$.

6. $r_n \cdot U_{(i+n,j),A} \subseteq U_{(i,j),A}$ and $U_{(i+n,j),A} \cdot r_n \subseteq U_{(i,j),A}$ for any natural numbers i, j, n and any set A of natural numbers.

We prove the inclusion **5** by induction on the number j.

In fact, if $j = 2$, then, from the definition of sets $U_{(i,j),A}$, Statements 2.4.1 and 2.4.4, it follows:

$$U_{(i+1,2),A} \cdot U_{(i+1,2),A} = F_1(S_{i+2};U_{i+2,A}) \cdot F_1(S_{i+2};U_{i+2,A}) \subseteq$$
$$F_1\big(S_{i+1};F_1(S_{i+2};U_{i+2,A})\big) \subseteq F_1(S_{i+1};U_{i+1,A} \cup F_1(S_{i+2};U_{i+2,A})) =$$

$$F_2(S_{i+1}, S_{i+2}; U_{i+1,A}, U_{i+2,A}) = U_{(i,2),A}$$

and

$$U_{(i+1,2),A} + U_{(i+1,2),A} = F_1(S_{i+2}; U_{i+2,A}) + F_1(S_{i+2}; U_{i+2,A}) \subseteq$$
$$F_1\big(S_{i+1}; U_{i+1,A} \cup F_1(S_{i+2}; U_{i+2,A})\big) = F_1\big(S_{i+1}; F_1(S_{i+2}; U_{i+2,A})\big) \subseteq$$
$$F_2(S_{i+1}, S_{i+2}; U_{i+1,A}, U_{i+2,A}) = U_{(i,2),A}$$

for any natural number i and any set A of natural numbers.

Assume that the required inclusion is proved for natural number $j = n \geq 2$ and any natural number i. Then

$$U_{(i+1,i+n+1),A} \cdot U_{(i+1,i+n+1),A} = F_n(S_{i+2}, \ldots, S_{i+n+1};$$
$$U_{i+2,A}, \ldots, U_{i+n+1,A}) \cdot F_n(S_{i+2}, \ldots, S_{i+n+1}; U_{i+2,A}, \ldots, U_{i+n+1,A}) \subseteq$$
$$F_1(S_{i+1}; U_{i+1} \cup F_n(S_{i+2}, \ldots, S_{i+n+1}; U_{i+2,A}, \ldots, U_{i+n+1,A})) \subseteq$$
$$F_1\big(S_{i+1}; U_{i+1,A} \cup F_n(S_{i+2} \ldots, S_{i+n+1}; U_{i+2,A}, \ldots, U_{i+n+1,A})\big) =$$
$$F_{n+1}(S_{i+1} \ldots, S_{i+n+1}; U_{i+1,A}, \ldots, U_{i+n+1,A}) = U_{(i,n+1),A}$$

and

$$U_{(i+1,i+n+1),A} + U_{(i+1,i+n+1),A} =$$
$$F_n(S_{i+2}, \ldots, S_{i+n+1}; U_{i+2,A}, \ldots, U_{i+n+1,A}) +$$
$$F_n(S_{i+2}, \ldots, S_{i+n+1}; U_{i+2,A}, \ldots, U_{i+n+1,A}) \subseteq$$
$$F_1(S_{i+1}; U_{i+1} \cup F_n(S_{i+2}, \ldots, S_{i+n+1}; U_{i+2,A}, \ldots, U_{i+n+1,A})) \subseteq$$
$$F_1(S_{i+1}; U_{i+1,A} \cup F_n(S_{i+2}, \ldots, S_{i+n+1}; U_{i+2,A}, \ldots, U_{i+n+1,A})) =$$
$$F_{n+1}(S_{i+1} \ldots, S_{i+n+1}; U_{i+1,A}, \ldots, U_{i+n+1,A}) = U_{(i,n+1),A}.$$

Proof of inclusion **6**. In fact,

$$r_n \cdot U_{(i+n,j),A} \subseteq S_{i+n} \cdot F_{n+i+j}(S_{n+i+1}, \ldots, S_{n+i+j}; U_{n+i+1,A}, \ldots, U_{n+i+j,A}) \subseteq$$
$$F_1(S_{n+i}; U_{n+i,A} \cup F_{n+i+j}(S_{n+i+1}, \ldots, S_{n+i+j}; U_{n+i+1,A}, \ldots, U_{n+i+j,A})) =$$

