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Limits of solutions to the singularly perturbed abstract hyperbolic-parabolic system

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Abstract. We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + A(t)u_\varepsilon(t) = f_\varepsilon(t), & t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases}$$

in the Hilbert space H as $\varepsilon \to 0$, where $A(t), t \in (0,\infty)$, is a family of linear self-adjoint operators.

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1 Introduction

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$, and V is also a real Hilbert space endowed with the norm $||\cdot||$. Let $A(t) : V \subset H \to H$, $t \in [0,\infty)$, be a family of linear self-adjoint operators. Consider the following Cauchy problem:

$$\begin{cases} \varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + A(t)u_\varepsilon(t) = f_\varepsilon(t), & t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter ($\varepsilon \ll 1$), $u_\varepsilon, f_\varepsilon : [0,T) \to H$.

We investigate the behavior of solutions u_ε to the problems (P_ε) when $u_{0\varepsilon} \to u_0$, $f_\varepsilon \to f$ as $\varepsilon \to 0$. We establish a relationship between solutions to the problems (P_ε) and the corresponding solution to the following unperturbed problem:

$$\begin{cases} v'(t) + A(t)v(t) = f(t), & t \in (0,T), \\ v(0) = u_0. \end{cases} \tag{P_0}$$

If in some topology the solutions u_ε to the perturbed problems (P_ε) tend to the corresponding solution v to the unperturbed problem (P_0) as $\varepsilon \to 0$, then the problem (P_0) is called *regularly perturbed*. In the opposite case the problem (P_0) is called *singularly perturbed*. In the last case a subset of $[0,\infty)$ arises in which solutions u_ε have a singular behavior relative to ε. This subset is called *the boundary layer*. The function which defines the singular behavior of solution u_ε within the boundary layer is called *the boundary layer function*.

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In Theorems 5.1 and 5.2 we prove that solutions u_ε to the perturbed problem (P_ε) tend to the solution v to the unperturbed problem P_0 in the norm of the space $C([0,T];H)$, as $\varepsilon \to 0$. At the same time in the space $C^1([0,T];H)$ the solution u_ε has a singular behavior relative to parameter ε in the neighbourhood of $t=0$.

The problem (P_ε) is an abstract model of singularly perturbed problems of hyperbolic-parabolic type. Such kind of problems arises in the mathematical modeling of elasto-plasticity phenomena.

A large class of works is dedicated to the study of singularly perturbed Cauchy problems for differential equations of second order. Without pretending to a complete analysis of these works, we will mention some of them, which contain a rich bibliography. In [9, 10, 17], some asymptotic expansions of the solutions to linear wave equations and their derivatives have been obtained. In [1, 2, 4, 8, 15, 16] nonlinear problems of hyperbolic-parabolic type have been studied. Nonlinear abstract problems of hyperbolic-parabolic type have been studied in [5–7, 12].

Unlike other methods, our approach is based on two key points. The first one is the relationship between solutions to the problems (P_ε) and (P_0) in the linear case. The second key point consists of *a priori* estimates of solutions to the unperturbed problem, which are uniform with respect to small parameter ε. Moreover, the problem (P_ε) is studied for a larger class of functions f_ε, i. e. $f_\varepsilon \in W^{1,p}(0,T;H)$. Also we obtain the convergence rate, as $\varepsilon \to 0$.

In what follows we will need some notations. Let $k \in N^*$, $1 \le p \le +\infty$, $(a,b) \subset (-\infty,+\infty)$ and X be a Banach space. By $W^{k,p}(a,b;X)$ denote the Banach space of vectorial distributions $u \in D'(a,b;X)$, $u^{(j)} \in L^p(a,b;X)$, $j = 0,1,\dots,k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)} \right)^{\frac{1}{p}} \quad \text{for} \quad p \in [1,\infty),$$

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0\le j\le k} \|u^{(j)}\|_{L^\infty(a,b;X)} \quad \text{for} \quad p=\infty.$$

In the particular case $p=2$ we put $W^{k,2}(a,b;X) = H^k(a,b;X)$. If X is a Hilbert space, then $H^k(a,b;X)$ is also a Hilbert space with the scalar product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \left(u^{(j)}(t), v^{(j)}(t) \right)_X dt.$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$ define the Banach spaces

$$W^{k,p}_s(a,b;H) = \{ f : (a,b) \to H; f^{(l)}(\cdot)e^{-st} \in L^p(a,b;X),\, l = 0,\dots,k \},$$

with the norms

$$\|f\|_{W^{k,p}_s(a,b;X)} = ||fe^{-st}||_{W^{k,p}(a,b;X)}.$$

The framework of our paper will be determined by the following conditions:

(H1) *V is separable and $V \subset H$ densely and continuously,* i. e.

$$|u|^2 \leq \gamma ||u||^2, \quad \forall u \in V;$$

(H2) *For each $u, v \in V$ the function $t \mapsto (A(t)u, v)$ is continuously differentiable on $(0, \infty)$ and*

$$\big|(A'(t)u, v)\big| \leq a_0 |u||v|, \quad \forall u, v \in V, \quad \forall t \in [0, \infty);$$

(H3) *The operators $A(t) : V \subset H \to H, t \in [0, \infty)$ are linear, self-adjoint and positive definite, i.e. there exists $\omega > 0$ such that*

$$(A(t)u, u) \geq \omega \, ||u||^2, \quad \forall u \in V, \quad \forall t \in [0, \infty).$$

(H4) *For each $u, v \in V$ the function $t \mapsto (A(t)u, v)$ is twice continuously differentiable on $(0, \infty)$ and*

$$\big|(A''(t)u, v)\big| \leq a_1 |u||v|, \quad \forall u, v \in V, \quad \forall t \in [0, \infty).$$

2 Existence of solutions to problems (P_ε) and (P_0)

In [11] the following results concerning the solvability of problems (P_ε) and (P_0) are proved.

Theorem 2.1. *Let $T > 0$. Let us assume that the conditions* **(H1)**, **(H2)** *and* **(H3)** *are fulfilled. If $u_{0\varepsilon} \in V$, $u_{1\varepsilon} \in H$ and $f_\varepsilon \in L^2(0, T; H)$, then there exists the unique function $u_\varepsilon \in W^{2,2}(0, T; H) \bigcap L^2(0, T; V)$, $A(\cdot)u_\varepsilon \in L^2(0, T; H)$ (strong solution) which satisfies the equation a.e. on $(0, T)$ and the initial conditions from (P_ε).*

If, in addition, $u_{1\varepsilon} \in V$, $f_\varepsilon(0) - A(0)u_{0\varepsilon} \in V$, $f_\varepsilon \in W^{2,1}(0, T; H)$, then $A(\cdot)u_\varepsilon \in W^{1,2}(0, T; H)$ and $u_\varepsilon \in W^{3,2}(0, T; H) \bigcap W^{1,2}(0, T; H)$.

