BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(76), 2014, Pages 65–79
ISSN 1024–7696

On LCA groups with locally compact rings of continuous endomorphisms. I

Valeriu Popa

Abstract. We determine the discrete abelian groups and the compact abelian groups with the property that their rings of continuous endomorphisms are locally compact in the compact-open topology.

Mathematics subject classification: 16W80.
Keywords and phrases: LCA groups, rings of continuous endomorphisms, compact-open topology.

1 Introduction

Let X be an LCA group, and let $A(X)$ denote the group of all topological automorphisms of X, taken with the Birkhoff topology. As is well known, $A(X)$ is a Hausdorff topological group [3, Ch. IV]. M. Levin in [9], O. Mel'nikov in [10], P. Plaumann in [12], and L. Robertson in [13] have investigated (among many other things) various types of LCA groups X with the property that their group $A(X)$ is locally compact.

By analogy, one may ask for a description of LCA groups X with the property that the ring $E(X)$ of continuous endomorphisms of X is locally compact in the compact-open topology. Here we answer this question for the case of discrete abelian groups and for the case of compact abelian groups.

2 Notation

Throughout the following, $\mathbb{N}$ is the set of natural numbers (including zero), $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$, and $\mathbb{P}$ is the set of prime numbers.

The groups of which we shall make constant use are the reals modulo one $\mathbb{T}$, the p-adic integers $\mathbb{Z}_p$ (all with their usual topologies), the rationals $\mathbb{Q}$, the quasi-cyclic groups $\mathbb{Z}(p^\infty)$ and the cyclic groups $\mathbb{Z}(n)$ of order n (all with the discrete topology), where $p \in \mathbb{P}$ and $n \in \mathbb{N}_0$.

We denote by $\mathcal{L}$ the class of all locally compact abelian groups. For $X \in \mathcal{L}$, we let $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and X^* denote, respectively, the connected component of zero in X, the maximal divisible subgroup of X, the subgroup of compact elements of X, the smallest closed subgroup K of X such that the quotient group X/K is torsion-free, the torsion subgroup of X, and the character group of X. Further, we denote by $E(X)$ the ring of continuous endomorphisms of X and by $H(X,Y)$, where $Y \in \mathcal{L}$, the group of continuous homomorphisms from X to Y, both endowed with the compact-open topology. It is well known that $H(X,Y)$ is a topological group and $E(X)$ is a topological ring. Recall that the compact-open topology on $H(X,Y)$ is generated by the sets

$$\Omega_{X,Y}(K,U) = \{h \in H(X,Y) \mid h(K) \subset U\},$$

where K is a compact subset of X and U is an open subset of Y. We write $\Omega_X(K,U)$ for $\Omega_{X,X}(K,U)$.

For $p \in \mathbb{P}$, $n \in \mathbb{N}$ and $X \in \mathcal{L}$, X_p is the topological p-primary component of X, $t_p(X)$ is the p-primary component of $t(X)$, $X[n] = \{x \in X \mid nx = 0\}$, $nX = \{nx \mid x \in X\}$, and $S(X) = \{q \in \mathbb{P} \mid \big(k(X)/c(X)\big)_q \neq 0\}$.

For $a \in X$ and $S \subset X$, $o(a)$ is the order of a, $\langle S\rangle$ is the subgroup of X generated by S, $\overline{S}$ is the closure of S in X, and $A(X^*, S) = \{\gamma \in X^* \mid \gamma(x) = 0 \text{ for all } x \in S\}$. If $(A_i)_{i\in I}$ is an indexed collection of subgroups of X, $\sum_{i\in I} A_i$ stands for $\langle\bigcup_{i\in I} A_i\rangle$.

Further, if A is a closed subgroup of X, then A_* denotes the smallest pure subgroup of X containing A, and X/A the quotient of X modulo A, taken with the quotient topology. Also, we write $X = A \oplus B$ in case X is a topological direct sum of its subgroups A and B.

Let $(X_i)_{i\in I}$ be an indexed collection of groups in $\mathcal{L}$. We write $\prod_{i\in I} X_i$ for the direct product of the groups X_i with the product topology. In case each X_i is discrete, $\bigoplus_{i\in I} X_i$ denotes the external direct sum of the groups X_i, taken with the discrete topology. If each $X_i = X$ for some fixed X, we write X^I for $\prod_{i\in I} X_i$ and $X^{(I)}$ for $\bigoplus_{i\in I} X_i$.

The symbol $\cong$ denotes topological group (ring) isomorphism.

We close this section by mentioning a few facts, which will be used frequently in the sequel. The first one is the famous theorem of Ascoli, whose proof can be found in [2, Ch. X, §2, Theorem 2, Corollary 3].

Theorem 1 (**Ascoli**). *Let X be a locally compact topological space, let Y be a uniform space, and let $C(X,Y)$ be the space of continuous mappings from X into Y, endowed with the compact-open topology. A subset Ω of $C(X,Y)$ is relatively compact in $C(X,Y)$ if ant only if the following conditions hold:*

(i) *Ω is equicontinuous.*

(ii) *For each $a \in X$, the orbit $\Omega a = \{f(a) \mid f \in \Omega\}$ is relatively compact in Y.*

Lemma 1. *For any $X \in \mathcal{L}$, the mapping $f \to f^*$, where the endomorphism $f^* \in E(X^*)$ is defined by $f^*(\gamma) = \gamma \circ f$ for all $\gamma \in X^*$, is a topological ring anti-isomorphism from $E(X)$ onto $E(X^*)$.*

Proof. The assertion follows from [11, Ch. IV, Theorem 4.2, Corollary 2]. □

Lemma 2. *Let X be a group in $\mathcal{L}$ admitting the decomposition $X = A \oplus B$ for some closed subgroups A and B of X. Then $E(X) \cong \begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$, where the matrix ring $\begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$ is taken with the usual addition and multiplication, and carries the product topology.*

