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The endomorphism semigroup of a free dimonoid of rank 1

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Abstract. We describe all endomorphisms of a free dimonoid of rank 1 and construct a semigroup which is isomorphic to the endomorphism semigroup of this free dimonoid. Also, we give an abstract characteristic for the endomorphism semigroup of a free dimonoid of rank 1.

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1 Introduction

The notion of a dimonoid was introduced by Jean-Louis Loday in [1]. An algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ is called *a dimonoid* if for all $x, y, z \in D$ the following conditions hold:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z).$$

Algebras of dimension one play a special role in studying different properties of algebras of an arbitrary dimension. For example, free triods of rank 1 were described in [2] and used for building free trialgebras. Semigroups of cohomological dimension one are used in algebraic topology [3]. With the help of properties of free dimonoids (in particular, of rank 1), free dialgebras were described and a cohomology of dialgebras was investigated [1]. More general information on dimonoids and examples of some dimonoids can be found, e. g., in [1, 4–6].

Observe that if the operations of a dimonoid coincide, then the dimonoid becomes a semigroup. For a free dimonoid the operations are distinct, however every free semigroup can be obtained from the free dimonoid by a suitable factorization. It is well known that the endomorphism semigroup of a free semigroup (free monoid) of rank one is isomorphic to the multiplicative semigroup of positive (nonnegative) integers. Endomorphism semigroups of a free monoid and a free semigroup of a nontrivial rank were described by G. Mashevitzky and B. M. Schein [7]. The structure of the endomorphism semigroup of a free group was investigated by E. Formanek [8]. In this paper, we study the endomorphism semigroup of a free dimonoid of rank 1.

The paper is organized as follows. In Section 2, we give necessary definitions and auxiliary assertions. In Section 3, we describe all endomorphisms of a free dimonoid of rank 1 and construct a semigroup which is isomorphic to the endomorphism semigroup of the given free dimonoid. In Section 4, we prove that free dimonoids of rank 1 are determined by their endomorphism semigroups and give an abstract characteristic for the endomorphism semigroup of a free dimonoid of rank 1.

2 Preliminaries

Let $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ be arbitrary dimonoids. A mapping $\varphi : D_1 \to D_2$ is called *a homomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$ if for all $x, y \in D_1$ we have:

$$(x \dashv_1 y)\varphi = x\varphi \dashv_2 y\varphi, \quad (x \vdash_1 y)\varphi = x\varphi \vdash_2 y\varphi.$$

A bijective homomorphism $\varphi : D_1 \to D_2$ is called *an isomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$. In this case dimonoids $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are called *isomorphic* .

Let X be an arbitrary set and $\overline{X} = \{\overline{x} \mid x \in X\}$. Define two binary operations on the set

$$Fd\,(X) = \overline{X} \cup \left(\overline{X} \times X\right) \cup \left(X \times \overline{X}\right) \cup$$

$$\cup (\overline{X} \times X \times X) \cup (X \times \overline{X} \times X) \cup (X \times X \times \overline{X}) \cup \ldots$$

as follows:

$$(x_1, \ldots, \overline{x_i}, \ldots, x_k) \dashv (y_1, \ldots, \overline{y_j}, \ldots, y_l) = (x_1, \ldots, \overline{x_i}, \ldots, x_k, y_1, \ldots, y_l)\,,$$

$$(x_1, \ldots, \overline{x_i}, \ldots, x_k) \vdash (y_1, \ldots, \overline{y_j}, \ldots, y_l) = (x_1, \ldots, x_k, y_1, \ldots, \overline{y_j}, \ldots, y_l)\,.$$

The algebra $(Fd(X), \dashv, \vdash)$ is *a free dimonoid* (see [1]). Elements of $Fd(X)$ are called *words* and $\overline{X}$ is *the generating set* of $(Fd\,(X)\,, \dashv, \vdash)$. By $|\omega|$ we denote *the length* of a word $\omega \in Fd(X)$.

Note that a free dimonoid can be defined in another way. Let X^+ be the free semigroup on an alphabet X and $|w|$ be the length of $w \in X^+$. By N we denote the set of all positive integers.

On the set

$$\tilde{F}[X] = \{(w; m) \in X^+ \times N : |w| \geq m\}$$

we define operations $\dashv', \vdash'$ by the rule:

$$(w_1, m_1) \dashv' (w_2, m_2) = (w_1 w_2, m_1),$$

$$(w_1, m_1) \vdash' (w_2, m_2) = (w_1 w_2, |w_1| + m_2).$$

Proposition 1 (see [9, Lemma 3])**.** *The free dimonoid* $(Fd(X), \dashv, \vdash)$ *is isomorphic to the dimonoid* $(\tilde{F}[X], \dashv', \vdash')$.