Theorem 2.2. *Let $T > 0$. Let us assume that the conditions* **(H1)**, **(H2)** *and* **(H3)** *are fulfilled. If $u_{0\varepsilon} \in H$, and $f_\varepsilon \in L^2(0, T; H)$, then there exists the unique function $u_\varepsilon \in W^{2,2}(0, T; H) \bigcap L^2(0, T; V)$ which satisfies a. e. on $(0, T)$ the equation and the initial conditions from (P_0).*

3 A priori estimates for solutions to the problem (P_ε)

In what follows, we will give some *a priori* estimates of solutions to the problem (P_ε).

Lemma 3.1. *Let us assume that conditions* **(H1)**, **(H2)** *and* **(H3)** *are fulfilled. If $u_{0\varepsilon} \in V$, $u_{1\varepsilon} \in H$ and $f_\varepsilon \in L^2(0, \infty; H)$, then there exists a constant $C = C(\gamma, a_0, \omega) > 0$ such that for every solution u_ε to the problem (P_ε) the estimate*

$$||u_\varepsilon||_{C([0,t];H)} + ||A^{1/2}(\cdot)u_\varepsilon||_{L^2([0,t];H)} \leq C \; M_{0\varepsilon}, \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall t \geq 0 \qquad (3.1)$$

is valid, where

$$M_{0\varepsilon} = |A^{1/2}(0)u_{0\varepsilon}| + \varepsilon|u_{1\varepsilon}| + ||f_\varepsilon||_{L^2(0,\infty;H)}, \quad \varepsilon_0 = \min\Big\{1, \frac{\omega}{2\gamma\, a_0}\Big\}.$$

If, in addition, $u_{1\varepsilon} \in V$ *and* $f_\varepsilon \in W^{1,2}(0,\infty;H)$ *then*

$$||u_\varepsilon'||_{C([0,\,t];H)} + ||A^{1/2}(\cdot)u_\varepsilon'||_{L^2([0,\,t];H)} \le C\ M_\varepsilon, \quad \forall \varepsilon \in (0,\varepsilon_0), \quad \forall t \ge 0, \tag{3.2}$$

$$M_\varepsilon = |A(0)u_{0\varepsilon}| + |A^{1/2}(0)u_{1\varepsilon}| + ||f_\varepsilon||_{W^{1,2}(0,\infty;H)}.$$

Proof. Proof of estimate (3.1). Denote by

$$E(u,t) = \varepsilon^2|u'(t)|^2 + \frac{1}{2}|u(t)|^2 + \varepsilon\Big(A(t)u(t), u(t)\Big) + \varepsilon\int_0^t |u'(\tau)|^2 d\tau$$

$$+\varepsilon\Big(u(t), u'(t)\Big) + \int_0^t \Big(A(\tau)u(\tau), u(\tau)\Big) d\tau. \tag{3.3}$$

For every strong solution u_ε to problem (P_ε) we have

$$\frac{d}{dt}E(u_\varepsilon,t) = 2\varepsilon^2\big(u_\varepsilon''(t), u_\varepsilon'(t)\big) + \big(u_\varepsilon'(t), u_\varepsilon(t)\big) + 2\varepsilon\big(A(t)u_\varepsilon(t), u_\varepsilon'(t)\big)$$

$$+\varepsilon\big(A'(t)u_\varepsilon(t), u_\varepsilon(t)\big) + \varepsilon|u_\varepsilon(t)|^2 + \varepsilon|u_\varepsilon'(t)|^2 + \varepsilon\big(u_\varepsilon''(t), u_\varepsilon(t)\big) + \big(A(t)u_\varepsilon(t), u_\varepsilon(t)\big)$$

$$= 2\varepsilon\big(u_\varepsilon'(t), f_\varepsilon(t) - u_\varepsilon'(t) - A(t)u_\varepsilon(t)\big) + \big(u_\varepsilon'(t), u_\varepsilon(t)\big)$$

$$+2\varepsilon\big(A(t)u_\varepsilon(t), u_\varepsilon'(t)\big) + \varepsilon\big(A'(t)u_\varepsilon(t), u_\varepsilon(t)\big) + 2\varepsilon|u_\varepsilon'(t)|^2 + \big(A(t)u_\varepsilon(t), u_\varepsilon(t)\big)$$

$$+\big(u_\varepsilon(t), f_\varepsilon(t) - u_\varepsilon'(t) - A(t)u_\varepsilon(t)\big)$$

$$= \big(f_\varepsilon(t), u_\varepsilon(t) + 2\varepsilon u_\varepsilon'(t)\big) + \varepsilon\big(A'(t)u_\varepsilon(t), u_\varepsilon(t)\big), \quad \forall t \ge 0.$$

Thus

$$\frac{d}{dt}E(u_\varepsilon,t) = \big(f_\varepsilon(t), u_\varepsilon(t) + 2\varepsilon u_\varepsilon'(t)\big) + \varepsilon\big(A'(t)u_\varepsilon(t), u_\varepsilon(t)\big), \quad \forall t \ge 0.$$

Integrating on $(0,t)$ we get

$$E(u_\varepsilon,t) = E(u_\varepsilon,0) + \int_0^t \big(f_\varepsilon(\tau), u_\varepsilon(\tau) + 2\varepsilon u_\varepsilon'(\tau)\big)\, d\tau$$

$$+\varepsilon\int_0^t \big(A'(\tau)u_\varepsilon(\tau), u_\varepsilon(\tau)\big) d\tau, \quad \forall t \ge 0.$$

Let us observe that

$$\int_0^t \big|f_\varepsilon(\tau)||u_\varepsilon(\tau)\big|\, d\tau \le \frac{1}{2}\int_0^t \big(A(\tau)u_\varepsilon(\tau), u_\varepsilon(\tau)\big) d\tau + \frac{\gamma}{2\omega}\int_0^t \big|f_\varepsilon(\tau)|^2 d\tau,$$

$$2\varepsilon\int_0^t \big|f_\varepsilon(\tau)||u_\varepsilon'(\tau)\big|\,d\tau \le \varepsilon^2\int_0^t \big|u_\varepsilon'(\tau)\big|^2 d\tau + \int_0^t \big|f_\varepsilon(\tau)|^2 d\tau,$$

$$\varepsilon\int_0^t \Big|\big(A'(\tau)u_\varepsilon(\tau),u_\varepsilon(\tau)\big)\Big|d\tau \le \frac{a_0\gamma}{\omega}\,\varepsilon\int_0^t \big(A(\tau)u_\varepsilon(\tau),u_\varepsilon(\tau)\big)d\tau, \quad \forall t\ge 0.$$

Thus

$$E(u_\varepsilon,t) \le C(\gamma,a_0,\omega)\Big[E(u_\varepsilon,0)+||f_\varepsilon||^2_{L^2(0,t;H)}\Big],\ \forall t\ge 0,\ \forall 0<\varepsilon<\varepsilon_0=\min\Big\{1,\frac{\omega}{2\gamma\,a_0}\Big\}.$$

Using the Brézis' Lemma (see, e. g., [13]), the estimate (3.1) is a simple consequence of the last inequality.