$U_{(i+n-1,j),A} \subseteq U_{(i,j),A}$ and

$$U_{(i+n,j),A} \cdot r_n \subseteq F_{n+i+j}(S_{n+i+1}, \ldots, S_{n+i+j}; U_{n+i+1,A}, \ldots, U_{n+i+j,A}) \cdot S_{i+n} \subseteq$$
$$F_1(S_{n+i}; U_{n+i,A} \cup F_{n+i+j}(S_{n+i+1}, \ldots, S_{n+i+j}; U_{n+i+1,A}, \ldots, U_{n+i+j,A})) =$$

$U_{(i+n-1,j),A} \subseteq U_{(i,j),A}$ for any natural numbers i, j, n, and any set A of natural numbers.

Step III. For every infinite set A of natural numbers and any natural number i we take $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(ij),A}$ and show that the set $\{\hat{U}_i(A) | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 2.1, and hence, this set is a basis of the filter of neighborhoods of zero for a ring topology $\tau(A)$ on the ring R.

In fact, since

$$U_{(i,n+1),A} = F_{n+1}(S_{i+1}, \dots, S_{i+n+1}; U_{i+1,A}, \dots, U_{i+n+1,A}) \subseteq$$

$$F_{n+1}(S_{i+1} \dots, S_{i+n+1}; V_{i+1}, \dots, V_{i+n+1}) \subseteq V_i$$

for any natural numbers i and n, then $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(ij),A} \subseteq V_i$. Then $\{0\} \subseteq \bigcap\limits_{i=1}^{\infty} \hat{U}_i(A) \subseteq \bigcap\limits_{i=1}^{\infty} V_i = \{0\}$, and hence, the condition 1 of Theorem 2.1 is satisfied.

From inclusions 2 and 3 (see Step II), it follows

$$\hat{U}_i(A) \bigcap \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} (U_{(i,j),A}) \bigcap (\bigcup_{l=1}^{\infty} U_{(k,l),A}) =$$

$$\bigcup_{j=1}^{\infty} \bigcup_{l=1}^{\infty} (U_{(i,j),A} \bigcap U_{(k,l),A}) = \bigcup_{j=1}^{\infty} U_{(t,j),A} = \hat{U}_t(A),$$

where $t = \max\{i, k\}$, and hence, the condition 2 of Theorem 2.1 is satisfied.

From inclusions 2 and 5 (see Step II) it follows

$$\hat{U}_i(A) + \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} U_{(i,j),A}) + (\bigcup_{l=1}^{\infty} U_{(i,l),A}) =$$

$$\bigcup_{j=1} \bigcup_{j=1} (U_{(i,j),A} + U_{(i,l),A}) = \bigcup_{t=1}^{\infty} U_{(i-1,t),A} = \hat{U}_{i-1}(A)$$

and

$$\hat{U}_i(A) \cdot \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} U_{(i,j),A}) \cdot (\bigcup_{l=1}^{\infty} U_{(i,l),A}) =$$

$$\bigcup_{j=1}^{\infty} \bigcup_{j=1}^{\infty} (U_{(i,j),A} \cdot U_{(i,l),A}) = \bigcup_{t=1}^{\infty} U_{(i-1,t),A} = \hat{U}_{i-1}(A)$$

for any natural number $i > 1$, and hence, the conditions 3 and 6 of Theorem 2.1 are satisfied.

From inclusion 3 (see Step II) it follows

$$-\hat{U}_i(A) = -(\bigcup_{j=1}^{\infty} U_{(i,j),A}) = \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{\infty} (-U_{(i,j),A}) = \bigcup_{j=1}^{\infty} U_{j,A} = \hat{U}_i(A)$$

for any natural number i, and hence, the condition 4 of Theorem 2.1 is satisfied.

Now, let $r \in R$.

If $r = 0$, then $r \cdot \hat{U}_i(A) = \{0\} \subseteq \hat{U}_i(A)$ and $\hat{U}_i(A) \cdot r = \{0\} \subseteq \hat{U}_i(A)$ for any natural number i and any set A of natural numbers.