We denote the free dimonoid $(Fd(X), \dashv, \vdash)$ on an n-element set X by Fd_n. Now we consider the structure of the free dimonoid Fd_1.

Let $X = \{x\}$, then

$$Fd_1 = \{\overline{x}, (\overline{x}, x), (x, \overline{x}), (\overline{x}, x, x), (x, \overline{x}, x), (x, x, \overline{x}), ...\}.$$

Define on the set

$$P = \{(a; b) \in N \times N \mid a \geq b\}$$

two binary operations $\prec$ and $\succ$ as follows:

$$(a; b) \prec (c; d) = (a + c; b),$$

$$(a; b) \succ (c; d) = (a + c; a + d).$$

Proposition 2 (see [9, Lemma 4])**.** *The free dimonoid $(Fd_1, \dashv, \vdash)$ of rank 1 is isomorphic to the dimonoid $(P, \prec, \succ)$.*

Further we will identify elements of the free dimonoid $(Fd_1, \dashv, \vdash)$ with respective elements of the dimonoid $(P, \prec, \succ)$.

3 Endomorphisms of a free dimonoid of rank 1

For an arbitrary dimonoid $\mathfrak{D} = (D, \dashv, \vdash)$, by $End(\mathfrak{D})$ we denote the semigroup of all endomorphisms of the dimonoid $\mathfrak{D}$. First of all, we describe the structure of endomorphisms of a free dimonoid of rank 1.

Theorem 1. *For any $(k; l) \in P$ a transformation $\xi_{k,l}$ of the free dimonoid $(Fd_1, \dashv, \vdash)$ defined by $(a; b)\xi_{k,l} = (ak; (b-1)k + l)$ is a monomorphism. Also every endomorphism of $(Fd_1, \dashv, \vdash)$ has the above form.*

Proof. Fix an arbitrary pair $(k; l) \in P$ and take $(a; b), (c; d) \in Fd_1$. Then

$$((a; b) \prec (c; d))\xi_{k,l} = (a + c; b)\xi_{k,l} =$$

$$= ((a + c)k; (b - 1)k + l) =$$

$$= (ak; (b - 1)k + l) \prec (ck; (d - 1)k + l) =$$

$$= (a; b)\xi_{k,l} \prec (c; d)\xi_{k,l}$$

and

$$((a; b) \succ (c; d))\xi_{k,l} = (a + c; a + d)\xi_{k,l} =$$

$$= ((a + c)k; (a + d - 1)k + l) =$$

$$= (ak; (b - 1)k + l) \succ (ck; (d - 1)k + l) =$$

$$= (a; b)\xi_{k,l} \succ (c; d)\xi_{k,l}.$$

Therefore, $\xi_{k,l} \in End(Fd_1)$ for all $(k; l) \in P$.

Suppose that $(a;b)\xi_{k,l} = (c;d)\xi_{k,l}$ for some $(a;b), (c;d) \in Fd_1$. Then

$$(ak;(b-1)k+l) = (ck;(d-1)k+l),$$

whence $a = c, b = d$. Thus, $\xi_{k,l}$ is a monomorphism for all $(k;l) \in P$.

Now, let ξ be an arbitrary endomorphism of $(Fd_1, \dashv, \vdash)$ and $(1;1)\xi = (k;l)$. By induction on a, we show that

$$(a;a)\xi = (ak;(a-1)k+l)$$

for all $(a;a) \in Fd_1$.

For $a = 1$ this is obvious and for $a = 2$ we have

$$(2;2)\xi = ((1;1) \succ (1;1))\xi = (k;l) \succ (k;l) = (2k;k+l).$$

Assume that $(n;n)\xi = (nk;(n-1)k+l)$ for some $n \in N, n \geq 3$. Then for $a = n+1$ we obtain

$$(n+1;n+1)\xi = ((n;n) \succ (1;1))\xi =$$

$$= (n;n)\xi \succ (1;1)\xi = (nk;(n-1)k+l) \succ (k;l) =$$

$$= ((n+1)k;nk+l).$$

So, by induction $(a;a)\xi = (ak;(a-1)k+l)$ for all $a \in N$.

Finally, for all $(a;b) \in Fd_1$, where $a > b$,

$$(a;b)\xi = ((b;b) \prec (a-b;a-b))\xi =$$

$$= (b;b)\xi \prec (a-b;a-b)\xi =$$

$$= (bk;(b-1)k+l) \prec ((a-b)k;(a-b-1)k+l) =$$

$$= (ak;(b-1)k+l).$$

Thus, $\xi = \xi_{k,l}$ and the theorem is proved. □

Further we consider a binary operation $\circ$ on N defined as follows:

$$(a;b) \circ (c;d) = (ac;(b-1)c+d).$$

Note that if $a \geq b, c \geq d$, then $ac = (a-1)c + c \geq (b-1)c + d$. Therefore, the operation $\circ$ is completed on the set $P = \{(a;b) \in N \times N \mid a \geq b\}$ and so the algebra $(P, \circ)$ is a semigroup.

