BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(76), 2014, Pages 3–12
ISSN 1024–7696

Estimates for the number of vertices with an interval spectrum in proper edge colorings of some graphs

R. R. Kamalian

Abstract. For an undirected, simple, finite, connected graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. A function $\varphi : E(G) \to \{1, 2, \ldots, t\}$ is called a proper edge t-coloring of a graph G if all colors are used and no two adjacent edges receive the same color. An arbitrary nonempty subset of consecutive integers is called an interval. The set of all proper edge t-colorings of G is denoted by $\alpha(G, t)$. The minimum value of t for which there exists a proper edge t-coloring of a graph G is denoted by $\chi'(G)$. Let

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G, t).$$

If G is a graph, $\varphi \in \alpha(G)$, $x \in V(G)$, then the set of colors of edges of G incident with x is called a spectrum of the vertex x in the coloring φ of the graph G and is denoted by $S_G(x, \varphi)$. If $\varphi \in \alpha(G)$ and $x \in V(G)$, then we say that φ is interval (persistent-interval) for x if $S_G(x, \varphi)$ is an interval (an interval with 1 as its minimum element). For an arbitrary graph G and any $\varphi \in \alpha(G)$, we denote by $f_{G,i}(\varphi)(f_{G,pi}(\varphi))$ the number of vertices of the graph G for which φ is interval (persistent-interval). For any graph G, let us set

$$\eta_i(G) \equiv \max_{\varphi \in \alpha(G)} f_{G,i}(\varphi), \quad \eta_{pi}(G) \equiv \max_{\varphi \in \alpha(G)} f_{G,pi}(\varphi).$$

For graphs G from some classes of graphs, we obtain lower bounds for the parameters $\eta_i(G)$ and $\eta_{pi}(G)$.

Mathematics subject classification: 05C15.
Keywords and phrases: Proper edge coloring, interval spectrum.

1 Introduction

We consider undirected, simple, finite, connected graphs. For a graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. For any $x \in V(G)$, $d_G(x)$ denotes the degree of the vertex x in G. For a graph G, we denote by $\Delta(G)$ the maximum degree of a vertex of G. A function $\varphi : E(G) \to \{1, 2, \ldots, t\}$ is called a proper edge t-coloring of a graph G if all colors are used and no two adjacent edges receive the same color. The set of all proper edge t-colorings of G is denoted by $\alpha(G, t)$. The minimum value of t for which there exists a proper edge t-coloring of a graph G is called a chromatic index [22] of G and is denoted by $\chi'(G)$.

Let us also define the set $\alpha(G)$ of all proper edge colorings of the graph G

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G,t).$$

If G is a graph, $\varphi \in \alpha(G)$, $x \in V(G)$, then the set of colors of edges of G incident with x is called a spectrum of the vertex x in the coloring φ of the graph G and is denoted by $S_G(x,\varphi)$.

An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p,q]$. An interval D is called an h-interval if $|D| = h$.

For any real number ξ, we denote by $\lfloor \xi \rfloor$ ($\lceil \xi \rceil$) the maximum (minimum) integer which is less (greater) than or equal to ξ.

If G is a graph, $\varphi \in \alpha(G)$, and $x \in V(G)$, then we say that φ is interval (persistent-interval) for x if $S_G(x,\varphi)$ is a $d_G(x)$-interval (a $d_G(x)$-interval with 1 as its minimum element). For an arbitrary graph G and any $\varphi \in \alpha(G)$, we denote by $f_{G,i}(\varphi)(f_{G,pi}(\varphi))$ the number of vertices of the graph G for which φ is interval (persistent-interval). For any graph G, let us [17] set

$$\eta_i(G) \equiv \max_{\varphi\in\alpha(G)} f_{G,i}(\varphi), \quad \eta_{pi}(G) \equiv \max_{\varphi\in\alpha(G)} f_{G,pi}(\varphi).$$

The terms and concepts that we do not define can be found in [23].