Proof. Since the canonical projections $\pi_A : X \to A$, $\pi_B : X \to B$ and the canonical injections $\eta_A : A \to X$, $\eta_B : B \to X$ are continuous, the mapping

$$\xi : E(X) \to \begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix},\ f \to \begin{pmatrix} \pi_A f \eta_A & \pi_A f \eta_B \\ \pi_B f \eta_A & \pi_B f \eta_B \end{pmatrix},$$

is an isomorphism of rings with the inverse ξ^{-1} given by

$$\xi^{-1}(\begin{pmatrix} f_A & f_{B,A} \\ f_{A,B} & f_B \end{pmatrix}) = \eta_A f_A \pi_A + \eta_A f_{B,A} \pi_B + \eta_B f_{A,B} \pi_A + \eta_B f_B \pi_B$$

[6, Proposition 106.1]. Now, if K_A (resp., K_B) is a compact subset of A (resp., B) and U_A (resp., U_B) is an open neighborhood of zero in A (resp., B), then $K_A + K_B$ is a a compact subset of X, $U_A + U_B$ is an open neighborhood of zero in X, and

$$\xi\big(\Omega_X(K_A + K_B, U_A + U_B)\big) \subset \begin{pmatrix} \Omega_A(K_A, U_A) & \Omega_{B,A}(K_B, U_A) \\ \Omega_{A,B}(K_A, U_B) & \Omega_B(K_B, U_B) \end{pmatrix}.$$

Since the sets $\begin{pmatrix} \Omega_A(K_A, U_A) & \Omega_{B,A}(K_B, U_A) \\ \Omega_{A,B}(K_A, U_B) & \Omega_B(K_B, U_B) \end{pmatrix}$ form a fundamental system of neighborhoods of zero in $\begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$, it follows that ξ is continuous. To see that ξ is also open, pick any compact subset K of X and any open neighborhood U of zero in X. Further, choose an open neighborhood V of zero in X such that $V + V \subset U$. Since $X = A \oplus B$, we can consider that $V = \pi_A(V) + \pi_B(V)$. Then

$$\xi\big(\Omega_X(K, U)\big) \supset \begin{pmatrix} \Omega_A(\pi_A(K), \pi_A(V)) & \Omega_{B,A}(\pi_B(K), \pi_A(V)) \\ \Omega_{A,B}(\pi_A(K), \pi_B(V)) & \Omega_B(\pi_B(K), \pi_B(V)) \end{pmatrix},$$

proving that ξ is open. □

3 Discrete torsion groups

We begin our study on local compactness of the topological ring $E(X)$ with the case of discrete torsion groups $X \in \mathcal{L}$. The description of these groups, will permit also to answer our question for compact, totally disconnected groups in $\mathcal{L}$.

First we establish a preparatory fact.

Lemma 3. *Let $p \in \mathbb{P}$, and let X be a discrete p-group in $\mathcal{L}$. If $E(X)$ is locally compact, then X is isomorphic to a group of the form $\mathbb{Z}(p^\infty)^{n_0} \times \prod_{i=1}^m \mathbb{Z}(p^{n_i})$, where $m, n_0, \ldots, n_m \in \mathbb{N}$.*

Proof. Let $E(X)$ be locally compact. We can write $X = A \oplus B$, where $A = d(X)$ and B is a reduced subgroup of X. Since

$$E(X) \cong \begin{pmatrix} E(A) & H(B,A) \\ 0 & E(B) \end{pmatrix},$$

it follows that $E(A)$ and $E(B)$ are locally compact. To determine the structure of A, let Ω_A be a compact neighborhood of zero in $E(A)$. Since A is discrete, there is a finite subset K_A of A such that $\Omega_A(K_A, \{0\}) \subset \Omega_A$. Consequently, $\Omega_A(K_A, \{0\})$ is compact in $E(A)$. Let $F_A = \langle K_A \rangle$. Then F_A is finite because A is torsion [5, Theorem 15.5], and A/F_A is divisible because A is divisible [5, (D), p. 98]. It follows that if $A \neq \{0\}$, then $A/F_A \neq \{0\}$ too, and hence we can write $A/F_A = D \oplus D'$, where $D \cong \mathbb{Z}(p^\infty)$ [5, Theorem 23.1]. Let α be the canonical projection of A onto A/F_A and φ the canonical projection of A/F_A onto D with kernel D'. Fix a non-zero $a \in D[p]$ and any $a' \in A$ such that $(\varphi \circ \alpha)(a') = a$. Further, choose an arbitrary $y \in A[p]$, and denote by ξ_y the extension to D of the group homomorphism from $\langle a \rangle$ into A, which transports a to y [5, Theorem 21.1]. Then $(\xi_y \circ \varphi \circ \alpha) \in \Omega_A(K_A, \{0\})$ and $(\xi_y \circ \varphi \circ \alpha)(a') = y$. Since $y \in A[p]$ was chosen arbitrarily, it follows that

$$A[p] \subset \Omega_A(K_A, \{0\})a'.$$

But $\Omega_A(K_A, \{0\})a'$ is finite since $\Omega_A(K_A, \{0\})$ is compact and A is discrete. It follows that $A[p]$ is finite, and hence $A \cong \mathbb{Z}(p^\infty)^{n_0}$ for some $n_0 \in \mathbb{N}$ [5, Theorem 25.1].

Next we determine the structure of B. As in the case of A, there is a finite subset K_B of B such that $\Omega_B(K_B, \{0\})$ is compact in $E(B)$. Let $F_B = \langle K_B \rangle$. Then F_B is finite, and hence if $B = F_B$, there is nothing to prove. Assume $B \neq F_B$. First observe that B/F_B cannot be divisible. For, if it were, then it would follow that $B = pB + F_B$. Choosing $n \in \mathbb{N}$ such that $p^n F_B = \{0\}$, we would obtain $p^n B = p^{n+1} B$, which would imply that $p^n B$ is divisible. Since B is reduced, it would follow that $p^n B = \{0\}$, and hence B/F_B would be of bounded order as well. This is in contradiction with the fact that B/F_B is non-zero and divisible. Consequently, B/F_B is not divisible, and hence its socle contains elements of finite height [5, (C), p. 98]. It follows that B/F_B admits a non-zero cyclic direct summand [5, Corollary 27.2]. Write

$$B/F_B = \langle b \rangle \oplus C$$

for some non-zero $b \in B/F_B$, and let β be the canonical projection of B onto B/F_B, and ψ the canonical projection of B/F_B onto $\langle b \rangle$ with kernel C. Further, let $b' \in B$ be such that $(\psi \circ \beta)(b') = b$. Given any $z \in B[p]$, define $\eta_z \in H(\langle b \rangle, B)$ by setting $\eta_z(b) = z$. Then $\eta_z \circ \psi \circ \beta \in \Omega_B(K_B, \{0\})$ and $(\eta_z \circ \psi \circ \beta)(b') = z$. Since $z \in B[p]$ was chosen arbitrarily, it follows that

$$B[p] \subset \Omega_B(K_B, \{0\})b',$$

so $B[p]$ is finite, and hence $B \cong \prod_{i=1}^{m} \mathbb{Z}(p^{n_i})$ for some $m, n_1, \ldots, n_m \in \mathbb{N}$ [5, Theorem 25.1]. □

