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Closure operators in the categories of modules. Part IV (Relations between the operators and preradicals)

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Abstract. In this work (which is a continuation of [1–3]) the relations between the class $\mathbb{CO}$ of the closure operators of a module category R-Mod and the class $\mathbb{PR}$ of preradicals of this category are investigated. The transition from $\mathbb{CO}$ to $\mathbb{PR}$ and backwards is defined by three mappings $\Phi : \mathbb{CO} \to \mathbb{PR}$ and $\Psi_1, \Psi_2 : \mathbb{CO} \to \mathbb{PR}$. The properties of these mappings are studied.

Some monotone bijections are obtained between the preradicals of different types (idempotent, radical, hereditary, cohereditary, etc.) of $\mathbb{PR}$ and the closure operators of $\mathbb{CO}$ with special properties (weakly hereditary, idempotent, hereditary, maximal, minimal, cohereditary, etc.).

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1 Introduction. Preliminary notions and results

The purpose of this work is the investigation of the relations between the preradicals of the module category R-Mod and the closure operators of this category. For that three known mappings are used, which provide the connection between the closure operators and preradicals of R-Mod. We will study the properties of these mappings for different classes of preradicals and of closure operators of R-Mod.

This article is a continuation of [1–3], where the necessary notions are indicated. Nevertheless, for completeness and independence of this part, we would remind shortly the main notions and results which are used in continuation.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. For every module $M \in R$-Mod we denote by $\mathbb{L}(M)$ the lattice of submodules of M.

A *preradical* of R-Mod is a subfunctor r of the identity functor of R-Mod, i.e. $r(M) \subseteq M$ for every $M \in R$-Mod and $f\big(r(M)\big) \subseteq r(M')$ for every R-morphism $f : M \to M'$. We denote by $\mathbb{PR}$ the class of all preradicals of the category R-Mod. We remind the *principal types* of preradicals [4–6]. The preradical $r \in \mathbb{PR}$ is called:

– *idempotent* if $r\big(r(M)\big) = r(M)$ for every $M \in R$-Mod;
– *radical* if $r\big(M/\,r(M)\big) = O$ for every $M \in R$-Mod;
– *hereditary* (*pretorsion*) if $r(N) = r(M) \cap N$ for every $N \subseteq M$;
– *cohereditary* if $r(M/\,N) = \big(r(M) + N\big)/\,N$ for every $N \subseteq M$;

– *torsion* if it is a hereditary radical;

– *cotorsion* if it is idempotent and cohereditary.

It is well known that some types of preradicals of R-Mod can be characterized by special ring constructions, such as preradical filters (i.e. linear topologies), radical filters, ideals, idempotent ideals, etc. (see [4–6]). Some results of such type are included in the following statement.

Proposition 1.1. *There exist the bijections between:*

1) *the* ***pretorsions*** *of R-Mod and preradical filters of R;*
2) *the* ***torsions*** *of R-Mod and radical filters of R;*
3) *the* ***cohereditary preradicals*** *of R-Mod and ideals of R;*
4) *the* ***cotorsions*** *of R-Mod and idempotent ideals of R;*
5) *the* ***hereditary and cohereditary*** *preradicals of R-Mod and still ideals of R, i.e. the ideals with the condition* (a) : $a \in Ia$ *for every* $a \in I$ *(see* [4]*, p.* 12; [5]*, p.* 467*).* $\square$

A *closure operator* of R-Mod is a mapping C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule of M denoted by $C_M(N)$ which satisfies the conditions:

(c_1) $N \subseteq C_M(N)$;

(c_2) if $N_1 \subseteq N_2$ for $N_1, N_2 \in \mathbb{L}(M)$, then $C_M(N_1) \subseteq C_M(N_2)$ (the monotony);

(c_3) for every R-morphism $f : M \to M'$ and $N \in \mathbb{L}(M)$ we have

$$f\big(C_M(N)\big) \subseteq C_{M'}(N)\big(f(M)\big) \text{ (the continuity) [6–9].}$$

We denote by $\mathbb{CO}$ the class of all closure operators of R-Mod.