Theorem 2. *The endomorphism semigroup $End(Fd_1)$ of the free dimonoid $(Fd_1, \dashv, \vdash)$ is isomorphic to the semigroup $(P, \circ)$.*

Proof. Define a mapping Θ of $End(Fd_1)$ into $(P,\circ)$ by $\xi_{k,l}\Theta = (k;l)$ for all $\xi_{k,l} \in End(Fd_1)$. Obviously, Θ is a bijection.

Let $\xi_{k,l}, \xi_{p,q} \in End(Fd_1)$ and $(a;b) \in P$. Then

$$(a;b)\xi_{k,l}\xi_{p,q} = (ak;(b-1)k+l)\xi_{p,q} =$$
$$= (akp; bkp - kp + lp - p + q) =$$
$$= (akp; ((b-1)k + l - 1)p + q) =$$
$$= (a;b)\xi_{kp,lp-p+q}.$$

Hence $\xi_{k,l}\xi_{p,q} = \xi_{kp,lp-p+q}$ and so

$$(\xi_{k,l}\xi_{p,q})\Theta = \xi_{kp,lp-p+q}\Theta = (kp; lp - p + q) =$$
$$= (k;l) \circ (p;q) = \xi_{k,l}\Theta \circ \xi_{p,q}\Theta.$$

□

We will identify elements of $End(Fd_1)$ with respective elements of $(P,\circ)$.

It is obvious that the automorphism group of $(Fd_1, \dashv, \vdash)$ is trivial.

4 Characteristics of the monoid $End(Fd_1)$

There is a number of algebras properties of which are determined by properties of their endomorphism semigroups. Definability conditions and some other characteristics for the endomorphism semigroup of a free semigroup (free monoid) and a free group were obtained in [7] and [8], respectively. An abstract characteristic for the endomorphism semigroup of a free group was described by V. M. Usenko [10].

Theorem 3. *Let X be a singleton set, Y be an arbitrary set and there exists an isomorphism $\Theta : End(Fd(X)) \to End(Fd(Y))$. Then free dimonoids $(Fd(X)\ \dashv, \vdash)$ and $(Fd(Y), \dashv, \vdash)$ are isomorphic.*

Proof. Assume that $|\overline{Y}| \geq 2$ and $y \in \overline{Y}$. By $E(Fd(X))$ and $E(Fd(Y))$ we denote the set of all idempotents of $End(Fd(X))$ and, respectively, $End(Fd(Y))$. Since Θ is an isomorphism, then $E(Fd(X))\Theta = E(Fd(Y))$.

For arbitrary $(k;l) \in Fd(X)$, we have $(k;l) \in E(Fd(X))$ if and only if $(k^2; (l-1)k + l) = (k;l)$, whence $k = l = 1$. Thus, $|E(Fd(X))| = 1$.

Define a transformation φ_y of $\overline{Y}$ by the rule: $t\varphi_y = y$ for all $t \in \overline{Y}$. Further we extend φ_y to a transformation Φ_y of the dimonoid $(Fd(Y), \dashv, \vdash)$ which is defined as follows:

$$(u_1, ..., \overline{u_i}, ..., u_k)\Phi_y = u_1\varphi_y \vdash ... \vdash \overline{u_i\varphi_y} \dashv ... \dashv u_k\varphi_y$$

for all $(u_1, ..., \overline{u_i}, ..., u_k) \in Fd(Y)$.

Thus, for all $t \in \overline{Y}$ we have

$$t\Phi_y^2 = (t\Phi_y)\Phi_y = y\Phi_y = y = t\Phi_y,$$

whence $\Phi_y^2 = \Phi_y$. Taking into account the identity automorphism of $End(Fd(Y))$, we obtain $|E(Fd(Y))| \geq 3$ that contradicts the condition $E(Fd(X))\Theta = E(Fd(Y))$. So, $|\overline{Y}| = 1$ and $(Fd(X) \dashv, \vdash)$, $(Fd(Y), \dashv, \vdash)$ are isomorphic dimonoids. □

In addition, the semigroup $End(Fd_1)$ can be represented as a semilattice of some their subsemigroups. Indeed, let

$$P_1 = \{(1;1)\}, P_2 = (N \setminus \{1\}) \times \{1\},$$

$$P_3 = \{(n;n) | n \in N, n \neq 1\}$$

and

$$P_4 = \{(m;n) \in N \times N | m > n, n \neq 1\}.$$

It is obvious that each of sets $P_i, 1 \leq i \leq 4$, is a subsemigroup of the monoid $End(Fd_1)$. We put $\Omega = \{1, 2, 3, 4\}$ and define on Ω the following operation:

$$1 \cdot i = i = i \cdot 1, \;\; 4 \cdot i = 4 = i \cdot 4 \;\text{ and}$$

$$i \cdot i = i, \;\; 2 \cdot 3 = 4 = 3 \cdot 2 \;\; (1 \leq i \leq 4).$$

It is easy to see that $(\Omega, \cdot)$ is a commutative semigroup of idempotents, that is a semilattice.