Now we can prove

Theorem 2. *For a discrete torsion group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(X)$ is compact.*

(ii) *$E(X)$ is locally compact.*

(iii) *For each $p \in S(X)$, $t_p(X)$ is isomorphic with $\mathbb{Z}(p^\infty)^{n_0(p)} \times \prod_{i=1}^{m(p)} \mathbb{Z}(p^{n_i(p)})$, where $m(p), n_0(p), \ldots, n_{m(p)}(p) \in \mathbb{N}$.*

Proof. The fact that (i) and (iii) are equivalent is proved in [6, Proposition 107.4]. It is also clear that (i) implies (ii). Assume (ii), and pick an arbitrary $p \in S(X)$. We can write

$$X = t_p(X) \oplus t_p(X)^{\#},$$

where $t_p(X)^{\#} = \sum_{q \in S(X)\setminus\{p\}} t_q(X)$. It follows from Lemma 2 that $E\big(t_p(X)\big)$ is locally compact, so (ii) implies (iii) by Lemma 3. □

By utilizing duality, we obtain the solution to the considered problem in the case of compact, totally disconnected groups in $\mathcal{L}$.

Corollary 1. *For a compact, totally disconnected group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(X)$ is compact.*

(ii) *$E(X)$ is locally compact.*

(iii) *For each $p \in S(X)$, X_p is topologically isomorphic with $\mathbb{Z}_p^{n_0(p)} \times \prod_{i=1}^{m(p)} \mathbb{Z}(p^{n_i(p)})$, where $m(p), n_0(p), \ldots, n_{m(p)}(p) \in \mathbb{N}$.*

4 Discrete torsion-free groups

In this section, we consider the case of discrete torsion-free groups in $\mathcal{L}$ and the case of their duals, the compact connected groups in $\mathcal{L}$. We have

Theorem 3. *For a discrete torsion-free group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(X)$ is discrete.*

(ii) *$E(X)$ is locally compact.*

(iii) *There is a finitely generated subgroup F of X such that X contains no subgroup isomorphic to a group of the form X/L, where L is a proper, pure subgroup of X containing F.*

Proof. Clearly, (i) implies (ii). Assume (ii), and let Ω be a compact neighborhood of zero in $E(X)$. Since X is discrete, there exists a finite subset K of X such that $\Omega_X(K, \{0\}) \subset \Omega$, so $\Omega_X(K, \{0\})$ is compact in $E(X)$. We claim that $\Omega_X(K, \{0\}) = \{0\}$. For, if there existed a nonzero endomorphism $f \in \Omega_X(K, \{0\})$, we would have $f(x) \neq 0$ for some $x \in X \backslash F$. It would then follow that the orbit $\Omega_X(K, \{0\})x$ contains the infinite group $\langle f(x) \rangle$. This is a contradiction because $\Omega_X(K, \{0\})x$ must be finite by Ascoli's theorem. Thus $\Omega_X(K, \{0\}) = \{0\}$, and hence (ii) implies (i).

Next we show that (i) and (iii) are equivalent. Assume (i), and let K be a finite subset of X such that $\Omega_X(K, \{0\}) = \{0\}$. Set $F = \langle K \rangle$. Given any proper, pure subgroup L of X such that $F \subset L$, let λ be the canonical projection of X onto X/L. If there existed an isomorphism η from X/L into X, we would have $\eta \circ \lambda \in \Omega_X(K, \{0\})$. This is a contradiction, because $X/L \neq \{0\}$, and hence $\eta \circ \lambda \neq 0$. Consequently, (i) implies (iii). Now assume (iii), and let F be a finitely generated subgroup of X with the property that X contains no subgroup isomorphic to a group of the form X/L, where L is a proper, pure subgroup of X containing F. Fix a finite set K of generators of F. We claim that $\Omega_X(K, \{0\}) = \{0\}$. Indeed, if there existed a non-zero $f \in \Omega_X(K, \{0\})$, we would have $\ker(f) \neq X$, $F \subset \ker(f)$, and $X/\ker(f) \cong \operatorname{im}(f)$. Moreover, since $\operatorname{im}(f)$ is torsion-free, $\ker(f)$ would also be pure in X [5, (d), p. 114]. This contradiction shows that $\Omega_X(K, \{0\}) = \{0\}$, so (iii) implies (i). □

We mention for later use the following

Corollary 2. *Let $X \in \mathcal{L}$ be discrete, divisible, and torsion-free. The ring $E(X)$ is locally compact if and only if $X \cong \mathbb{Q}^r$ for some $r \in \mathbb{N}$.*

Proof. Let $E(X)$ be locally compact. It follows from Theorem 3 that there is a finitely generated subgroup F of X such that X contains no subgroup isomorphic to a group of the form X/L, where L is a proper, pure subgroup of X containing F. Since any pure subgroup of a divisible group is divisible, we can write $X = F_* \oplus G$ for some subgroup G of X [5, Theorem 21.2]. As $F \subset F_*$ and $G \cong X/F_*$, we must have $X = F_*$, so $X \cong \mathbb{Q}^r$ for some $r \in \mathbb{N}$.

The converse is clear. □

Dualizing Theorem 3 gives the following characterization of compact connected groups $X \in \mathcal{L}$ whose ring $E(X)$ is locally compact.