A closure operator $C \in \mathbb{CO}$ is called:

– *weakly hereditary* if $C_{C_M(N)}(N) = C_M(N)$ for every $N \subseteq M$;

– *idempotent* if $C_M\big(C_M(N)\big) = C_M(N)$ for every $N \subseteq M$;

– *hereditary* if $C_N(L) = C_M(L) \cap N$ for every $L \subseteq N \subseteq M$;

– *cohereditary* if $\big(C_M(N) + K\big)/K = C_{M/K}\big((N + K)/K\big)$ for every $K, N \in \mathbb{L}(M)$;

– *maximal* if $C_M(N)/N = C_{M/N}(\bar{0})$ for every $N \subseteq M$ (or: $C_M(N)/K = C_{M/K}(N/K)$ for every $K \subseteq N \subseteq M$);

– *minimal* if $C_M(N) = C_M(O)+N$ for every $N \subseteq M$ (or: $C_M(N) = C_M(L)+N$ for every $L \subseteq N \subseteq M$).

The investigations of the present work are based on the following mappings between the classes $\mathbb{CO}$ and $\mathbb{PR}$ [7–9]:

1) $\Phi : \mathbb{CO} \to \mathbb{PR}$, where we denote $\Phi(C) = r_{_C}$, for every $C \in \mathbb{CO}$, and define:

$$r_{_C}(M) = C_M(O) \tag{1.1}$$

for every $M \in R$-Mod;

2) $\Psi_1 : \mathbb{PR} \to \mathbb{CO}$, where $\Psi_1(r) = C^r$ for every $r \in \mathbb{PR}$ and

$$[(C^r)_M(N)]/N = r(M/N) \tag{1.2}$$

for every $N \subseteq M$;

3) $\Psi_2 : \mathbb{PR} \to \mathbb{CO}$, where $\Psi_2(r) = C_r$ for every $r \in \mathbb{PR}$ and

$$(C_r)_M(N) = N + r(M) \tag{1.3}$$

for every $N \subseteq M$.

Proposition 1.2. (See [7,8]). *Let $r \in \mathbb{PR}$. Then:*

a) $\Phi(C^r) = r$ *and C^r is the **largest** closure operator $C \in \mathbb{CO}$ with the property $\Phi(C) = r$;*

b) $\Phi(C_r) = r$ *and C_r is the **least** closure operator $C \in \mathbb{CO}$ with the property $\Phi(C) = r$.* □

Let $r \in \mathbb{PR}$. Then for the closure operator $C \in \mathbb{CO}$ we have:

$$\Phi(C) = r \Leftrightarrow C_r \leq C \leq C^r \quad (\text{i.e. } \Phi^{-1}(r) = [C_r, C^r]).$$

The closure operators of the from C^r, where $r \in \mathbb{PR}$, are exactly the *maximal* closure operators and, similarly, the closure operators of the form C_r coincide with the *minimal* closure operators of R-Mod. We denote by $\mathbb{M}ax(\mathbb{CO})$ the class of all maximal closure operators of R-Mod, and by $\mathbb{M}in(\mathbb{CO})$ the class of all minimal closure operators of R-Mod.

For every closure operator $C \in \mathbb{CO}$ we have the maximal closure operator C^{r_C} associated to C, as well as the associated minimal closure operator C_{r_C}. The previous facts in other form can be expressed as follows. In the class $\mathbb{CO}$ we define the binary relation by the rule:

$$C \sim D \Leftrightarrow \Phi(C) = \Phi(D) \quad (\text{i.e. } r_C = r_D).$$

Then we obtain an equivalence in $\mathbb{CO}$ such that every closure operator $C \in \mathbb{CO}$ defines the equivalence class $[C_r, C^r]$. If we denote by $\mathbb{CO} / \sim$ the family of equivalence classes of $\mathbb{CO}$, then it is clear that $\mathbb{PR} \cong \mathbb{CO} / \sim$.