Proposition 3. *The monoid $End(Fd_1)$ is a semilattice $(\Omega, \cdot)$ of semigroups P_i, $i \in \Omega$. Moreover, P_2 and P_3 are isomorphic to the free commutative semigroup with the countably infinite set of free generators that are prime numbers.*

Proof. Define a mapping η of $End(Fd_1)$ onto $(\Omega, \cdot)$ as follows:

$$(a;b)\eta = i, \;\text{ if }\; (a;b) \in P_i.$$

A direct check shows that η is a homomorphism and semigroups P_2, P_3 and the multiplicative semigroup $(N \setminus \{1\}, \cdot)$ are isomorphic. □

Further we describe an abstract characteristic of $End(Fd_1)$. Recall that an ideal I of a semigroup S is called *densely embedded* (see [11, 12]) if every nontrivial homomorphism (that is not an isomorphism) of S induces a nontrivial homomorphism of I, and for every semigroup T such that $S \subset T$ and I is an ideal of T, there exists a nontrivial homomorphism of T which induces an isomorphism on I.

A semigroup S is called *reductive on the left* if for $a, b \in S$ the condition $ua = ub$ for all $u \in S$, implies $a = b$. If for a semigroup S there do not exist distinct a_1, a_2 such that $a_1x = a_2x, xa_1 = xa_2$ for all $x \in S$, then S is called *a weakly reductive semigroup*. It is clear that every reductive on the left semigroup is weakly reductive.

Let $\Im(X)$ be the symmetric semigroup of all transformations on X and S be an arbitrary semigroup. As is well known, a homomorphism

$$\rho : S \to \Im(S) : t \mapsto \rho_t,$$

where $u\rho_t = ut$ for all $u \in S$, is called *a regular representation* of S.

For every ideal I of a semigroup S, a regular representation ρ induces the following representation:

$$\rho^I : S \to \Im(I) : t \mapsto \rho_t|_I,$$

where $\rho_t|_I$ is the restriction of $\rho_t \in \Im(S)$ to I.

An ideal I of a semigroup S is called *essential* [10] if the induced regular representation ρ^I is an injective mapping.

Proposition 4 (see [10])**.** *For an ideal I of a semigroup S the following conditions are equivalent:*

(i) I is essential;

(ii) I is densely embedded and reductive on the left.

For an arbitrary semigroup S, by $T(S)$ we denote the translation hull (see, e. g., [12]) of S and by $T_0(S)$ the inner part of $T(S)$. If S is a weakly reductive semigroup, then S is isomorphic to $T_0(S)$.

Finally, we obtain an abstract characteristic for the endomorphism semigroup of a free dimonoid of rank 1.

Theorem 4. *An arbitrary semigroup S is isomorphic to the endomorphism semigroup $End(Fd_1)$ of the free dimonoid $(Fd_1, \dashv, \vdash)$ if and only if S contains a densely embedded ideal isomorphic to the semigroup P_4.*

Proof. Let $S \cong End(Fd_1)$, then S contains up to isomorphism the semigroup P_4. From Proposition 3 it follows that P_4 is an ideal of $End(Fd_1)$. We show that the ideal P_4 is essential.

Suppose there exist $(a; b), (c; d) \in End(Fd_1)$ for which $(a; b)\rho^{P_4} = (c; d)\rho^{P_4}$, that is $\rho_{(a;b)}|_{P_4} = \rho_{(c;d)}|_{P_4}$.

Then for all $(x; y) \in P_4$ we have

$$(a; b)(x; y) = (c; d)(x; y),$$

$$(ax; (b-1)x + y) = (cx; (d-1)x + y),$$

whence $a = c$, $b = d$ by reductivity of the multiplicative monoid $(N, \cdot)$. Therefore, ρ^{P_4} is injective and then P_4 is an essential ideal. Thus, by Proposition 4, P_4 is a densely embedded ideal for the semigroup $End(Fd_1)$.

Conversely, let a semigroup S contain a densely embedded ideal I isomorphic to P_4. Then as is known [11], $T(P_4) \cong End(Fd_1)$ and $T(I) \cong S$, whence S and $End(Fd_1)$ are isomorphic. □

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