It is clear that if for any graph G $\eta_{pi}(G) = |V(G)|$, then $\chi'(G) = \Delta(G)$. For a regular graph G, these two conditions are equivalent: $\eta_{pi}(G) = |V(G)| \Leftrightarrow \chi'(G) = \Delta(G)$. It is known [15, 19] that for a regular graph G, the problem of deciding whether or not the equation $\chi'(G) = \Delta(G)$ is true is NP-complete. It means that for a regular graph G, the problem of deciding whether or not the equation $\eta_{pi}(G) = |V(G)|$ is true is also NP-complete. For any tree G, some necessary and sufficient condition for fulfilment of the equation $\eta_{pi}(G) = |V(G)|$ was obtained in [8]. In this paper, for an arbitrary regular graph G, we obtain a lower bound for the parameter $\eta_{pi}(G)$.

If G is a graph, $R_0 \subseteq V(G)$, and the coloring $\varphi \in \alpha(G)$ is interval (persistent-interval) for any $x \in R_0$, then we say that φ is interval (persistent-interval) on R_0.

$\varphi \in \alpha(G)$ is called an interval coloring of a graph G if φ is interval on $V(G)$.

We define the set $\mathfrak{N}$ as the set of all graphs for which there is an interval coloring. Clearly, for any graph G, $G \in \mathfrak{N}$ if and only if $\eta_i(G) = |V(G)|$.

The notion of an interval coloring was introduced in [6]. In [6, 7, 16] it is shown that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. For a regular graph G, these two conditions are equivalent: $G \in \mathfrak{N} \Leftrightarrow \chi'(G) = \Delta(G)$ [6, 7, 16]. Consequently, for a regular graph G, four conditions are equivalent: $G \in \mathfrak{N}$, $\chi'(G) = \Delta(G)$, $\eta_i(G) = |V(G)|$, $\eta_{pi}(G) = |V(G)|$. It means that for any regular graph G,

1) the problem of deciding whether G has or not an interval coloring is NP-complete,

2) the problem of deciding whether the equation $\eta_i(G) = |V(G)|$ is true or not is NP-complete.

In this paper, for an arbitrary regular graph G, we obtain a lower bound for the parameter $\eta_i(G)$.

We also obtain some results for bipartite graphs. The complexity of the problem of existence of an interval coloring for bipartite graphs is investigated in [3, 9, 21]. In [16] it is shown that for a bipartite graph G with bipartition (X, Y) and $\Delta(G) = 3$ the problem of existence of a proper edge 3-coloring which is persistent-interval on $X \cup Y$ (or even only on Y [6, 16]) is NP-complete.

Suppose that G is an arbitrary bipartite graph with bipartition (X, Y) [3]. Then $\eta_i(G) \geq \max\{|X|, |Y|\}$.

Suppose that G is a bipartite graph with bipartition (X, Y) for which there exists a coloring $\varphi \in \alpha(G)$ persistent-interval on Y. Then $\eta_{pi}(G) \geq 1 + |Y|$.

Some attention is paid to (α, β)-biregular bipartite graphs [4, 13, 14, 18] in the case when $|\alpha - \beta| = 1$.

We show that if G is a $(k-1, k)$-biregular bipartite graph, $k \geq 4$, then

$$\eta_i(G) \geq \frac{k-1}{2k-1} \cdot |V(G)| + \left\lceil \frac{k}{\lceil \frac{k}{2} \rceil \cdot (2k-1)} \cdot |V(G)| \right\rceil.$$

We show that if G is a $(k-1, k)$-biregular bipartite graph, $k \geq 3$, then

$$\eta_{pi}(G) \geq \frac{k}{2k-1} \cdot |V(G)|.$$

2 Results

Theorem 1 (see [17]). *If G is a regular graph with $\chi'(G) = 1 + \Delta(G)$, then*

$$\eta_{pi}(G) \geq \left\lceil \frac{|V(G)|}{1 + \Delta(G)} \right\rceil.$$

Proof. Suppose that $\beta \in \alpha(G, 1 + \Delta(G))$. For any $j \in [1, 1 + \Delta(G)]$, define

$$V_{G,\beta,j} \equiv \{x \in V(G)/j \notin S_G(x, \beta)\}.$$

For arbitrary integers j', j'', where $1 \leq j' < j'' \leq 1 + \Delta(G)$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \varnothing \quad \text{and} \quad \bigcup_{j=1}^{1+\Delta(G)} V_{G,\beta,j} = V(G).$$