The following statements in continuation will serve as starting point of our investigation [7–9].

Proposition 1.3. *The mappings (Φ, Ψ_1) define a monotone bijection between the **maximal** closure operators of $\mathbb{CO}$ and preradicals of R-Mod:* $\mathbb{M}ax(\mathbb{CO}) \cong \mathbb{PR}$. □

Proposition 1.4. *The mappings (Φ, Ψ_2) define a monotone bijection between the **minimal** closure operators of $\mathbb{CO}$ and preradicals of R-Mod:* $\mathbb{M}in(\mathbb{CO}) \cong \mathbb{PR}$. □

2 The mappings (Φ, Ψ_1) and their effects

The restriction of the bijection of Proposition 1.3, defined by the mappings (Φ, Ψ_1), leads to some new monotone bijections which connect the closure operators of special types of $\mathbb{M}ax(\mathbb{CO})$ with the preradicals of $\mathbb{PR}$ which possess the respective properties.

We begin with a preliminary statement which shows the relations between some properties of the closure operators of R-Mod ([2], Lemmas 5.2 and 6.2).

Lemma 2.1. *Every minimal closure operator is idempotent. The closure operator $C \in \mathbb{CO}$ is cohereditary if and only if it is maximal and minimal.* □

Remark. If the closure operator $C \in \mathbb{CO}$ is cohereditary, then for the respective preradical $r = r_C$ we have $C^r = C_r$, therefore the corresponding equivalence class of $\mathbb{CO}$ consists of only one element. Obviously, the condition $C^r = C_r$ means that this closure operator is cohereditary.

In continuation we consider the monotone bijection of Proposition 1.3, defined by Φ and Ψ_1, analyzing its effect on some important classes of closure operators and of preradicals.

Proposition 2.2. *If the closure operator $C \in \mathbb{CO}$ is* ***weakly hereditary****, then the preradical $\Phi(C) = r_C$ is idempotent. If the preradical $r \in \mathbb{PR}$ is idempotent, then the associated maximal closure operator C^r is weakly hereditary. Therefore the mappings (Φ, Ψ_1) define a monotone bijection between the maximal weakly hereditary closure operators of $\mathbb{CO}$ and idempotent preradicals of R-Mod.*

Proof. Let $C \in \mathbb{CO}$ be a weakly hereditary closure operator. Then $C_{C_M(O)}(O) = C_M(O)$ for every module $M \in R$-Mod, therefore

$$r_C\big(r_C(M)\big) = r_C\big(C_M(O)\big) = C_{C_M(O)}(O) = C_M(O) = r_C(M),$$

i.e. r_C is an idempotent preradical.

Conversely, let $r \in \mathbb{PR}$ be an idempotent preradical. Then the associated maximal closure operator $\Psi_1(r) = C^r$, defined by the rule $[(C^r)_M(N)]/N = r(M/N)$ for every $N \subseteq M$, possesses the property:

$$[(C^r)_{(C^r)_M(N)}(N)]\,/\,N = r\big[\big((C^r)_M(N)\big)/\,N\big] =$$

$$= r\big(r(M/N)\big) = r(M/N) = [(C^r)_M(N)]\,/\,N.$$

Therefore $(C^r)_{(C^r)_M(N)}(N) = (C^r)_M(N)$ for every $N \subseteq M$, which means that the operator C^r is weakly hereditary. The last statement now follows from Proposition 1.3. □

Corollary 2.3. *If the closure operator $C \in \mathbb{CO}$ is weakly hereditary, then the associated maximal closure operator C^{r_C} also is weakly hereditary.*

With the intention to study the closure operators of $\mathbb{CO}$ which correspond to *radicals* of R-Mod, we introduce the following notion.

Definition 2.1. The closure operator $C \in \mathbb{CO}$ will be called **zero-idempotent** if $C_M\big(C_M(O)\big) = C_M(O)$ for every $M \in R$-Mod.