Corollary 3. *Let $X \in \mathcal{L}$ be compact and connected. The following statements are equivalent:*

(i) *$E(X)$ is discrete.*

(ii) *$E(X)$ is locally compact.*

(iii) *There is a closed subgroup G of X satisfying the conditions:*

  (1) *X/G has no small subgroups;*

(2) *No non-zero quotient of X by a closed subgroup is topologically isomorphic to a pure, closed subgroup of X contained in G.*

Proof. As is well known, X is compact and connected iff X^* is discrete and torsion-free [7, (23.17) and (24.25)]. Now, since $E(X)$ and $E(X^*)$ are topologically isomorphic, it is clear that $E(X)$ is discrete iff $E(X^*)$ is discrete, and $E(X)$ is locally compact iff $E(X^*)$ is locally compact. In particular, it follows from Theorem 3 that (i) and (ii) are equivalent. To finish, it remains to observe that a subgroup F of X^* is finitely generated iff the quotient $X/A(X,F)$ has no small subgroups [1, Proposition 7.9]. Further, a subgroup L of X^* is pure in X^* iff $A(X,L)$ is pure in X [1, Corolary 7.6] and $F \subset L$ iff $A(X,L) \subset A(X,F)$. Finally, the existence of a monomorphism $f : X^*/L \to X^*$ is equivalent to the existence of a continuous epimorphism $f^* : X \to A(X,L)$ [7, (24.40)]. □

Corollary 4. *Let $X \in \mathcal{L}$ be compact, connected, and torsion-free. The ring $E(X)$ is locally compact if and only if $X \cong (\mathbb{Q}^*)^r$ for some $r \in \mathbb{N}$.*

Proof. Follows from Corollary 2 by duality. □

5 Discrete mixed and reduced groups

For discrete mixed groups in $\mathcal{L}$ the situation is more complicated. In this section we examine the case of reduced groups. We also examine the case of duals of such groups.

We begin by recalling the following

Definition 1. Let X be an abelian group. For $p \in \mathbb{P}$ and $a \in X$, the p-height of a in X is defined by:

$$h_p^X(a) = \begin{cases} n, & \text{if } a \in p^n X \text{ but } a \notin p^{n+1}X; \\ \infty, & \text{if } a \in \bigcap_{i\in\mathbb{N}} p^i X. \end{cases}$$

Theorem 4. *Let $X \in \mathcal{L}$ be discrete, mixed, and reduced. The ring $E(X)$ is locally compact if and only if X has a finitely generated subgroup F satisfying the following conditions:*

(i) *For each proper subgroup L of X such that $F \subset L$ and $X/L \neq t(X/L)$, X contains no subgroup isomorphic to X/L.*

(ii) *For each $p \in S(X)$, either X/F is p-divisible or $t_p(X)$ is finite.*

(iii) *For each non-zero $a \in X$, the set*

$$S_a = \left\{p \in S(X) \mid h_p^{X/F}(a+F) < \infty \ \text{ and } \ t_p(X) \neq X[p^{h_p^{X/F}(a+F)}]\right\}$$

is finite.

Proof. Let $E(X)$ be locally compact. Since X is discrete, there is a finite subset K of X such that $\Omega_X(K,\{0\})$ is compact in $E(X)$. Given $x \in X$, we deduce from the Ascoli's theorem that $\Omega_X(K,\{0\})x$ is finite. Consequently, for any $f \in \Omega_X(K,\{0\})$, $f(x)$ is a torsion element of X, and hence $\operatorname{im}(f) \subset t(X)$. Set $F = \langle K \rangle$.

To see that (i) holds, let L be a proper subgroup of X such that $F \subset L$ and $t(X/L) \neq X/L$, and let $\lambda : X \to X/L$ be the canonical projection. If there existed an isomorphism η from X/L into X, we would have $\eta \circ \lambda \in \Omega_X(K,\{0\})$, a contradiction because $\operatorname{im}(\eta \circ \lambda) \not\subset t(X)$. This proves (i).

To see that (ii) holds, pick any $p \in S(X)$ such that X/F is not p-divisible. Then $F + pX \neq X$, so we can write

$$X/(F + pX) = A \oplus B,$$

where $A \cong \mathbb{Z}(p)$. Let π be the canonical projection of X onto $X/(F + pX)$ and φ the canonical projection of $X/(F + pX)$ onto A with kernel B. Fix a generator g of A, and let $g' \in X$ be such that $(\varphi \circ \pi)(g') = g$. Further, pick any $y \in X[p]$, and define $\xi_y \in H(A, X)$ by setting $\xi_y(g) = y$. It is clear that $\xi_y \circ \varphi \circ \pi \in \Omega_X(K,\{0\})$, so

$$y = (\xi_y \circ \varphi \circ \pi)(g') \in \Omega_X(K,\{0\})g'.$$

Since $y \in X[p]$ was chosen arbitrarily, it follows that

$$X[p] \subset \Omega_X(K,\{0\})g',$$

so $X[p]$ is finite. Since X is reduced, this proves (ii) [5, Theorem 25.1].

To see that (iii) holds, pick any non-zero $a \in X$ and any $p \in S_a$, and set $n = h_p^{X/F}(a+F)$. Then $a \in F + p^n X$ and $a \notin F + p^{n+1}X$, so $X/(F + p^{n+1}X)$ is a non-zero p-group of bounded order and $a + (F + p^{n+1}X)$ has order p in $X/(F + p^{n+1}X)$. Let $g' \in X$ be such that $a - p^n g' \in F$. Then

$$a + (F + p^{n+1}X) = p^n g' + (F + p^{n+1}X).$$

By [6, Corollary 27.1], we can write

$$X/(F + p^{n+1}X) = A' \oplus B',$$

where $A' = \langle g' + (F + p^{n+1}X) \rangle$ and B' is a subgroup of $X/(F + p^{n+1}X)$. Clearly, $g' + (F + p^{n+1}X)$ has order p^{n+1} in $X/(F + p^{n+1}X)$. Let π' be the canonical projection of X onto $X/(F + p^{n+1}X)$ and φ' the canonical projection of $X/(F + p^{n+1}X)$ onto A' with kernel B'. Given any $y \in X[p^{n+1}]$, define $\xi'_y \in H(A, X)$ by setting

$$\xi'_y\big(g' + (F + p^{n+1}X)\big) = y.$$

Then $\xi'_y \circ \varphi' \circ \pi' \in \Omega_X(K,\{0\})$, so

$$p^n y = (\xi'_y \circ \varphi' \circ \pi')(a) \in \Omega_X(K,\{0\})a.$$

Since, by the definition of S_a, $t_p(X)$ contains elements y of order p^{n+1}, it follows that $\Omega_X(K,\{0\})a$ contains at least one non-zero p-element. Thus, if S_a were infinite, it would follow that $\Omega_X(K,\{0\})a$ is infinite as well, a contradiction. This proves (iii).