Hence, there exists $j_0 \in [1, 1 + \Delta(G)]$ for which

$$|V_{G,\beta,j_0}| \geq \left\lceil \frac{|V(G)|}{1 + \Delta(G)} \right\rceil.$$

Set $R_0 \equiv V_{G,\beta,j_0}$.

Case 1. $j_0 = 1 + \Delta(G)$.

Clearly, β is persistent-interval on R_0.

Case 2. $j_0 \in [1, \Delta(G)]$.

Define a function $\varphi : E(G) \to [1, 1+\Delta(G)]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} \beta(e) & \text{if } \beta(e) \notin \{j_0, 1+\Delta(G)\}, \\ j_0 & \text{if } \beta(e) = 1+\Delta(G), \\ 1+\Delta(G) & \text{if } \beta(e) = j_0. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, 1+\Delta(G))$ and φ is persistent-interval on R_0. □

Corollary 1 (see [17]). *If G is a cubic graph, then there exists a coloring from $\alpha(G, \chi'(G))$ which is persistent-interval for at least $\left\lceil \frac{|V(G)|}{4} \right\rceil$ vertices of G.*

Theorem 2 (see [17]). *If G is a regular graph with $\chi'(G) = 1 + \Delta(G)$, then*

$$\eta_i(G) \geq \left\lceil \frac{|V(G)|}{\left\lceil \frac{1+\Delta(G)}{2} \right\rceil} \right\rceil.$$

Proof. Suppose that $\beta \in \alpha(G, 1+\Delta(G))$. For any $j \in [1, 1+\Delta(G)]$, define

$$V_{G,\beta,j} \equiv \{x \in V(G) / j \notin S_G(x, \beta)\}.$$

For arbitrary integers j', j'', where $1 \leq j' < j'' \leq 1+\Delta(G)$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \varnothing \quad \text{and} \quad \bigcup_{j=1}^{1+\Delta(G)} V_{G,\beta,j} = V(G).$$

For any $i \in [1, \lceil \frac{1+\Delta(G)}{2} \rceil]$, let us define the subset $V(G, \beta, i)$ of the set $V(G)$ as follows:

$$V(G,\beta,i) \equiv \begin{cases} V_{G,\beta,2i-1} \cup V_{G,\beta,2i} & \text{if } \Delta(G) \text{ is odd and } i \in [1, \frac{1+\Delta(G)}{2}] \\ & \text{or } \Delta(G) \text{ is even and } i \in [1, \frac{\Delta(G)}{2}], \\ V_{G,\beta,1+\Delta(G)} & \text{if } \Delta(G) \text{ is even and } i = 1 + \frac{\Delta(G)}{2}. \end{cases}$$

For arbitrary integers i', i'', where $1 \leq i' < i'' \leq \lceil \frac{1+\Delta(G)}{2} \rceil$, we have

$$V(G,\beta,i') \cap V(G,\beta,i'') = \varnothing \quad \text{and} \quad \bigcup_{i=1}^{\lceil \frac{1+\Delta(G)}{2} \rceil} V(G,\beta,i) = V(G).$$

Hence, there exists $i_0 \in \left[1, \lceil \frac{1+\Delta(G)}{2} \rceil\right]$ for which

$$|V(G,\beta,i_0)| \geq \left\lceil \frac{|V(G)|}{\left\lceil \frac{1+\Delta(G)}{2} \right\rceil} \right\rceil.$$

Set $R_0 \equiv V(G, \beta, i_0)$.

Case 1. $i_0 = \lceil \frac{1+\Delta(G)}{2} \rceil$.

Case 1.a. $\Delta(G)$ is even.