Proposition 2.4. *If r is a* ***radical*** *of $\mathbb{PR}$, then the associated maximal closure operator C^r is zero-idempotent. If the operator $C \in \mathbb{CO}$ is maximal and zero-idempotent, then the corresponding preradical $\Phi(C) = r_{_C}$ is a radical. Therefore the mappings (Φ, Ψ_1) define a monotone bijection between the radicals of R-Mod and the maximal zero-idempotent closure operators of $\mathbb{CO}$.*

Proof. Let r be a radical of $\mathbb{PR}$ and C^r be the corresponding maximal closure operator, i.e. $[(C^r)_M(N)]/N = r(M/N)$ for every $N \subseteq M$. If $N = r(M)$, then $\big[(C^r)_M\big(r(M)\big)\big]/r(M) = r\big(M/r(M)\big) = \bar{0}$, since r is a radical. But $r(M) = (C^r)_M(O)$, and so

$$\big[(C^r)_M\big((C^r)_M(O)\big)\big] \;/\; [(C^r)_M(O)] = \bar{0},$$

therefore $(C^r)_M\big((C^r)_M(O)\big) = (C^r)_M(O)$, which means that C^r is a zero-idempotent closure operator.

Let now C be an arbitrary maximal zero-idempotent closure operator of $\mathbb{CO}$. Then $C_M\big(C_M(O)\big) = C_M(O)$ for every $M \in R$-Mod, i.e. $C_M\big(r_{_C}(M)\big) = r_{_C}(M)$ and $C_M\big(r_{_C}(M)\big)/\,r_{_C}(M) = \bar{0}$.

From the other hand, since C is maximal, by definition $\big[C_M\big(r_{_C}(M)\big)\big]/r_{_C}(M) = C_{M/r_{_C}(M)}(\bar{0})$. Therefore $C_{M/r_{_C}(M)}(\bar{0}) = \bar{0}$, i.e. $r_{_C}\big(M/\,r_{_C}(M)\big) = \bar{0}$, which means that $r_{_C}$ is a radical.

The proof is finished by the application of Proposition 1.3. $\square$

Corollary 2.5. *If the operator $C \in \mathbb{CO}$ is maximal and idempotent, then $\Phi(C) = r_{_C}$ is a radical of R-Mod.*

Combining Propositions 2.2 and 2.4 we obtain

Corollary 2.6. *The mappings (Φ, Ψ_1) define a monotone bijection between the* ***idempotent radicals*** *of R-Mod and the maximal, weakly hereditary, zero-idempotent closure operators of $\mathbb{CO}$.*

Now we will show the closure operators of $\mathbb{CO}$ which correspond to hereditary preradicals (pretorsions) of R-Mod.

Proposition 2.7. *If the closure operator $C \in \mathbb{CO}$ is* ***hereditary****, then the preradical $\Phi(C) = r_{_C}$ is hereditary. If $r \in \mathbb{PR}$ is a hereditary preradical (pretorsion) of R-Mod, then the associated maximal closure operator $\Psi_1(r) = C^r$ is hereditary. Therefore the mappings (Φ, Ψ_1) define a monotone bijection between the pretorsions of R-Mod and the maximal hereditary closure operators of $\mathbb{CO}$.*

Proof. Let $C \in \mathbb{CO}$ be a hereditary closure operator. Then in the sitution $L \subseteq N \subseteq M$ we have $C_N(L) = C_M(L) \cap N$. For $L = O$ we obtain $C_N(O) = C_M(O) \cap N$, i.e. $r_{_C}(N) = r_{_C}(M) \cap N$, which means that the preradical $r_{_C}$ is hereditary.