To show the converse, let F be a finitely generated subgroup of X satisfying the conditions (i), (ii) and (iii), and let K be a finite set of generators of F. We claim that $\Omega_X(K,\{0\})$ is compact in $E(X)$. Indeed, since X is discrete, $\Omega_X(K,\{0\})$ acts equicontinuously on X. Hence, by the Ascoli's theorem, we need only to show that $\Omega_X(K,\{0\})$ acts with finite orbits. Fix an arbitrary $a \in X$. We first show that

$$\Omega_X(K,\{0\})a \subset \sum_{p \in S_a} t_p(X).$$

Pick an arbitrary $f \in \Omega_X(K,\{0\})$. If $f(a)=0$, there is nothing to prove. Assume $f(a) \neq 0$. Since $F \subset \ker(f)$ and $X/\ker(f) \cong \operatorname{im}(f)$, it follows from (i) that $X/\ker(f)$ is torsion, and hence $\operatorname{im}(f) \subset t(X)$. Consequently, we can write

$$f(a) = b_1 + \cdots + b_m \tag{1}$$

with nonzero $b_1 \in t_{p_1}(X), \ldots, b_m \in t_{p_m}(X)$ for some $m \in \mathbb{N}_0$ and some distinct $p_1, \ldots, p_m \in S(X)$. We must show that $p_1, \ldots, p_m \in S_a$. By way of contradiction, suppose there is $j = 1, \ldots, m$ such that $p_j \notin S_a$. First observe that X/F cannot be p_j-divisible. For, if it were, it would follow from [5, (D), p. 98] that $\operatorname{im}(f)$ is p_j-divisible, so the projection $\operatorname{im}(f)_{p_j}$ of $\operatorname{im}(f)$ into $t_{p_j}(X)$ would be p_j-divisible. As $\operatorname{im}(f)_{p_j}$ is a p_j-group, it would follow that $\operatorname{im}(f)_{p_j}$ is divisible [5, p. 98], which would imply $\operatorname{im}(f)_{p_j} = \{0\}$ because X is reduced. Hence we would have $b_j = 0$, a contradiction. This proves that X/F cannot be p_j-divisible, and hence $t_{p_j}(X)$ must be finite by (ii). Now, since by assumption $p_j \notin S_a$, we must have either $h_{p_j}^{X/F}(a+F) = \infty$, or else $n = h_{p_j}^{X/F}(a+F) < \infty$ and $t_{p_j}(X) = X[p_j^n]$. In the former case, we clearly have $h_{p_j}^X\big(f(a)\big) = \infty$ [5, (g), p. 98]. Hence, given any $i \in \mathbb{N}$, we can write $f(a) = p_j^i y_i$ with $y_i \in X$. Further, since the numbers $p_1, \ldots, p_m$ are distinct, we can write $b_k = p_j^i c_{i,k}$ for all $k \neq j$ [5, p. 98]. It follows from (1) that

$$b_j = p_j^i (y_i - c_{i,1} - \cdots - c_{i,j-1} - c_{i,j+1} - \ldots - c_{i,m}).$$

Since $t_{p_j}(X)$ is p_j-pure in X, we conclude that $b_j = p_j^i c_{i,j}$ for some $c_{i,j} \in t_{p_j}(X)$ [5, p. 98]. Thus $h_{p_j}^{t_{p_j}(X)}(b_j) = \infty$, whence $b_j = 0$ because $t_{p_j}(X)$ is finite. This contradicts the assumption that $b_j \neq 0$. In the second case, one can show in a similar way that $h_{p_j}^{t_{p_j}(X)}(b_j) \geq n$, which again gives $b_j = 0$ because, in this case, $t_{p_j}(X)$ cannot have non-zero elements b with $h_{p_j}^{t_{p_j}(X)}(b) \geq n$. This proves that $p_1, \ldots, p_m \in S_a$. Since $f \in \Omega_X(K,\{0\})$ was chosen arbitrarily, we conclude that $\Omega_X(K,\{0\})a \subset \sum_{p \in S_a} t_p(X)$. Finally, in order to show that $\Omega_X(K,\{0\})$ acts with finite orbits, it is enough to show by the finiteness of S_a that for each $p \in S_a$, $t_p(X)$ is finite. But, if $p \in S_a$, then $n = h_p^{X/F}(a+F) < \infty$, so $a \in F + p^n X$ and $a \notin F + p^{n+1}X$. It follows that $X \neq F + p^{n+1}X$, so $X/(F + p^{n+1}X)$, being a non-zero p-group of bounded order, cannot be p-divisible [5, p. 98]. As

$$X/(F + p^{n+1}X) \cong \big(X/F\big)/\big((F + p^{n+1}X)/F\big),$$

X/F cannot be p-divisible too, so $t_p(X)$ is finite by (ii). The proof is complete. □

To state the dual analog of Theorem 4, several additional concepts must be introduced.

Definition 2. A group $X \in \mathcal{L}$ is said to be

(i) comixed if either $\bigcap_{n\in\mathbb{N}} \overline{nX}$ is a non-trivial subgroup of X, i.e. $\{0\} \subsetneq \bigcap_{n\in\mathbb{N}} \overline{nX} \subsetneq X$, or $\bigcap_{n\in\mathbb{N}} \overline{nX} = \{0\}$ and none of the subgroups $\overline{nX}$ with $n \in \mathbb{N}_0$ is compact.

(ii) coreduced if $m(X) = X$.

Definition 3. Let $X \in \mathcal{L}$. A closed subgroup C of X is said to be submaximal in X if X/C is topologically isomorphic to a closed subgroup of $\mathbb{T}$.