Clearly, β is interval on R_0.

Case 1.b. $\Delta(G)$ is odd.

Define a function $\varphi : E(G) \to [1, 1+\Delta(G)]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e)+1)(\mathrm{mod}(1+\Delta(G))) & \text{if } \beta(e) \neq \Delta(G), \\ 1+\Delta(G) & \text{if } \beta(e) = \Delta(G). \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, 1+\Delta(G))$ and φ is interval on R_0.

Case 2. $1 \le i_0 \le \lceil \frac{\Delta(G)-1}{2} \rceil$.

Define a function $\varphi : E(G) \to [1, 1+\Delta(G)]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e)+2+\Delta(G)-2i_0)(\mathrm{mod}(1+\Delta(G))) & \text{if } \beta(e) \neq 2i_0-1, \\ 1+\Delta(G) & \text{if } \beta(e) = 2i_0-1. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, 1+\Delta(G))$ and φ is interval on R_0. □

Corollary 2 (see [17])**.** *If G is a cubic graph, then there exists a coloring from $\alpha(G, \chi'(G))$ which is interval for at least $\frac{|V(G)|}{2}$ vertices of G.*

Theorem 3 (see [6,7,16])**.** *Let G be a bipartite graph with bipartition (X, Y). Then there exists a coloring $\varphi \in \alpha(G, |E(G)|)$ which is interval on X.*

Corollary 3. *Let G be a bipartite graph with bipartition (X, Y). Then $\eta_i(G) \ge \max\{|X|, |Y|\}$.*

Theorem 4 (see [1,6,7])**.** *Let G be a bipartite graph with bipartition (X, Y) where $d_G(x) \le d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Then there exists a coloring $\varphi_0 \in \alpha(G, \Delta(G))$ which is persistent-interval on Y.*

Theorem 5. *Suppose G is a bipartite graph with bipartition (X, Y), and there exists a coloring $\varphi_0 \in \alpha(G, \Delta(G))$ which is persistent-interval on Y. Then, for an arbitrary vertex $x_0 \in X$, there exists $\psi \in \alpha(G, \Delta(G))$ which is persistent-interval on $\{x_0\} \cup Y$.*

Proof. Case 1. $S_G(x_0, \varphi_0) = [1, d_G(x_0)]$. In this case ψ is φ_0.

Case 2. $S_G(x_0, \varphi_0) \neq [1, d_G(x_0)]$.

Clearly, $[1, d_G(x_0)] \backslash S_G(x_0, \varphi_0) \neq \emptyset$, $S_G(x_0, \varphi_0) \backslash [1, d_G(x_0)] \neq \emptyset$. Since $|S_G(x_0, \varphi_0)| = |[1, d_G(x_0)]| = d_G(x_0)$, there exists $\nu_0 \in [1, d_G(x_0)]$ satisfying the condition $|[1, d_G(x_0)] \backslash S_G(x_0, \varphi_0)| = |S_G(x_0, \varphi_0) \backslash [1, d_G(x_0)]| = \nu_0$.

Now let us construct the sequence $\Theta_0, \Theta_1, \ldots, \Theta_{\nu_0}$ of proper edge $\Delta(G)$-colorings of the graph G, where for any $i \in [0, \nu_0]$, Θ_i is persistent-interval on Y.

Set $\Theta_0 \equiv \varphi_0$.

Suppose that for some $k \in [0, \nu_0 - 1]$, the subsequence $\Theta_0, \Theta_1, \ldots, \Theta_k$ is already constructed.

Let

$$t_k \equiv \max(S_G(x_0, \Theta_k)\backslash[1, d_G(x_0)]),$$

$$s_k \equiv \min([1, d_G(x_0)]\backslash S_G(x_0, \Theta_k)).$$

Clearly, $t_k > s_k$. Consider the path $P(k)$ in the graph G of maximum length with the initial vertex x_0 whose edges are alternatively colored by the colors t_k and s_k. Let Θ_{k+1} be obtained from Θ_k by interchanging the two colors t_k and s_k along $P(k)$.