Conversely, let r be a hereditary preradical of R-Mod, i.e. $r(N) = r(M) \cap N$ for every $N \subseteq M$. In the situation $L \subseteq N \subseteq M$ by heredity of r we have:

$$\big[\big((C^r)_M(L)\big) \cap N\big] / L = \big[\big((C^r)_M(L)\big) / L\big] \cap (N/L) =$$
$$= [r(M/L)] \cap (N/L) = r(N/L) = [(C^r)_N(L)] / L.$$

Therefore $[(C^r)_M(L)] \cap N = (C^r)_N(L)$, i. e. the operator C^r is hereditary. $\square$

Corollary 2.8. *If the closure operator $C \in \mathbb{CO}$ is hereditary, then the associated maximal closure operator C^{r_C} also is hereditary.*

Taking into account the description of hereditary preradicals by the preradical filters of R (Proposition 1.1, 1)), from Proposition 2.7 follows

Corollary 2.9. *There exists a bijection between the preradical filters of the ring R and the maximal hereditary closure operators of $\mathbb{CO}$.*

More concretely, if $\mathcal{E}$ is a preradical filter of R (see [6]), then it defines a pretorsion $r_{\mathcal{E}}$ in R-Mod by the rule:

$$r_{\mathcal{E}}(M) = \{m \in M \mid (0 : m) \in \mathcal{E}\},$$

for every $M \in R$-Mod. The corresponding maximal closure operator $C^{r_{\mathcal{E}}}$ is defined as

$$(C^{r_{\mathcal{E}}})_M(N) = \{m \in M \mid (N : m) \in \mathcal{E}\}$$

for every $N \subseteq M$, where $(N : m) = \{a \in R \mid am \in N\}$.

From the other hand, if C is a maximal hereditary closure operator of $\mathbb{CO}$, then the associated preradical filter is $\mathcal{F}_1({}_RR) = \{I \in \mathbb{L}({}_RR) \mid C_R(I) = R\}$, i. e. the set of C-*dense* left ideals of R.

A very important type of preradicals of R-Mod are the *torsions* of this category and now we will indicate the closure operators of $\mathbb{CO}$ which correspond to the torsions of R-Mod. Since the torsions are hereditary radicals, the result follows by combining Propositions 2.4 and 2.7.

Corollary 2.10. *The mappings (Φ, Ψ_1) define a monotone bijection between the **torsions** of R-Mod and maximal, zero-idempotent, hereditary closure operators of $\mathbb{CO}$.*

The torsions of R-Mod are described by the radical filters of the ring R (Proposition 1.1, 2)), therefore is true

Corollary 2.11. *There exists a bijection between the **radical filters** of the ring R and the maximal, zero-idempotent, hereditary closure operators of $\mathbb{CO}$.*

The cases related to the cohereditary preradicals are considered in the following section, since for such preradicals the mappings Ψ_1 and Ψ_2 coincide.

3 The mappings (Φ, Ψ_2) and their effects

Now we will study the relations between the classes $\mathbb{CO}$ and $\mathbb{PR}$, defined by the mappings $\Phi : \mathbb{CO} \to \mathbb{PR}$ and $\Psi_2 : \mathbb{PR} \to \mathbb{CO}$, where $\Psi_2(r) = C_r$ and $(C_r)_M(N) = N + r(M)$. Every minimal closure operator is idempotent (Lemma 2.1). By Proposition 1.4 we have the monotone bijection $\mathbb{M}in(\mathbb{CO}) \cong \mathbb{PR}$ defined by (Φ, Ψ_2). We will restrict this bijection, considering various types of preradicals and showing properties of the corresponding closure operators.

We begin with the idempotent preradicals of R-Mod. Firstly we remind that by Proposition 2.2 if $C \in \mathbb{CO}$ is weakly hereditary, then the preradical $\Phi(C) = r_C$ is idempotent. Now we verify the inverse transition: from $r \in \mathbb{PR}$ to $\Psi_2(r) = C_r$.