The *proof of estimate* (3.2) is similar to the proof of (3.1) if we denote by $y_\varepsilon = u_\varepsilon'$, which is the solution to the problem

$$\begin{cases} \varepsilon y_\varepsilon''(t) + y_\varepsilon'(t) + A(t)y_\varepsilon(t) = f_\varepsilon'(t) - A'(t)u_\varepsilon(t), \quad t\in(0,\infty),\\ y_\varepsilon(0) = u_{1\varepsilon}, \quad y_\varepsilon'(0) = \dfrac{1}{\varepsilon}\Big(f_\varepsilon(0) - u_{1\varepsilon} - A(0)u_{0\varepsilon}\Big). \end{cases}$$

□

Let u_ε be the strong solution to the problem (P_ε) and let us denote by

$$z_\varepsilon(t) = u_\varepsilon'(t) + h_\varepsilon e^{-t/\varepsilon}, \quad h_\varepsilon = f_\varepsilon(0) - u_{1\varepsilon} - A(0)u_{0\varepsilon}. \tag{3.4}$$

Similarly to Lemma 3.1, for the function z_ε we obtain the following result:

Lemma 3.2. *Let us assume that conditions* **(H1)**—**(H4)** *are fulfilled. If* $f_\varepsilon(0) - A(0)u_{0\varepsilon},\ u_{1\varepsilon} \in V$ *and* $f_\varepsilon \in W^{1,2}(0,\infty;H)$, *then there exist constants* $C = C(\gamma,\omega,a_0,a_1) > 0$ *and* $\varepsilon_0 = \varepsilon_0(\gamma,\omega,a_0,a_1)\in(0;1)$ *such that for* z_ε, *defined by* (3.4), *the estimate*

$$\big|\big|A^{1/2}(\cdot)z_\varepsilon\big|\big|_{C(0,\,t;\,H)} + \big|\big|z_\varepsilon'\big|\big|_{L^2(0,\,t;\,H)} \le C\,M_{1\varepsilon}, \quad \forall\varepsilon\in(0,\varepsilon_0), \quad \forall t\ge 0, \tag{3.5}$$

is valid, where

$$M_{1\varepsilon} = |A^{1/2}(0)\big(f_\varepsilon(0) - A(0)u_{0\varepsilon}\big)| + |A^{1/2}(0)u_{1\varepsilon}| + ||A(t)h_\varepsilon||_{L^2(0,\infty;H)} + ||f_\varepsilon||_{W^{2,2}(0,\infty;H)}.$$

4 The relationship between the solutions to the problems (P_ε) and (P_0) in the linear case

Now we are going to present the relationship between solutions to the problem (P_ε) and the corresponding solutions to the problem (P_0). This relationship was established in the work [14]. To this end we define the kernel of the transformation which realizes this relationship.

For $\varepsilon > 0$ denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi\varepsilon}}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon)=\exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_2(t,\tau,\varepsilon)=\exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t,\tau,\varepsilon)=\exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big),\quad \lambda(s)=\int_s^\infty e^{-\eta^2}\,d\eta.$$

The properties of kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 4.1. *The function $K(t,\tau,\varepsilon)$ possesses the following properties:*

- **(i)** $K\in C([0,\infty)\times[0,\infty))\cap C^2((0,\infty)\times(0,\infty))$;
- **(ii)** $K_t(t,\tau,\varepsilon)=\varepsilon K_{\tau\tau}(t,\tau,\varepsilon)-K_\tau(t,\tau,\varepsilon),\quad \forall t>0,\quad \forall\tau>0$;
- **(iii)** $\varepsilon K_\tau(t,0,\varepsilon)-K(t,0,\varepsilon)=0,\quad \forall t\geq 0$;
- **(iv)** $K(0,\tau,\varepsilon)=\dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\},\quad \forall\tau\geq 0$;
- **(v)** *For every $t>0$ fixed and every $q,s\in\mathbb{N}$ there exist constants $C_1(q,s,t,\varepsilon)>0$ and $C_2(q,s,t)>0$ such that*

$$\big|\partial_t^s\partial_\tau^q K(t,\tau,\varepsilon)\big|\leq C_1(q,s,t,\varepsilon)\exp\{-C_2(q,s,t)\tau/\varepsilon\},\quad \forall\tau>0;$$

Moreover, for $\gamma\in\mathbb{R}$ there exist C_1,C_2 and ε_0, all of them positive and depending on γ, such that the following estimates are fulfilled:

$$\int_0^\infty e^{\gamma\tau}\,\big|K_t(t,\tau,\varepsilon)\big|\,d\tau\leq C_1\,\varepsilon^{-1}\,e^{C_2t},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\geq 0,$$

$$\int_0^\infty e^{\gamma\tau}\,\big|K_\tau(t,\tau,\varepsilon)\big|\,d\tau\leq C_1\,\varepsilon^{-1}\,e^{C_2t},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\geq 0,$$

$$\int_0^\infty e^{\gamma\tau}\,\big|K_{\tau\tau}(t,\tau,\varepsilon)\big|\,d\tau\leq C_1\,\varepsilon^{-2}\,e^{C_2t},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\geq 0;$$

- **(vi)** $K(t,\tau,\varepsilon)>0,\quad \forall t\geq 0,\quad \forall\tau\geq 0$;
- **(vii)** *For every continuous function $\varphi:[0,\infty)\to H$ with $|\varphi(t)|\leq M\exp\{\gamma t\}$ the following equality is true:*

$$\lim_{t\to 0}\Big|\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau-\int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau\Big|=0,\ \textit{for every}\ \varepsilon\in\big(0,(2\gamma)^{-1}\big);$$

- **(viii)**

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau=1,\quad \forall t\geq 0,$$

(ix) *Let $\gamma > 0$ and $q \in [0,1]$. There exist C_1, C_2 and ε_0 all of them positive and depending on γ and q, such that the following estimates are fulfilled:*

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau} |t-\tau|^q \, d\tau \le C_1\, e^{C_2 t}\, \varepsilon^{q/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t > 0.$$

If $\gamma \le 0$ and $q \in [0,1]$, then

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau}\, |t-\tau|^q \, d\tau \le C\, \varepsilon^{q/2} \left(1+\sqrt{t}\right)^q, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0;$$