It is not difficult to see that Θ_{ν_0} is persistent-interval on $\{x_0\} \cup Y$. Set $\psi \equiv \Theta_{\nu_0}$. □

Corollary 4. *Let G be a bipartite graph with bipartition (X, Y) where $d_G(x) \le d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Let x_0 be an arbitrary vertex of X. Then there exists a coloring $\varphi_0 \in \alpha(G, \Delta(G))$ which is persistent-interval on $\{x_0\} \cup Y$.*

Corollary 5 (see [17])**.** *Let G be a bipartite graph with bipartition (X, Y) where $d_G(x) \le d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Then $\eta_{pi}(G) \ge 1 + |Y|$.*

Remark 1. Notice that the complete bipartite graph $K_{n+1,n}$ for an arbitrary positive integer n satisfies the conditions of Corollary 5. Is is not difficult to see that $\eta_{pi}(K_{n+1,n}) = 1 + n$. It means that the bound obtained in Corollary 5 is sharp since in this case $|Y| = n$.

Remark 2. Let G be a bipartite $(k-1, k)$-biregular graph with bipartition (X, Y), where $k \ge 3$. Assume that all vertices in X have the degree $k-1$ and all vertices in Y have the degree k. Then the numbers $\frac{|X|}{k}$, $\frac{|Y|}{k-1}$, and $\frac{|V(G)|}{2k-1}$ are integer. It follows from the equalities $\gcd(k-1, k) = 1$ and $|E(G)| = |X| \cdot (k-1) = |Y| \cdot k$.

Theorem 6 (see [17])**.** *Let G be a bipartite $(k-1, k)$-biregular graph, where $k \ge 4$. Then*

$$\eta_i(G) \ge \frac{k-1}{2k-1} \cdot |V(G)| + \left\lceil \frac{k}{\lceil \frac{k}{2} \rceil \cdot (2k-1)} \cdot |V(G)| \right\rceil.$$

Proof. Suppose that (X, Y) is a bipartition of G. Without loss of generality we assume that all vertices in X have the degree $k-1$ and all vertices in Y have the degree k. Clearly, $\chi'(G) = \Delta(G) = k$. Suppose that $\beta \in \alpha(G, k)$. For any $j \in [1, k]$, define:

$$V_{G,\beta,j} \equiv \{x \in X / j \notin S_G(x, \beta)\}.$$

For arbitrary integers j', j'', where $1 \le j' < j'' \le k$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \varnothing \quad \text{and} \quad \bigcup_{j=1}^{k} V_{G,\beta,j} = X.$$

For any $i \in [1, \lceil \frac{k}{2} \rceil]$, let us define the subset $V(G, \beta, i)$ of the set X as follows:

$$V(G,\beta,i) \equiv \begin{cases} V_{G,\beta,2i-1} \cup V_{G,\beta,2i} & \text{if } k \text{ is odd and } i \in [1, \frac{k-1}{2}] \\ & \text{or } k \text{ is even and } i \in [1, \frac{k}{2}], \\ V_{G,\beta,k} & \text{if } k \text{ is odd and } i = \frac{1+k}{2}. \end{cases}$$

For arbitrary integers i', i'', where $1 \leq i' < i'' \leq \lceil \frac{k}{2} \rceil$, we have

$$V(G,\beta,i') \cap V(G,\beta,i'') = \varnothing \quad \text{and} \quad \bigcup_{i=1}^{\lceil \frac{k}{2} \rceil} V(G,\beta,i) = X.$$

Hence, there exists $i_0 \in \left[1, \lceil \frac{k}{2} \rceil\right]$ for which

$$|V(G,\beta,i_0)| \geq \left\lceil \frac{|X|}{\lceil \frac{k}{2} \rceil} \right\rceil.$$

Set $R_0 \equiv Y \cup V(G,\beta,i_0)$.
It is not difficult to verify that

$$|R_0| \geq \frac{k-1}{2k-1} \cdot |V(G)| + \left\lceil \frac{k}{\lceil \frac{k}{2} \rceil \cdot (2k-1)} \cdot |V(G)| \right\rceil.$$