Proposition 3.1. *Let $r \in \mathbb{PR}$ be an **idempotent preradical** of R-Mod. Then the associated minimal closure operator $\Psi_2(r) = C_r$ is weakly hereditary. Therefore the mappings (Φ, Ψ_2) define a monotone bijection between the idempotent preradicals of R-Mod and the minimal weakly hereditary closure operators of $\mathbb{CO}$.*

Proof. If a preradical $r \in \mathbb{PR}$ is idempotent and $N \subseteq M$, then $r\big(r(M)\big) = r(M)$ and by definitions we have:

$$(C_r)_{(C_r)_M(N)}(N) = N + r[(C_r)_M(N)] =$$

$$= N + r\big(N + r(M)\big) \supseteq N + r\big(r(M)\big) = N + r(M) = (C_r)_M(N).$$

Therefore $(C_r)_{(C_r)_M(N)}(N) \supseteq (C_r)_M(N)$, and the inverse inclusion follows from the monotony of C_r, since $(C_r)_M(N) \subseteq M$. So we have $(C_r)_{(C_r)_M(N)}(N) = (C_r)_M(N)$, i.e. the minimal closure operator C_r is weakly hereditary.

Taking into account the first statement of Proposition 2.2, from Proposition 1.4 now we obtain the indicated monotone bijection. ☐

Corollary 3.2. *If the closure operator $C \in \mathbb{CO}$ is weakly hereditary, then the associated minimal closure operator C_{r_C} also is weakly hereditary.*

We consider in continuation the *radicals* of R-Mod and look for the effect of the mapping Ψ_2 on the preradicals of such type. For that we need the following notion.

Definition 3.1. An operator $C \in \mathbb{CO}$ will be called **zero-radical** closure operator if $C_{M/C_M(O)}(\bar{0}) = \bar{0}$ for every $M \in R$-Mod.

Proposition 3.3. *If $r \in \mathbb{PR}$ is a **radical** of R-Mod, then the associated minimal closure operator $\Psi_2(r) = C_r$ is a zero-radical operator. If $C \in \mathbb{CO}$ is a zero-radical operator, then $\Phi(C) = r_C$ is a radical. Therefore the mappings (Φ, Ψ_2) define a monotone bijection between the radicals of R-Mod and the minimal, zero-radical closure operators of $\mathbb{CO}$.*

Proof. If $r \in \mathbb{PR}$ is a radical of R-Mod, then for every $M \in R$-Mod we have:

$$(C_r)_{M/[(C_r)_M(O)]}(\bar{0}) = (C_r)_{M/r(M)}(\bar{0}) = \bar{0} + r\big(M/r(M)\big) = \bar{0},$$

so C_r is a zero-radical closure operator.

Conversely, if $C \in \mathbb{CO}$ is a zero-radical closure operator, then by definition $r_C\big(M/\,r_C(M)\big) = C_{M/\,C_M(O)}(\bar{0}) = \bar{0}$ for every $M \in R$-Mod, i.e. r_C is a radical.

The conclusion of our statement now follows from Proposition 1.4. □

Remark. If $C \in \mathbb{CO}$ is a zero-radical closure operator, then each operator of the interval $[C_{r_C}, C^{r_C}]$ also is zero-radical, since the corresponding preradical coincides with r_C, which is a radical.

Combining Propositions 3.1 and 3.3 now we obtain

Corollary 3.4. *The mappings (Φ, Ψ_2) define a monotone bijection between the* ***idempotent radicals*** *of R-Mod and the minimal, weakly hereditary, zero-radical closure operators of $\mathbb{CO}$.*

The following step of our investigation is the consideration of the *hereditary preradicals* (*pretorsions*) of R-Mod. We remind that if an operator $C \in \mathbb{CO}$ is hereditary, then the preradical $\Phi(C) = r_C$ is hereditary (Proposition 2.7).