(x) *Let $p \in (1,\infty]$ and $f : [0,\infty) \to H$, $f(t) \in W^{1,p}_\gamma(0,\infty;H)$. If $\gamma > 0$, then there exist C_1, C_2 and ε_0 all of them positive and depending on γ and p, such that*

$$\Big| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big|$$
$$\le C_1\, e^{C_2 t}\, ||f'||_{L^p_\gamma(0,\infty;H)}\, \varepsilon^{(p-1)/2p}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0.$$

If $\gamma \le 0$, then

$$\Big| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big|$$
$$\le C(\gamma,p)\, \| f\,'\|_{L^p_\gamma(0,\infty;H)} \left(1+\sqrt{t}\right)^{\frac{p-1}{p}} \varepsilon^{(p-1)/2p}, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0.$$

(xi) *For every $q > 0$ and $\alpha \ge 0$ there exists a constant $C(q,\alpha) > 0$ such that*

$$\int_0^t \int_0^\infty K(\tau,\theta,\varepsilon)\, e^{-q\,\theta/\varepsilon}\, |\tau-\theta|^\alpha \, d\theta\, d\tau \le C(q,\alpha)\, \varepsilon^{1+\alpha}, \ \forall \varepsilon > 0, \ \forall t \ge 0;$$

(xii) *Let $f \in W^{1,\infty}_\gamma(0,\infty;H)$ with $\gamma \ge 0$. There exist positive constants C_1, C_2 and ε_0, depending on γ, such that*

$$\Big| \int_0^\infty K_t(t,\tau,\varepsilon) f(\tau) d\tau \Big| \le C_1\, e^{C_2 t} \|f'\|_{L^\infty_\gamma(0,\infty;H)}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0.$$

Theorem 4.1. *Let us assume that operators $A(t), t \in [0,\infty)$, verify conditions* **(H1)–(H3)** *and $f_\varepsilon \in L^\infty_\gamma(0,\infty;H)$ for some $\gamma \ge 0$. If u_ε is the strong solution to the problem (P_ε), with $u_\varepsilon \in W^{2,\infty}_\gamma(0,\infty;H) \cap L^\infty_\gamma(0,\infty;H)$, $Au_\varepsilon \in L^\infty_\gamma(0,\infty;H)$, then for every $0 < \varepsilon < (4\gamma)^{-1}$ the function w_ε, defined by*

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\, u_\varepsilon(\tau)\, d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} w_\varepsilon'(t) + A(t) w_\varepsilon(t) = F_0(t,\varepsilon) + \displaystyle\int_0^\infty K(t,\tau,\varepsilon) \big[A(t) - A(\tau)\big] u_\varepsilon(\tau)\, d\tau, \text{ a. e. } t > 0, \\ w_\varepsilon(0) = \varphi_\varepsilon, \end{cases}$$

where

$$F_0(t,\varepsilon)=\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1+\int_0^\infty K(t,\tau,\varepsilon)\,f_\varepsilon(\tau)\,d\tau,$$

$$\varphi_\varepsilon=\int_0^\infty e^{-\tau}\,u_\varepsilon(2\varepsilon\tau)\,d\tau.$$

5 Limits of solutions to the problem (P_ε) as $\varepsilon \to 0$

In this section we will prove the convergence estimates for the difference of solutions to the problems (P_ε) and (P_0). These estimates will be uniform relative to small values of the parameter ε.

Theorem 5.1. *Let $T>0$. Let us assume that operators $A(t)$, $t\in[0,\infty)$, satisfy conditions* **(H1)**–**(H3)**. *If u_0, $u_{0\varepsilon}$, $u_{1\varepsilon}\in V$ and f, $f_\varepsilon\in W^{1,2}(0,T;H)$, then there exist constants $C=C(T,\gamma,a_0,\omega)>0$, $\varepsilon_0=\varepsilon_0(\gamma,a_0,\omega)$, $\varepsilon_0\in(0,1)$, such that*

$$||u_\varepsilon-v||_{C([0,T];H)}$$

$$\leq C\left(M(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^{1/4}+|u_{0\varepsilon}-u_0|+||f_\varepsilon-f||_{L^2(0,T;H)}\right),\forall\varepsilon\in(0,\varepsilon_0),\quad(5.1)$$

where u_ε and v are strong solutions to problems (P_ε) and (P_0) respectively,

$$M(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)=|A(0)u_{0\varepsilon}|+|A^{1/2}(0)u_{1\varepsilon}|+||f_\varepsilon||_{W^{1,2}(0,T;H)}.$$

Proof. During the proof we will agree to denote constants $C=C(T,\gamma,a_0,\omega)$, $M(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$ and $\varepsilon_0=\varepsilon_0(\gamma,a_0,\omega)$ by C, M and ε_0, respectively.

If f, $f_\varepsilon\in W^{k,p}(0,T;H)$ with $k\in\mathbb{N}$ and $p\in(1,\infty]$, then f, $f_\varepsilon\in C([0,T];H)$ (see, for example, [3]). Moreover, there exist extensions $\tilde f$, $\tilde f_\varepsilon\in W^{k,p}(0,\infty;H)$ such that

$$\begin{cases}||\tilde f||_{C([0,\infty);H)}+||\tilde f||_{W^{k,p}(0,\infty;H)}\leq C(T,p)\,||f||_{W^{k,p}(0,T;H)},\\ ||\tilde f_\varepsilon||_{C([0,\infty);H)}+||\tilde f_\varepsilon||_{W^{k,p}(0,\infty;H)}\leq C(T,p)\,||f_\varepsilon||_{W^{k,p}(0,T;H)}.\end{cases}\quad(5.2)$$

Let us denote by $\tilde u_\varepsilon$ the unique strong solution to the problem (P_ε), defined on $(0,\infty)$ instead of $(0,T)$ and $\tilde f_\varepsilon$ instead of f_ε.