Case 1. $i_0 = \lceil \frac{k}{2} \rceil$.
Case 1.a. k is odd.
Clearly, β is interval on R_0.
Case 1.b. k is even.
Define a function $\varphi : E(G) \to [1, k]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e)+1)(\operatorname{mod} k) & \text{if } \beta(e) \neq k-1, \\ k & \text{if } \beta(e) = k-1. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, k)$ and φ is interval on R_0.
Case 2. $i_0 \in \left[1, \lceil \frac{k}{2} \rceil - 1\right]$.
Define a function $\varphi : E(G) \to [1, k]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e)+1+k-2i_0)(\operatorname{mod} k) & \text{if } \beta(e) \neq 2i_0 - 1, \\ k & \text{if } \beta(e) = 2i_0 - 1. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, k)$ and φ is interval on R_0. □

Corollary 6 (see [17]). *Let G be a bipartite $(k-1,k)$-biregular graph, where k is even and $k \geq 4$. Then*

$$\eta_i(G) \geq \frac{k+1}{2k-1} \cdot |V(G)|.$$

Corollary 7 (see [17]). *Let G be a bipartite $(3,4)$-biregular graph. Then there exists a coloring from $\alpha(G,4)$ which is interval for at least $\frac{5}{7}|V(G)|$ vertices of G.*

Remark 3. For an arbitrary bipartite graph G with $\Delta(G) \leq 3$, there exists an interval coloring of G [10–12]. Consequently, if G is a bipartite $(2,3)$-biregular graph, then $\eta_i(G) = |V(G)|$.

Remark 4. Some sufficient conditions for existence of an interval coloring of a $(3,4)$-biregular bipartite graph were obtained in [2, 5, 20].

Theorem 7 (see [17]). *Let G be a bipartite $(k-1,k)$-biregular graph, where $k \geq 3$. Then*

$$\eta_{pi}(G) \geq \frac{k}{2k-1} \cdot |V(G)|.$$

Proof. Suppose that (X,Y) is a bipartition of G. Without loss of generality we assume that all vertices in X have the degree $k-1$ and all vertices in Y have the degree k. Clearly, $\chi'(G) = \Delta(G) = k$. Suppose that $\beta \in \alpha(G,k)$.

For any $j \in [1,k]$, define:

$$V_{G,\beta,j} \equiv \{x \in X / j \notin S_G(x,\beta)\}.$$

For arbitrary integers j', j'', where $1 \leq j' < j'' \leq k$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \varnothing \quad \text{and} \quad \bigcup_{j=1}^{k} V_{G,\beta,j} = X.$$

Hence, there exists $j_0 \in [1,k]$ for which

$$|V_{G,\beta,j_0}| \geq \frac{|X|}{k}.$$

Set $R_0 \equiv Y \cup V_{G,\beta,j_0}$.

It is not difficult to verify that

$$|R_0| \geq \frac{k}{2k-1} \cdot |V(G)|.$$

Case 1. $j_0 = k$.

Clearly, β is persistent-interval on R_0.

Case 2. $j_0 \in [1,k-1]$.

Define a function $\varphi : E(G) \to [1,k]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} \beta(e) & \text{if } \beta(e) \notin \{j_0,k\}, \\ j_0 & \text{if } \beta(e) = k, \\ k & \text{if } \beta(e) = j_0. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G,k)$ and φ is persistent-interval on R_0. $\square$

Corollary 8 (see [17])**.** *Let G be a bipartite $(3,4)$-biregular graph. Then there exists a coloring from $\alpha(G,4)$ which is persistent-interval for at least $\frac{4}{7}|V(G)|$ vertices of G.*

Acknowledgment. The author thanks professors A. S. Asratian and P. A. Petrosyan for their attention to this work.

References

[1] Asratian A. S. *Investigation of some mathematical model of Scheduling Theory.* Doctoral dissertation, Moscow University, 1980 (in Russian).