If $r \neq 0$, then $r = r_n$ or $r = -r_n$ for some natural number n. Then, from the inclusion of 6, it follows $r_n \cdot \hat{U}_{i+n}(A) \subseteq \hat{U}_i(A)$ and $\hat{U}_{i+n}(A) \cdot r_n \subseteq \hat{U}_i(A)$ for any natural number i, and hence, the condition 5 of Theorem 2.1 is satisfied.

Thus, we have shown that the set $\{\hat{U}_i(A) | i \in \mathbb{N}\}$ satisfies conditions 1 – 6 of Theorem 2.1, and hence, this set is a basis of the filter neighborhoods of zero for a ring topology $\tau(A)$ on the ring R.

Since $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(i,j),A} \subseteq V_i$ for any natural number i, then $\tau_0 \leq \tau(A)$.

Thus Statement 3.1.1 is proved.

Proof of Statement 3.1.2.

For any subset $A \in \tilde{\mathbb{N}}$ (definition of the set $\tilde{\mathbb{N}}$ see in Statement 2.10.1) we consider the ring topology $\tau(A)$, constructed in the proof of Step III of Statement 3.1.1.

Since the set $\tilde{\mathbb{N}}$ has cardinality of continuum, then to finish the proof of this Statement, it remains to verify that $\sup\{\tau(A), \tau(B)\}$ is the discrete topology for different sets $A, B \in \tilde{\mathbb{N}}$.

Let $A, B \in \mathbb{N}$. Then there exists a natural number n such that

$$(A \backslash \{1, \ldots, n\}) \bigcap (B \backslash \{1, \ldots, n\}) = \emptyset.$$

If $A' = A \backslash \{1, \ldots, n\}$ and $B' = B \backslash \{1, \ldots, n\}$, then (see proof of Step I of Statement 3.1.1)

$$F_k(S_1, \ldots, S_k; U_{A',1}, \ldots, U_{A',k}) \bigcap F_k(S_1, \ldots, S_k; U_{B',1}, \ldots, U_{B',k}) = \{0\}$$

for any natural number k.

Since $U_{i,A} = U_{i,A'}$ and $U_{i,B} = U_{i,B'}$ for any natural number $i > n$, then $U_{(i,j),A} = U_{(i,j),A'}$ and $U_{(i,j),B} = U_{(i,j),B'}$ for any natural numbers i, j such that $i > n$, and hence, $U_{(i,j),A} \bigcap U_{(i,j),B} = U_{(i,j),A'} \bigcap U_{(i,j),B'} = \{0\}$ for any natural numbers i, j such that $i > n$. Then, from the inclusion of 2 (see step II), it follows that

$$\hat{U}_{n+1}(A) \bigcap \hat{U}_{n+1}(B) = \bigcup_{j=1}^{\infty} \big(U_{(n+1,j),A} \bigcap U_{(n+1,j),B}\big) = \{0\}.$$

Since $\hat{U}_{n+1}(A)$ and $\hat{U}_{n+1}(B)$ are neighborhoods of zero in topological rings $(R, \tau(A))$ and $(R, \tau(B))$, respectively, then $\{0\}$ is a neighborhood of zero in the topological ring $(R, \sup\{\tau(A), \tau(B)\})$, and hence, $\sup\{\tau(A), \tau(B)\}$ is the discrete topology.

Statement 3.1.2 is proved.

Proof of Statement 3.1.3. If $\mathbb{Q}$ is the set of all rational numbers and $\mathbb{N}$ is the set of all natural numbers, then there is a bijection $\xi : \mathbb{Q} \to \mathbb{N}$.

For each real number r we consider the infinite set $A_r = \{\xi(q) | q \in \mathbb{Q}, r \leq q\}$ of natural numbers and let $\tau(A_r)$ be the ring topology on the ring R, constructed in the proof of step III of Statement 3.1.1.

We show that the set $\{\tau(A_r) | r$ is a real number$\}$ of ring topologies is as required.

Since the set $\{\hat{U}_i(A_r) | i \in \mathbb{N}\}$ is a basis of the filter of neighborhoods of zero for the ring topology $\tau(A_r)$, then the topological ring $(R, \tau(A_r))$ has a countable basis of filter of neighborhoods of zero.

We show that for any two distinct real numbers r, r' topologies $\tau(A_r)$ and $\tau(A_{r'})$ are different and comparable.