From Lemma 3.1 it follows that $\tilde u_\varepsilon\in W^{2,\infty}(0,\infty;H)\cap L^\infty(0,\infty;H)$, $A(\cdot)\tilde u_\varepsilon\in L^\infty(0,\infty;H)$. Moreover, due to this lemma and inequalities (5.2), the following estimates hold

$$||u_\varepsilon||_{C([0,t];H)}+||A^{1/2}(\cdot)u_\varepsilon||_{L^2([0,t];H)}\leq C\,M,\quad\forall\varepsilon\in(0,\varepsilon_0],\quad\forall t\geq0,\qquad(5.3)$$

$$||u'_\varepsilon||_{C([0,t];H)}+||A^{1/2}(\cdot)u'_\varepsilon||_{L^2([0,t];H)}\leq C\,M,\quad\forall\varepsilon\in(0,\varepsilon_0],\quad\forall t\geq0.\qquad(5.4)$$

By Theorem 4.1, the function w_ε defined by

$$w_\varepsilon(t)=\int_0^\infty K(t,\tau,\varepsilon)\,\tilde u_\varepsilon(\tau)\,d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} w'_\varepsilon(t) + A(t)w_\varepsilon(t) = F(t,\varepsilon), \quad \text{a. e.} \qquad t>0, \\ w_\varepsilon(0) = w_0, \end{cases} \tag{5.5}$$

for every $\varepsilon \in (0,\varepsilon_0)$, where

$$F(t,\varepsilon) = f_0(t,\varepsilon)u_{1\varepsilon} + \int_0^\infty K(t,\tau,\varepsilon)\,\tilde f_\varepsilon(\tau)\,d\tau + \int_0^\infty K(t,\tau,\varepsilon)\,\big[A(t)-A(\tau)\big]u_\varepsilon(\tau)\,d\tau,$$

$$f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],$$

$$w_0 = \int_0^\infty e^{-\tau}\,u_\varepsilon(2\varepsilon\tau)\,d\tau.$$

Using properties (**vi**), (**viii**), (**x**) from Lemma 4.1, and the estimate (5.4), we obtain that

$$||\tilde u_\varepsilon - w_\varepsilon||_{C([0,t];\,H)} \le C\,M\,\varepsilon^{1/4}, \quad \forall\varepsilon\in(0,\varepsilon_0), \quad \forall t\ge 0. \tag{5.6}$$

Denote by $R(t,\varepsilon) = \tilde v(t) - w_\varepsilon(t)$, where $\tilde v$ is the strong solution to the problem (P_0) with $\tilde f$ instead of f, $T=\infty$ and w_ε is the solution of (5.5). Then

$$\begin{cases} R'(t,\varepsilon) + A(t)R(t,\varepsilon) = \mathcal{F}(t,\varepsilon) + \displaystyle\int_0^\infty K(t,\tau,\varepsilon)\,\big[A(t)-A(\tau)\big]u_\varepsilon(\tau)\,d\tau, \ \text{a.e. } t>0, \\ R(0,\varepsilon) = R_0, \end{cases}$$

where $R_0 = u_0 - w_0$ and

$$\mathcal{F}(t,\varepsilon) = \tilde f(t) - \int_0^\infty K(t,\tau,\varepsilon)\tilde f_\varepsilon(\tau)\,d\tau - f_0(t,\varepsilon)\,u_{1\varepsilon}.$$

Taking the inner product in H by R and then integrating, we obtain

$$|R(t,\varepsilon)|^2 + 2\int_0^t \Big|A^{1/2}(s)R(s,\varepsilon)\Big|^2\,ds \le |R(0,\varepsilon)|^2$$

$$+2\int_0^t |\mathcal{F}(s,\varepsilon)|\,|R(s,\varepsilon)|\,ds$$

$$+2\int_0^t\int_0^\infty K(s,\tau,\varepsilon)\Big(\big[A(s)-A(\tau)\big]u_\varepsilon(\tau), R(s,\varepsilon)\Big)d\tau ds, \quad \forall t\ge 0.$$

Using condition **(H2)** and the property (**ix**) from Lemma 4.1 we get

$$|R(t,\varepsilon)|^2 + 2\int_0^t \Big|A^{1/2}(s)R(s,\varepsilon)\Big|^2\,ds \le |R(0,\varepsilon)|^2$$

$$+2\int_0^t \Big(|\mathcal{F}(s,\varepsilon)| + C\,M\,\varepsilon^{1/2}\Big)|R(s,\varepsilon)|\,ds, \quad \forall t\ge 0, \quad \forall\varepsilon\in(0,\varepsilon_0). \tag{5.7}$$

Applying Brézis' Lemma to (5.7), we get

$$|R(t,\varepsilon)| + \Big(\int_0^t \Big|A^{1/2}(t)R(s,\varepsilon)\Big|^2\,ds\Big)^{1/2}$$

$$\leq \sqrt{2}\,|R(0,\varepsilon)| + \sqrt{2}\int_0^t \Big(|\mathcal{F}(s,\varepsilon)| + C\,M\,\varepsilon^{1/2}\Big)\,ds, \quad \forall t\geq 0, \quad \forall \varepsilon\in(0,\varepsilon_0). \quad (5.8)$$

From (5.4) it follows that

$$\big|R_0\big| \leq |u_{0\varepsilon}-u_0| + \int_0^\infty e^{-\tau}\big|\tilde{u}_\varepsilon(2\varepsilon\tau)-u_{0\varepsilon}\big|\,d\tau \leq |u_{0\varepsilon}-u_0| + \int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}\big|\tilde{u}'_\varepsilon(s)\big|\,ds\,d\tau$$

$$\leq |u_{0\varepsilon}-u_0| + C\,\varepsilon\,M\int_0^\infty \tau\,e^{-\tau+\gamma\,\varepsilon\,\tau}\,d\tau \leq |u_{0\varepsilon}-u_0| + C\,M\,\varepsilon, \quad \forall\varepsilon\in(0,\varepsilon_0). \quad (5.9)$$

In what follows we will estimate $\big|\mathcal{F}(t,\varepsilon)\big|$. Using the property (**x**) from Lemma 4.1 and (5.2), we have

$$\Big|\tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau\Big| \leq |\tilde{f}(t)-\tilde{f}_\varepsilon(t)| + \Big|\tilde{f}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau\Big|$$

$$\leq |\tilde{f}(t)-\tilde{f}_\varepsilon(t)| + C(T,p)\|f'_\varepsilon\|_{L^2(0,T\,;\,H)}\,\varepsilon^{1/4},\ \forall\varepsilon\in(0,\varepsilon_0),\ \forall t\in[0,T]. \quad (5.10)$$

Since

$$e^{\tau}\lambda(\sqrt{\tau}) \leq C, \quad \forall\tau\geq 0,$$

the estimates

$$\int_0^t \exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau \leq C\,\varepsilon\int_0^\infty e^{-\tau/4}\,d\tau \leq C\varepsilon, \quad \forall t\geq 0,$$

$$\int_0^t \lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\,d\tau \leq \varepsilon\int_0^\infty \lambda\Big(\frac{1}{2}\sqrt{\tau}\Big)\,d\tau \leq C\,\varepsilon, \quad \forall t\geq 0,$$

are true. Then

$$\Big|\int_0^t f_0(\tau,\varepsilon)\,d\tau\Big| \leq C\,\varepsilon, \quad \forall\varepsilon\in(0,\varepsilon_0), \quad \forall t\geq 0. \quad (5.11)$$