[2] Asratian A. S., Casselgren C. J. *A sufficient condition for interval edge colorings of* $(4,3)$*-biregular bipartite graphs.* Research report LiTH-MAT-R-2006-07, Linköping University, 2006.

[3] Asratian A. S., Casselgren C. J. *Some results on interval edge colorings of* (α,β)*-biregular bipartite graphs.* Research report LiTH-MAT-R-2006-09, Linköping University, 2006.

[4] Asratian A. S., Casselgren C. J. *On interval edge colorings of* (α,β)*-biregular bipartite graphs.* Discrete Math., 2007, **307**, 1951–1956.

[5] Asratian A. S., Casselgren C. J., Vandenbussche J., West D. B. *Proper path-factors and interval edge-coloring of* $(3,4)$*-biregular bigraphs.* J. of Graph Theory, 2009, **61**, 88–97.

[6] Asratian A. S., Kamalian R. R. *Interval colorings of edges of a multigraph.* Appl. Math., 1987, **5**, Yerevan State University, 25–34 (in Russian).

[7] Asratian A. S., Kamalian R. R. *Investigation of interval edge-colorings of graphs.* Journal of Combinatorial Theory, Series B, 1994, **62**, No. 1, 34–43.

[8] Caro Y., Schönheim J. *Generalized* 1*-factorization of trees.* Discrete Math., 1981, **33**, 319–321.

[9] Giaro K. *The complexity of consecutive* Δ*-coloring of bipartite graphs:* 4 *is easy,* 5 *is hard.* Ars Combin., 1997, **47**, 287–298.

[10] Giaro K. *Compact Task Scheduling on Dedicated Processors with no Waiting Periods.* Ph. D. Thesis, Technical University of Gdańsk, ETI Faculty, Gdańsk, 1999, (in Polish).

[11] Giaro K., Kubale M., Małafiejski M. *Compact scheduling in open shop with zero-one time operations.* INFOR, 1999, **37**, 37–47.

[12] Hansen H. *Scheduling with minimum waiting periods.* Master Thesis, Odense University, Odense, Denmark, 1992 (in Danish).

[13] Hanson D., Loten C. O. M. *A lower bound for Interval colouring bi-regular bipartite graphs.* Bulletin of the ICA, 1996, **18**, 69–74.

[14] Hanson D., Loten C. O. M., Toft B. *On interval colourings of bi-regular bipartite graphs.* Ars Combin., 1998, **50**, 23–32.

[15] Holyer I. *The* NP*-completeness of edge-coloring.* SIAM J. Comput., 1981, **10**, 718–720.

[16] Kamalian R. R. *Interval Edge Colorings of Graphs.* Doctoral dissertation, the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of USSR, Novosibirsk, 1990 (in Russian).

[17] Kamalian R. R. *On a number of vertices with an interval spectrum in proper edge colorings of some graphs.* Research report LiTH-MAT-R-2011/03-SE, Linköping University, 2011.

[18] Kostochka A. V. *Unpublished manuscript*, 1995.

[19] Leven D., Galil Z. *NP-completeness of finding the chromatic index of regular graphs.* J. Algorithms, 1983, **4**, 35–44.

[20] Pyatkin A. V. *Interval coloring of* $(3,4)$*-biregular bipartite graphs having large cubic subgraphs.* J. of Graph Theory, 2004, **47**, 122–128.

[21] Sevast'janov S. V. *Interval colorability of the edges of a bipartite graph.* Metody Diskret. Analiza, 1990, **50**, 61–72 (in Russian).

[22] Vizing V. G. *The chromatic index of a multigraph.* Kibernetika, 1965, **3**, 29–39.

[23] West D. B. *Introduction to Graph Theory.* Prentice-Hall, New Jersey, 1996.

R. R. Kamalian
Institute for Informatics and Automation Problems
National Academy of Sciences of RA
0014 Yerevan, Republic of Armenia
E-mail: *rrkamalian@yahoo.com*

Received April 30, 2013