In fact, if $r < r'$, then $A_r \backslash A_{r'}$ is an infinite set, and then for any natural number n there exists a natural number $k \in A_r \backslash A_{r'}$ such that $k > n$. Then $h_k \in U_{(k,1),A_r} \subseteq U_{(n,1),A_r}$ and $h_k \notin U_{(1,s),A_{r'}}$ for any natural number s, and hence, $h_k \in \hat{U}_n(A_r)$ and $h_k \notin \hat{U}_1(A_{r'})$. The arbitrariness of the number n implies that $\tau(A_r) \neq \tau(A_{r'})$, and hence, the set $\{\tau(A_r) | r \in R\}$ has the cardinality of continuum.

In addition, since $A_{r'} = \xi(\{q | q \in \mathbb{Q}, r' \leq q\} \subseteq \xi(\{q | q \in \mathbb{Q}, r \leq q\} = A_r$, then (see the definition of the sets $U_{(i,j),A}$ in the proof of Step II of Statement 3.1.1) $U_{(i,j),A_{r'}} \subseteq U_{(i,j),A_r}$ for any natural numbers i, j. Then $\hat{U}_{n,A_{r'}} \subseteq \hat{U}_{n,A_r}$ for any natural number n, and since the sets of $\{U_{n,A_{r'}} | n \in \mathbb{N}\}$ and $\{U_{n,A_r} | n \in \mathbb{N}\}$ are basis of the filters of neighborhoods of zero in topological rings $(R, \tau(A_{r'}))$ and $(R, \tau(A_r))$, respectively, then $\tau(A_r) \leq \tau(A_{r'})$.

Statement 3.1.3 is proved.

Proof of Statement 3.1.4. If (see Statement 2.10.2) $\hat{a} \in \beta\mathbb{N} \backslash \mathbb{N}$, then $\hat{U} \bigcap N$ is an infinite set of natural numbers for any neighborhood $\hat{U}$ of the element $\hat{a}$ in the topological space $(\beta\mathbb{N}, \tau)$.

Let $\tau(\hat{U} \bigcap \mathbb{N})$ be the ring topology, defined according to Statement 3.1.1, and let $\hat{\tau}_{\hat{a}} = \sup\Big\{\tau(\hat{U} \bigcap N) | \hat{U}$ is a neighborhood of element $\hat{a}$ in the topological space $(\beta\mathbb{N}, \tau)\Big\}$.

Since the cardinality of the set $\beta\mathbb{N} \backslash \mathbb{N}$ is equal to two to the power of the continuum, then it suffices to prove that $\sup\{\hat{\tau}_{\hat{a}}, \hat{\tau}_{\hat{b}}\}$ is a discrete topology for any deferent elements $\hat{a}, \hat{b} \in \beta\mathbb{N} \backslash \mathbb{N}$.

So, let $\hat{a}, \hat{b} \in \beta\mathbb{N} \backslash \mathbb{N}$ and $\hat{a} \neq \hat{b}$. Since the space $(\beta\mathbb{N}, \tau)$ is a Hausdorff space, then there exist neighborhoods $\hat{U}$ and $\hat{V}$ of elements $\hat{a}$ and $\hat{b}$ in the topological space $(\beta\mathbb{N}, \tau)$, respectively, such that $\hat{U} \bigcap \hat{V} = \emptyset$. Then, according to Statement 11.2, $\sup\{\tau(\mathbb{N} \cap \hat{U}), \tau(\mathbb{N} \cap \hat{V})\}$ is the discrete topology, and hence, $\sup\{\hat{\tau}_{\hat{a}}, \hat{\tau}_{\hat{b}}\}$ is the discrete topology.

Statement 3.1.4 is proved.

Proof of Statement 3.1.5. If T is the set of all non-discrete, ring topologies on the ring R, then by the theorem of Kuratowski-Zorn, for any non-discrete ring topology $\hat{\tau}_{\hat{a}}$, constructed in Statement 3.1.4, there is a maximal element $\tau^*_{\hat{a}}$ such that $\tau^*_{\hat{a}} \geq \hat{\tau}_{\hat{a}}$. Then the set $\{\tau^*_{\hat{a}} \ | \hat{a} \in \beta\mathbb{N} \backslash N\}$ is as required.

The theorem is proved. □

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