Using (5.2), (5.10) and (5.11) we obtain

$$\int_0^t \Big(|\mathcal{F}(s,\varepsilon)| + C\,M\,\varepsilon^{1/2}\Big)\,d\tau$$

$$\leq C\Big(M\,\varepsilon^{1/4} + ||f_\varepsilon - f||_{L^2(0,T;H)}\Big), \quad \forall\varepsilon\in(0,\varepsilon_0), \quad \forall t\in[0,T]. \quad (5.12)$$

From (5.8), using (5.9) and (5.12) we get the estimate

$$||R||_{C([0,\,t];\,H)} + ||A(\cdot)^{1/2}R||_{L^2(0,t;H)}$$

$$\leq C\left(M\,\varepsilon^{1/4}+|u_{0\varepsilon}-u_0|+||f_\varepsilon-f||_{L^p(0,T;H)}\right),\ \forall\varepsilon\in(0,\varepsilon_0),\quad\forall t\in[0,T].\quad(5.13)$$

In the consequence, from (5.6) and (5.13) we deduce

$$||\tilde{u}_\varepsilon-\tilde{v}||_{C([0,t];H)}\leq||\tilde{u}_\varepsilon-w_\varepsilon||_{C([0,t];H)}+||R||_{C([0,t];H)}$$

$$\leq C\left(M\,\varepsilon^{1/4}+|u_{0\varepsilon}-u_0|+||f_\varepsilon-f||_{L^2(0,T;H)}\right),\quad\forall\varepsilon\in(0,\varepsilon_0),\quad\forall t\in[0,T].\quad(5.14)$$

Since $u_\varepsilon(t)=\tilde{u}_\varepsilon(t)$ and $v(t)=\tilde{v}(t)$, for all $t\in[0,T]$, then the estimate (5.1) follows from (5.14). □

Theorem 5.2. *Let $T>0$. Let us assume that operators $A(t),t\in[0,\infty)$, satisfy conditions* **(H1)**–**(H4)**. *If u_0, $u_{0\varepsilon}$, $A(0)u_{0\varepsilon}$, $u_{1\varepsilon}, f_\varepsilon(0)\in V$ and f, $f_\varepsilon\in W^{2,2}(0,T;H)$, then there exist constants $C=C(T,\omega,\gamma,a_0,a_1)>0$, $\varepsilon_0=\varepsilon_0(\omega,\gamma,a_0,a_1)$, $\varepsilon_0\in(0,1)$, such that*

$$||u_\varepsilon'-v'+h_\varepsilon e^{-t/\varepsilon}||_{C([0,\,T]\,;H)}$$

$$\leq C\left(M_1(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^{1/4}+D_\varepsilon\right),\quad\forall\varepsilon\in(0,\varepsilon_0),\qquad(5.15)$$

where u_ε and v are strong solutions to problems (P_ε) and (P_0) respectively, $h_\varepsilon=f_\varepsilon(0)-u_{1\varepsilon}-A(0)u_{0\varepsilon}$,

$$M_1=|A^{1/2}(0)f_\varepsilon(0)|+|A^{3/2}(0)u_{0\varepsilon}|+|A^{1/2}(0)u_{1\varepsilon}|+||A(t)h_\varepsilon||_{L^2(0,\infty;H)}+||f_\varepsilon||_{W^{2,2}(0,\infty;H)},$$

$$D_\varepsilon=||f_\varepsilon-f||_{W^{1,\,2}(0,\,T;\,H)}+|A_0(u_{0\varepsilon}-u_0)|.$$

Proof. In the proof of this theorem, we will agree to denote the constants $C=C(T,\omega,\gamma,a_0,a_1)>0$, $\varepsilon_0=\varepsilon_0(\omega,\gamma,a_0,a_1)$ and $M_1(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$ by C, ε_0 and M_1 respectively. Also we preserve for $\tilde{v}(t)$, $\tilde{u}_\varepsilon(t)$, $\tilde{f}(t)$ and $\tilde{f}_\varepsilon(t)$ the same notations as in Theorem 5.1.

By Lemma 3.2, we have that the function

$$\tilde{z}_\varepsilon(t)=\tilde{u}_\varepsilon'(t)+h_\varepsilon e^{-t/\varepsilon},\ \text{ with }\ h_\varepsilon=f_\varepsilon(0)-u_{1\varepsilon}-A(0)u_{0\varepsilon},$$

is the solution to the problem

$$\begin{cases}\varepsilon\tilde{z}_\varepsilon''(t)+\tilde{z}_\varepsilon'(t)+A(t)\tilde{z}_\varepsilon(t)=\tilde{\mathcal{F}}(t,\varepsilon),\quad t>0,\\ \tilde{z}_\varepsilon(0)=f_\varepsilon(0)-A(0)u_{0\,\varepsilon},\quad\tilde{z}_\varepsilon'(0)=0,\end{cases}$$

where

$$\tilde{\mathcal{F}}(t,\varepsilon)=\tilde{f}_\varepsilon'(t)-A'(t)\tilde{u}_\varepsilon(t)+e^{-t/\varepsilon}\,A(t)h_\varepsilon$$

and

$$||A_0^{1/2}(\cdot)\tilde{z}_\varepsilon||_{C([0,\,t];\,H)}+||\tilde{z}_\varepsilon'||_{L^2(0,\,t;\,H)}\leq C\,M_1,\quad\forall t\geq0.\qquad(5.16)$$

Since $\tilde{z}_\varepsilon'(0)=0$, from Theorem 4.1, the function $w_{1\varepsilon}(t)$, defined by

$$w_{1\varepsilon}(t)=\int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)\,d\tau,\qquad(5.17)$$

satisfies in H the following conditions

$$\begin{cases} w'_{1\varepsilon}(t) + A(t)w_{1\varepsilon}(t) = F_1(t,\varepsilon), \quad \text{a. e.} \quad t > 0, \\ w_{1\varepsilon}(0) = \varphi_{1\varepsilon}, \end{cases}$$

for every $0 < \varepsilon < \varepsilon_0$, where

$$F_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\tilde{f}'_\varepsilon(\tau)\,d\tau - \int_0^\infty K(t,\tau,\varepsilon)\,A'(\tau)\tilde{u}_\varepsilon(\tau)\,d\tau$$

$$+ \int_0^\infty K(t,\tau,\varepsilon)\; e^{-\tau/\varepsilon} A(\tau)h_\varepsilon d\tau + \int_0^\infty K(t,\tau,\varepsilon)\big[A(t) - A(\tau)\big]\tilde{z}_\varepsilon(\tau)\,d\tau,$$

$$\varphi_{1\varepsilon} = \int_0^\infty e^{-\tau}\tilde{z}_\varepsilon(2\varepsilon\tau)\,d\tau.$$

Using (5.17), the properties (**vi**), (**viii**) and (**ix**) from Lemma 4.1 and (5.16), we get the estimate

$$\big|\tilde{z}_\varepsilon(t) - w_{1\varepsilon}(t)\big| \le \int_0^\infty K(t,\tau,\varepsilon)\big|\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau)\big|\,d\tau$$

$$\le \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^t \big|\tilde{z}'_\varepsilon(s)\big|\,ds\Big|\,d\tau \le C\,M_1 \int_0^\infty K(t,\tau,\varepsilon)\big|t-\tau\big|^{1/2}\,d\tau$$

$$\le C\,M_1\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0), \quad \forall t \ge 0,$$

which implies

$$\big|\big|\tilde{z}_\varepsilon - w_{1\varepsilon}\big|\big|_{C([0,t];\,H)} \le C\,M_1\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0), \quad \forall t \ge 0. \tag{5.18}$$

Let $v_1(t) = \tilde{v}'(t)$, where $\tilde{v}$ is the strong solution to the problem (P_0) with $\tilde{f}$ instead of f and $T = \infty$.