Definition 4. Let $X \in \mathcal{L}$, and let C and G be closed subgroups of X. For $p \in \mathbb{P}$, the p-coheight of C in G is defined by:

$$ch_p^G(C) = \begin{cases} n, & \text{if } G[p^n] \subset C \text{ but } G[p^{n+1}] \not\subset C; \\ \infty, & \text{if } t_p(G) \subset C. \end{cases}$$

We have

Corollary 5. *Let $X \in \mathcal{L}$ be compact, comixed, and coreduced. The ring $E(X)$ is locally compact if and only if X has a closed subgroup G satisfying the following conditions:*

(i) *X/G has no small subgroups.*

(ii) *No quotient of X by a closed subgroup is topologically isomorphic to a closed subgroup M of G with $c(M) \neq \{0\}$.*

(iii) *For each $p \in S(X)$, either $G[p] = \{0\}$ or $\bigcap_{n\in\mathbb{N}} p^nX$ is open in X.*

(iv) *For each submaximal subgroup C of X, the set*

$$S_C = \Big\{p \in S(X) \mid ch_p^G(C) < \infty \text{ and } \bigcap_{n\in\mathbb{N}} p^nX \neq p^{ch_p^G(C)}X\Big\}$$

is finite.

Proof. Clearly, X is compact, comixed, and coreduced iff X^* is discrete, mixed, and reduced. It follows from Lemma 1 that $E(X)$ is locally compact iff X^* satisfies the conditions of Theorem 4. It remains to translate these conditions in terms of X. We already mentioned in the proof of Corollary 3 that a subgroup F of X^* is finitely generated iff the quotient $X/A(X,F)$ has no small subgroups. Now, a subgroup L of X^* has the property that $X^*/L \neq t(X^*/L)$ iff its annihilator $A(X,L)$ has non-zero connected component. For,

$$c\big(A(X,L)\big) \cong c\big((X^*/L)^*\big) \cong A\big((X^*/L)^*, t(X^*/L)\big)$$

by [7, (23.25) and (24.20)]. It follows that X^* satisfies condition (i) of Theorem 4 iff X satisfies condition (ii) of this corollary. Further, given any $p \in S(X)$ and any subgroup F of X^*, we deduce from [7, (23.25) and (24.22)(i)] that

$$A(X,F)[p] \cong (X^*/F)^*[p] = A\big((X^*/L)^*, p(X^*/F)\big).$$

It follows that X^*/F is p-divisible iff $A(X,F)[p] = \{0\}$. It is also clear from [1, (e), p. 10] that $t_p(X^*)$ is finite in X^* iff $\cap_{n\in\mathbb{N}} p^n X$ is open in X. Consequently, X^* satisfies condition (ii) of Theorem 4 iff X satisfies condition (iii) of this corollary. Finally, we deduce from [7, (23.25)] that a closed subgroup C of X is submaximal in X iff $A(X^*,C)$ is cyclic in X^*. Letting $a \in X^*$ be a generator of $A(X^*,C)$, it is easy to see by use of duality that $ch_p^{A(X,F)}(C) = h_p^{X^*/F}(a+F)$, and $p^{ch_p^{A(X,F)}(C)}X \neq \cap_{n\in\mathbb{N}} p^n X$ iff $X^*[p^{h_p^{X^*/F}(a+F)}] \neq t_p(X^*)$. It follows that X^* satisfies condition (iii) of Theorem 4 iff X satisfies condition (iv) of our corollary. □

6 Discrete mixed and non-reduced groups

In this final section, we handle the remaining case of discrete, mixed, and non-reduced groups in $\mathcal{L}$. By duality, we obtain also the solution to our problem in the case of compact, comixid groups $X \in \mathcal{L}$ with $m(X) \neq X$.

Theorem 5. *Let X be a discrete, mixed, non-reduced group in $\mathcal{L}$, written in the form $X = d(X) \oplus Y$ for some reduced subgroup Y of X. The ring $E(X)$ is locally compact if and only if the following conditions hold:*

(i) $d(X) \cong \mathbb{Q}^r \times \big(\bigoplus_{p\in S(d(X))} \mathbb{Z}(p^\infty)^{n_p}\big)$, *where r and the n_p's are positive integers.*

(ii) *There is a finitely generated subgroup M of Y satisfying:*

  (1) $Y/M = t(Y/M)$.

  (2) *For each $p \in S(Y)$, either Y/M is p-divisible or $t_p(Y)$ is finite.*

  (3) *For each non-zero $a \in Y$, the set*

$$S_a = \big\{p \in S(Y) \mid h_p^{Y/M}(a+M) < \infty \;\; \text{and} \;\; t_p(Y) \neq Y[p^{h_p^{Y/M}(a+M)}]\big\}$$

  is finite.

Proof. Let $E(X)$ be locally compact. Since $X = d(X) \oplus Y$, we have

$$E(X) \cong \begin{pmatrix} E(d(X)) & H(Y, d(X)) \\ 0 & E(Y) \end{pmatrix},$$

so $E(d(X))$, $E(Y)$, and $H(Y, d(X))$ are locally compact. Further, we can write $d(X) = A \oplus B$, where $A = t(d(X))$ and $t(B) = \{0\}$. Then

$$E(d(X)) \cong \begin{pmatrix} E(A) & H(B, A) \\ 0 & E(B) \end{pmatrix},$$

so $E(A)$, $E(B)$, and $H(B, A)$ are locally compact as well. Now, from the local compactness of $E(A)$ it follows by Theorem 2 that for each $p \in S(d(X))$, $t_p(A) \cong \mathbb{Z}(p^\infty)^{n_p}$ for some $n_p \in \mathbb{N}_0$. Further, from the local compactness of $E(B)$ it follows by Corollary 2 that $B \cong \mathbb{Q}^r$ for some $r \in \mathbb{N}$. Hence (i) holds.