Let us denote by $R_1(t,\varepsilon) = v_1(t) - w_{1\varepsilon}(t)$. The function $R_1(t,\varepsilon)$ verifies in H the following problem

$$\begin{cases} R'_1(t,\varepsilon) + A(t)R_1(t,\varepsilon) = \mathcal{F}_1(t,\varepsilon), \quad t > 0, \\ R_1(0,\varepsilon) = R_{10}, \end{cases}$$

where

$$R_{10} = f(0) - A_0u_0 - \varphi_{1\varepsilon},$$

$$\mathcal{F}_1(t,\varepsilon) = \tilde{f}\,'(t) - \int_0^\infty K(t,\tau,\varepsilon)\tilde{f}'_\varepsilon(\tau)\,d\tau + \int_0^\infty K(t,\tau,\varepsilon)\,A'(\tau)\tilde{u}_\varepsilon(\tau)\,d\tau - A'(t)v(t)$$

$$- \int_0^\infty K(t,\tau,\varepsilon)\; e^{-\tau/\varepsilon} A(\tau)h_\varepsilon d\tau - \int_0^\infty K(t,\tau,\varepsilon)\big[A(t) - A(\tau)\big]\tilde{z}_\varepsilon(\tau)\,d\tau. \tag{5.19}$$

Using the properties (**viii**), (**ix**) from Lemma 4.1 and the inequalities (5.2), we get

$$\Big|\tilde{f}'(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}'_\varepsilon(\tau)\,d\tau\Big|$$

$$\leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)| + \int_0^\infty K(t,\tau,\varepsilon)\, \big|\tilde{f}'_\varepsilon(\tau) - \tilde{f}'_\varepsilon(t)\big|\, d\tau$$

$$\leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)| + ||\tilde{f}''_\varepsilon||_{L^2(0,\infty;\, H)} \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{1/2}\, d\tau \leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)|$$

$$+C(T)\, ||f''_\varepsilon||_{L^2(0,T;\, H)}\, \varepsilon^{1/2}, \quad \forall \varepsilon \in (0,\varepsilon_0), \quad \forall t \in [0,T]. \tag{5.20}$$

Taking the inner product in H by R_1 and then integrating, we obtain

$$|R_1(t,\varepsilon)|^2 + 2\int_0^t \Big|A^{1/2}(s) R_1(s,\varepsilon)\Big|^2\, ds = |R(0,\varepsilon)|^2$$

$$+2\int_0^t \Big(\mathcal{F}_1(s,\varepsilon), R_1(s,\varepsilon)\Big) ds, \quad \forall t \geq 0. \tag{5.21}$$

Using the properties (**viii**), (**ix**) from Lemma 4.1, conditions **(H2)**, **(H4)**, estimates (5.1), (5.3) and (5.4) we get

$$\Big(\int_0^\infty K(s,\tau,\varepsilon)\, A'(\tau)\tilde{u}_\varepsilon(\tau)\, d\tau - A'(s)v(s), R_1(s,\varepsilon)\Big)$$

$$= \int_0^\infty K(s,\tau,\varepsilon)\Big([A'(\tau) - A'(s)]\tilde{u}_\varepsilon(\tau)\, d\tau, R_1(s,\varepsilon)\Big) d\tau$$

$$+ \int_0^\infty K(s,\tau,\varepsilon)\Big(A'(s)[\tilde{u}_\varepsilon(\tau) - \tilde{u}_\varepsilon(s)]\,, R_1(s,\varepsilon)\Big) d\tau$$

$$+ \int_0^\infty K(s,\tau,\varepsilon)\Big(A'(s)[\tilde{u}_\varepsilon(s) - v(s)], R_1(s,\varepsilon)\Big) d\tau$$

$$\leq C\Big(M\,\varepsilon^{1/2} + M\varepsilon^{1/4} + |u_{0\varepsilon} - u_0| + ||f_\varepsilon - f||_{L^2(0,T;H)}\Big)\Big)\big|R_1(s,\varepsilon)\big|$$

$$\leq C\Big(M\,\varepsilon^{1/4} + |u_{0\varepsilon} - u_0| + ||f_\varepsilon - f||_{L^2(0,T;H)}\Big)\big|R_1(s,\varepsilon)\big|, \tag{5.22}$$

for every $\varepsilon \in (0,\varepsilon_0)$ and for all $s \in [0,t]$.

Using the property (**xi**) from Lemma 4.1, we can state

$$\int_0^t \int_0^\infty K(s,\tau,\varepsilon)\; e^{-\tau/\varepsilon} |A(\tau) h_\varepsilon| d\tau ds \leq C\, M_1 \varepsilon, \quad \forall \varepsilon > 0,\ \ \forall t \geq 0. \tag{5.23}$$

Using the properties (**viii**), (**ix**) from Lemma 4.1, condition **(H2)** and estimate (5.16) we get

$$\int_0^\infty K(s,\tau,\varepsilon)\Big(\,\big[A(s) - A(\tau)\big]\,\tilde{z}_\varepsilon(\tau), R_1(s,\varepsilon)\Big)\, d\tau$$

$$\leq C\, M_1\, \varepsilon^{1/2}\big|R_1(s,\varepsilon)\big|, \quad \forall \varepsilon \in (0,\varepsilon_0). \tag{5.24}$$

For R_{10}, due to (5.16), we have

$$|R_{10}| \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})| + \int_0^\infty e^{-\tau}\, |\tilde{z}_\varepsilon(2\varepsilon\tau) - \tilde{z}_\varepsilon(0)|\, d\tau$$

$$\leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})| + \int_0^\infty e^{-\tau} \int_0^{2\varepsilon\tau} |\tilde{z}'_\varepsilon(s)|\, ds\, d\tau$$
$$\leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})|$$
$$+C\, M_1\, \varepsilon \int_0^\infty \tau\, e^{-\tau + 2\,\gamma\,\varepsilon\,\tau}\, d\tau \leq C\, D_\varepsilon + C\, M_1\, \varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{5.25}$$