Next, from the local compactness of $H(Y, d(X))$, we deduce that there exists a finite subset K of Y such that $\Omega_{Y,d(X)}(K, \{0\})$ is compact in $H(Y, d(X))$. In addition, from the local compactness of $E(Y)$, we conclude that there is a finitely generated subgroup F of Y satisfying the conditions (i)-(iii) of Theorem 4. Fix a finite set K' of generators of F, and set $M = \langle K \cup K' \rangle$. Let us establish (1). Suppose the contrary, and pick an element $a \in Y$ with $o(a + M) = \infty$. We can define, for each $z \in d(X)$, a group homomorphism $\xi_z : \langle a + M \rangle \to d(X)$ by setting $\xi_z(a + M) = z$. Since $d(X)$ is divisible, ξ_z extends to a group homomorphism $\widehat{\xi_z} : Y/M \to d(X)$. Letting $\pi : Y \to Y/M$ denote the canonical projection, we have $\widehat{\xi_z} \circ \pi \in \Omega_{Y,d(X)}(K \cup K', \{0\})$ and $z = (\widehat{\xi_z} \circ \pi)(a)$, so $z \in \Omega_{Y,d(X)}(K \cup K', \{0\})a$. Since $z \in d(X)$ was chosen arbitrarily, it follows that

$$d(X) \subset \Omega_{Y,d(X)}(K \cup K', \{0\})a.$$

This is a contradiction because $\Omega_{Y,d(X)}(K \cup K', \{0\})a$ is finite by Ascoli's theorem and $d(X)$ is infinite. Thus Y/M must be torsion. To see that (2) holds, pick any $p \in S(Y)$. By the choice of F, either Y/F is p-divisible, or $t_p(Y)$ is finite. Since $Y/M \cong \big(Y/F\big)/\big(M/F\big)$, it is clear that if Y/M is not p-divisible, then Y/F is not p-divisible as well. Therefore (2) must hold. Finally, to see that (3) holds, pick any non-zero $a \in Y$. By the choice of F, the set

$$S_a^F = \big\{ p \in S(Y) \mid h_p^{Y/F}(a + F) < \infty \ \text{ and } \ t_p(Y) \neq Y[p^{h_p^{Y/F}(a+F)}] \big\}$$

is finite. But since Y/M is a homomorphic image of Y/F, we have

$$h_p^{Y/F}(a + F) \leq h_p^{Y/M}(a + M),$$

whence

$$Y[p^{h_p^{Y/F}(a+F)}] \subset Y[p^{h_p^{Y/M}(a+M)}]$$

for all $p \in S(Y)$. It follows that $S_a \subset S_a^F$, which proves (3).

For the converse, we first show that $E(d(X))$ is locally compact. Indeed, we can write $d(X) = A \oplus B$, where $A \cong \bigoplus_{p \in S\left(t\left(d(X)\right)\right)} \mathbb{Z}(p^\infty)^{n_p}$ and $B \cong \mathbb{Q}^r$. It is clear from Theorem 2 and Corollary 2 that $E(A)$ and $E(B)$ are locally compact. Since

$$E(d(X)) \cong \begin{pmatrix} E(A) & H(B,A) \\ 0 & E(B) \end{pmatrix},$$

it remains to show that $H(B,A)$ is locally compact. But $H(B,A) \cong H(\mathbb{Q},A)^r$ [7, (24.34)(c)], so it suffices to show that $H(\mathbb{Q},A)$ is locally compact. We claim that $\Omega_{\mathbb{Q},A}(\{1\},\{0\})$ is compact in $H(\mathbb{Q},A)$. Clearly, $\Omega_{\mathbb{Q},A}(\{1\},\{0\})$ is equicontinuous. Pick any $a \in \mathbb{Q}$, and let $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ denote the canonical projection. Since $\mathbb{Q}/\mathbb{Z}$ is torsion, we can write $\pi(a) = \sum_{p \in P_a} b_p$, where P_a is a finite subset of $\mathbb{P}$ and $b_p \in t_p(\mathbb{Q}/\mathbb{Z})$ for each $p \in P_a$. It is clear that $\pi(a) \in \sum_{p \in P_a} (\mathbb{Q}/\mathbb{Z})[p^{n_p}]$, where, for each $p \in P_a$, n_p denotes the exponent of b_p. Given any $f \in \Omega_{\mathbb{Q},A}(\{1\},\{0\})$, we can write $f = \widehat{f} \circ \pi$ for some $\widehat{f} \in H(\mathbb{Q}/\mathbb{Z}, A)$ [8, Theorem 5.6]. It follows that

$$f(a) = \widehat{f}(\pi(a)) \in \sum_{p \in P_a \cap S(A)} \widehat{f}\big((\mathbb{Q}/\mathbb{Z})[p^{n_p}]\big) \subset \sum_{p \in P_a \cap S(A)} A[p^{n_p}],$$

so

$$\Omega_{\mathbb{Q},A}(\{1\},\{0\})a \ \subset \sum_{p \in P_a \cap S(A)} A[p^{n_p}],$$

and hence $\Omega_{\mathbb{Q},A}(\{1\},\{0\})a$ is finite. This proves that $\Omega_{\mathbb{Q},A}(\{1\},\{0\})$ is compact, so $E(d(X))$ is locally compact. Next, since Y certainly satisfies the conditions of Theorem 4, we deduce that $E(Y)$ is locally compact as well. To finish, it is enough, in view of the isomorphism

$$E(X) \cong \begin{pmatrix} E(d(X)) & H(Y, d(X)) \\ 0 & E(Y) \end{pmatrix},$$

to show that $H(Y, d(X))$ is locally compact. Fix a finite set K of generators of M. We claim that $\Omega_{Y,d(X)}(K,\{0\})$ is compact in $H(Y,d(X))$. To see this, it suffices to show that $\Omega_{Y,d(X)}(K,\{0\})$ acts with finite orbits. Let $\pi' : Y \to Y/M$ be the canonical projection, and pick an arbitrary $a \in Y$. Since Y/M is torsion, we can write $\pi'(a) = \sum_{p \in P'_a} b'_p$, where P'_a is a finite subset of $\mathbb{P}$ and $b'_p \in t_p(Y/M)$ for each $p \in P'_a$. It is clear that $\pi'(a) \in \sum_{p \in P'_a} (Y/M)[p^{n_p}]$, where, for each $p \in P'_a$, n_p denotes the exponent of b'_p. As every $f \in \Omega_{Y,d(X)}(K,\{0\})$ can be written in the form $f = \widehat{f} \circ \pi'$ for some $\widehat{f} \in H(Y/M, d(X))$ [8, Theorem 5.6], we conclude that

$$\Omega_{Y,d(X)}(K,\{0\})a \ \subset \sum_{p \in P'_a \cap S(A)} A[p^{n_p}],$$

proving that $\Omega_{Y,d(X)}(K,\{0\})a$ is finite. Hence $H(Y,d(X))$ is locally compact. The proof is complete. $\square$

We end by stating the dual analog of Theorem 5.