Applying Lemma of Brézis to (5.21) and using estimates (5.22), (5.23), (5.24), (5.25), we get

$$|R_1(t,\varepsilon)| + ||A_0^{1/2} R_1||_{L^2(0,t;\,H)}$$
$$\leq C \Big(M_1(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\, \varepsilon^{1/4} + D_\varepsilon \Big), \quad \forall \varepsilon \in (0, \varepsilon_0), \tag{5.26}$$

which together with (5.18) implies (5.15). □

6 An example

Let $\Omega \subset R^n$ be an open bounded set with smooth $\partial\Omega$. In the real Hilbert space $L^2(\Omega)$ with the scalar product

$$(u, v) = \int_\Omega u(x)\, v(x)\, dx.$$

we will consider the following Cauchy problem

$$\begin{cases} \varepsilon \partial_t^2\, u_\varepsilon(x,t) + \partial_t\, u_\varepsilon(x,t) + A(x,t) u_\varepsilon(x,t) = f(x,t), & x \in \Omega, \ t > 0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad \partial_t\, u_\varepsilon(x,0) = u_{1\varepsilon}(x) \end{cases} \tag{6.1}$$

where $D(A(\cdot,t)) = H^2(\Omega) \cap H_0^1(\Omega), \quad t \in [0, \infty)$,

$$A(x,t)u(x) = -\sum_{i,j=1}^n \partial_{x_i}\big(a_{ij}(x,t)\partial_{x_j} u(x)\big) + a(x,t)u(x), \ u \in D(A(\cdot,t)), \ \forall t \in [0,\infty),$$

$$a_{ij}(\cdot,t) \in C^1(\overline{\Omega}), \ a(\cdot,t) \in C(\overline{\Omega}), \quad \forall t \in [0,\infty), \tag{6.2}$$

$$a(x,t) \geq 0, \ a_{ij}(x,t) = a_{ji}(x,t), \quad x \in \overline{\Omega}, \quad \forall t \in [0,\infty), \tag{6.3}$$

$$\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\, \xi_j \geq a_0\, |\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad a_0 > 0. \tag{6.4}$$

$a_{ij}(x,\cdot)$, $a(x,\cdot)$ are continuously differentiable on $(0,\infty)$, $\partial_t a_{ij}(x,\cdot), \partial_t a(x,\cdot)$ are bounded on $[0,\infty)$ and

$$\partial_t a_{ij}(\cdot,t) \in C^1(\overline{\Omega}), \ \partial_t a(\cdot,t) \in C(\overline{\Omega}), \quad \forall t \in [0,\infty), \tag{6.5}$$

$a_{ij}(x,\cdot), a(x,\cdot)$ are twice continuously differentiable on $(0,\infty)$, $\partial_t^2 a_{ij}(x,\cdot), \partial_t^2 a(x,\cdot)$ are bounded on $[0,\infty)$, and

$$\partial_t^2 a_{ij}(\cdot,t) \in C^1(\overline{\Omega}), \partial_t^2 a(\cdot,t) \in C(\overline{\Omega}), \quad \forall t \in [0,\infty). \tag{6.6}$$

In conditions (6.2)–(6.3) the operators $A(t)$, $\forall t \in [0,\infty)$, are positive and selfadjoint. Let us now consider the unperturbed problem associated to the problem (6.1)

$$\begin{cases} \partial_t v(x,t) + A(x,t)v = f(x,t), \quad x \in \Omega, \ t > 0, \\ v(x,0) = u_0(x). \end{cases} \tag{6.7}$$

Using Theorem 5.1 we obtain the following theorem.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth $\partial\Omega$. Let $T > 0$. Suppose that conditions* (6.2) – (6.5) *are fulfilled. If $u_0, u_{0\,\varepsilon}, u_{1\,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$, $f, f_\varepsilon \in W^{1,2}(0,T;L_2(\Omega))$, then there exist constants $\varepsilon_0 = \varepsilon_0(\gamma, a_0, \omega) \in (0,1)$ and $C = C(T,n,\gamma,a_0,\omega) > 0$ such that*

$$||u_\varepsilon - v||_{C([0,T];L_2(\Omega))}$$

$$\leq C\left(\widetilde{M}\,\varepsilon^{1/4} + |u_{0\,\varepsilon} - u_0| + \big|\big|f_\varepsilon - f\big|\big|_{L^2(0,T;L_2(\Omega))}\right), \quad \varepsilon \in (0,\varepsilon_0], \tag{6.8}$$

where u_ε and v are the strong solutions to problems (6.1) *and* (6.7), *respectively, and*

$$\widetilde{M} = \big|A(0)u_{0\,\varepsilon}\big| + |A^{1/2}(0)u_{1\,\varepsilon}| + ||f_\varepsilon||_{W^{1,2}(0,\infty;L_2(\Omega))}.$$

Using Theorem 5.2 we obtain the following theorem.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth $\partial\Omega$. Let $T > 0$. Suppose that conditions* (6.2) – (6.6) *are fulfilled. If*

$$u_0, u_{0\varepsilon}, A(0)u_{0\varepsilon}, u_{1\varepsilon}, f(0), f_\varepsilon(0) \in H^2(\Omega) \cap H_0^1(\Omega), \quad f, f_\varepsilon \in W^{2,2}(0,T;L_2(\Omega)),$$

then there exist constants $\varepsilon_0 = \varepsilon_0(\omega_0,\omega_1) \in (0,1)$ and $C = C(T,n,\omega_0,\omega_1) > 0$ such that

$$\big|\big|u'_\varepsilon - v' + h_\varepsilon\, e^{-\frac{t}{\varepsilon}}\big|\big|_{C([0,T];\,L_2(\Omega))} \leq C\left(\widetilde{M}_1\,\varepsilon^{(1/4} + \widetilde{D}_\varepsilon\right), \tag{6.9}$$

where v and u_ε are the strong solutions to problems (6.1) *and* (6.7), *respectively, $h_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(0)u_{0\,\varepsilon}$,*

$$\widetilde{D}_\varepsilon = \big|\big|f_\varepsilon - f\big|\big|_{W^{1,\,2}(0,\,T;\,H_0^1(\Omega))} + \big|A_0(u_{0\varepsilon} - u_0)\big|,$$

$$\widetilde{M}_1 = \big|A^{1/2}(0)f_\varepsilon(0)\big| + \big|A^{3/2}(0)u_{0\,\varepsilon}\big| + \big|A^{1/2}(0)u_{1\,\varepsilon}\big| + \big|A(t)h_\varepsilon\big| + \big|\big|f_\varepsilon\big|\big|_{W^{2,2}(0,\,\infty;\,H_0^1(\Omega))}.$$

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