Corollary 6. *Let X be a compact, comixed group in $\mathcal{L}$ with $m(X) \neq X$. The ring $E(X)$ is locally compact if and only if the following conditions hold:*

(i) $X/m(X) \cong (\mathbb{Q}^*)^r \times \prod_{p \in S(Y)} \mathbb{Z}_p^{n_p}$, *where r and the n_p's are positive integers.*

(ii) *There exists a closed totally disconnected subgroup L of $m(X)$ satisfying:*

(1) $m(X)/L$ *has no small subgroups.*

(2) *For each $p \in S(m(X))$, either $L[p] = \{0\}$ or $\bigcap_{n \in \mathbb{N}} p^n\big(m(X)\big)$ is open in $m(X)$.*

(3) *For each submaximal subgroup C of $m(X)$, the set*

$$S_C = \big\{p \in S(m(X)) \mid ch_p^L(C) < \infty \ \text{ and } \ \bigcap_{n \in \mathbb{N}} p^n\big(m(X)\big) \neq p^{ch_p^L(C)}\big(m(X)\big)\big\}$$

is finite.

Proof. Clearly, a group $X \in \mathcal{L}$ is compact, comixed, and satisfies $m(X) \neq X$ iff X^* is discrete, mixed, and non-reduced. In particular, $X^* = d(X^*) \oplus Y$ for some reduced subgroup Y of X^*, whence

$$X = m(X) \oplus A(X, Y).$$

Now, in view of Lemma 1, $E(X)$ is locally compact iff X^* satisfies the conditions of Theorem 5. It remains to translate these conditions in terms of X. Since

$$d(X^*) \cong \big(X/m(X)\big)^*,$$

it is clear that X^* satisfies condition (i) of Theorem 5 iff X satisfies condition (i) of this corollary. We next show that X^* satisfies condition (ii) of Theorem 5 iff X satisfies condition (ii) of our corollary. Given a subgroup M of Y, we have $A(X, M) \supset A(X, Y)$, so

$$A(X, M) = \big(m(X) \cap A(X, M)\big) \oplus A(X, Y).$$

Since

$$M^* \cong X/A(X, M) \cong m(X)/\big(m(X) \cap A(X, M)\big),$$

we conclude that M is finitely generated iff $m(X)/\big(m(X) \cap A(X, M)\big)$ has no small subgroups [1, Proposition 7.9]. Further, by [4, Exercise 3.8.7(b)], we have

$$\big(Y/M\big)^* \cong A(X, M)/A(X, Y) \cong m(X) \cap A(X, M),$$

so Y/M is torsion iff $m(X) \cap A(X, M)$ is totally disconnected [7, (24.26)]. It follows that Y has a finitely generated subgroup M satisfying condition (1) of Theorem 5 iff $m(X)$ has a closed totally disconnected subgroup L satisfying condition (1) of our corollary. Next, since

$$Y \cong X^*/d(X^*) \cong m(X)^*,$$

it is clear that $S(m(X)) = S(Y)$. Given any $p \in S(Y)$, we have

$$\begin{aligned} A\big((Y/M)^*, p(Y/M)\big) &\cong (Y/M)^*[p] \\ &\cong \big(A(X,M)/A(X,Y)\big)[p] \\ &\cong \big(m(X) \cap A(X,M)\big)[p], \end{aligned}$$

so Y/M is p-divisible iff $\big(m(X) \cap A(X,M)\big)[p] = \{0\}$. Taking account of the isomorphism $Y^* \cong m(X)$, we also see that $t_p(Y)$ is finite iff $\bigcap_{n\in\mathbb{N}} p^n\big(m(X)\big)$ is open in $m(X)$. Consequently, X^* satisfies condition (2) of Theorem 5 iff X satisfies condition (2) of this corollary. Finally, it follows from [7, (23.25)] that a closed subgroup C of $m(X)$ is submaximal in $m(X)$ iff $A(m(X)^*, C)$ is cyclic in $m(X)^*$. Since $m(X)^* \cong Y$, it is easy to see by use of duality that X^* satisfies condition (3) of Theorem 4 iff X satisfies condition (3) of this corollary. □

References

[1] Armacost D. L. *The structure of locally compact abelian groups.* Pure and Applied Mathematics Series, Vol. 68 (Marcel Dekker, ed.), New York, 1981.

[2] Bourbaki N. *Topologie generale, Chapter IX-X, Éléments de mathematique.* Nauka, Moscow, 1975.

[3] Braconnier J. *Sur les groupes topologiques localement compact.* J. Math. Pures Apl., 1948, **27**, No. 9, 1–85.

[4] Dikranjan D., Prodanov I., Stoyanov L. *Topological groups.* Pure and Applied Mathematics Series, Vol. 130 (Marcel Dekker, ed.), New York and Basel, 1990.

[5] Fuchs L. *Infinite abelian groups, Vol. 1.* Academic Press, New York and London, 1970.

[6] Fuchs L. *Infinite abelian groups, Vol. 2.* Academic Press, New York and London, 1973.

[7] Hewitt E., Ross K. *Abstract Harmonic Analysis, Vol. 1.* Academic Press, New York, 1963.

[8] Hungerford Th. W. *Algebra.* Springer-Verlag, New York, 1974.

[9] Levin M. *The automorphism group of a locally compact abelian group.* Acta Math., 1971, **127**, 259–278.

[10] Mel'nikov O. *Compactness conditions for groups of automorphisms of topological groups.* Mat. Zametki, 1976, **19**(5), 735–743.

[11] Moskowitz M. *Homological algebra in locally compact abelian groups.* Trans. Amer. Math. Soc., 1967, **127**, 361–404.

[12] Plaumann P. *Automorphism groups of locally compact abelian groups.* Lecture Notes Math., 1981, **874**, 272–282.

[13] Robertson L. *Connectivity, divisibility, and torsion.* Trans. Amer. Math. Soc., 1967, **128**, 482–505.

Valeriu Popa
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028, Chişinău
Moldova
E-mail: *vpopa@math.md*

Received August 25, 2014