Proposition 3.5. *If a preradical $r \in \mathbb{PR}$ is* ***hereditary****, then the associated minimal closure operator $\Psi_2(r) = C_r$ is hereditary. Therefore the mappings (Φ, Ψ_2) define a monotone bijection between the hereditery preradicals (pretorsions) of R-Mod and the minimal hereditary closure operators of $\mathbb{CO}$.*

Proof. Let $r \in \mathbb{PR}$ be a hereditary preradical of R-Mod. Then in the situation $L \subseteq N \subseteq M$ by definition we have:

$$(C_r)_N(L) = L + (C_r)_N(O) = L + r(N), \quad (C_r)_M(L) = L + (C_r)_M(O) = L + r(M).$$

By the modularity of $\mathbb{L}(M)$ and the inclusion $L \subseteq N$, we obtain:

$$\big(L + r(M)\big) \cap N = L + \big(r(M) \cap N\big),$$

and by the heredity of r we have $r(M) \cap N = r(N)$. Therefore

$$[(C_r)_M(L)] \cap N = \big(L + r(M)\big) \cap N = L + \big(r(M) \cap N\big) = L + r(N) = (C_r)_N(L),$$

hence the closure operator C_r is hereditary.

The conclusion of our statement now follows from Propositions 2.7 and 1.4. □

Corollary 3.6. *If the closure operator $C \in \mathbb{CO}$ is hereditary, then the associated minimal closure operator C_{r_C} also is hereditary.*

Using Proposition 1.1, 1), now from Proposition 3.5 follows

Corollary 3.7. *There exist a bijection between the* ***preradical filters*** *of the ring R and the minimal hereditary closure operators of $\mathbb{CO}$.*

Similarly, from Proposition 1.1, 2), using Propositions 3.3 and 3.5, we obtain

Corollary 3.8. *There exists a bijection between the* ***radical filters*** *of the ring* R *(i.e. the* ***torsions*** *of* R*-Mod) and the minimal, zero-radical, hereditary closure operators of* $\mathbb{CO}$.

In continuation we consider the similar questions for the *coheriditary preradicals* of R-Mod. As was mentioned above, for such preradicals the mappings Ψ_1 and Ψ_2 coincide.

Proposition 3.9. *If* r *is a cohereditary preradical of* R*-Mod, then* $\Psi_1(r) = \Psi_2(r)$ *(i.e.* $C^r = C_r$*) and this closure operator is cohereditary. If* $C \in \mathbb{CO}$ *is a cohereditary closure operator (i.e. is maximal and minimal), then the preradical* $\Phi(C) = r_C$ *is cohereditary. Therefore the mappings* (Φ, Ψ_1) $\big($*or* $(\Phi, \Psi_2)\big)$ *define a monotone bijection between the cohereditary preradicals of* R*-Mod and the cohereditary closure operators of* $\mathbb{CO}$.

Proof. If a preradical $r \in \mathbb{PR}$ is cohereditary, then by the definition of C^r we have:

$$[(C^r)_M(N)]\,/\,N = r(M/\,N) = \big(r(M) + N\big)/\,N,$$

hence $(C^r)_M(N) = r(M) + N = (C_r)_M(N)$ for every $N \subseteq M$, and so $C^r = C_r$. Since this closure operator is maximal and minimal, it is cohereditary (Lemma 2.1).

Conversely, if $C \in \mathbb{CO}$ is a cohereditary closure operator, then by the maximality of C we have $r_C(M/N) = C_{M/N}(\bar{0}) = C_M(N)/\,N$. Further, from the minimality of C it follows that $C_M(O) + N = C_M(N)$, therefore

$$\big(r_C(M) + N\big)/\,N = \big(C_M(O) + N\big)/\,N = C_M(N)/\,N$$

for every $N \subseteq M$. From the foregoing now follows that $r_C(M/N) = \big(r_C(M) {+} N\big)/\,N$, i.e. the preradical r_C is cohereditary.

Applying Proposition 1.3 (or 1.4) now we obtain the announced bijection. □

Using Proposition 1.1, 3), we have

Corollary 3.10. *There exists a bijection between the* ***ideals*** *of the ring* R *and the cohereditary closure operators of* $\mathbb{CO}$.

The case of *cotorsions* of R-Mod is reduced to the combination of Proposition 3.9 with Proposition 2.2 (or 3.1), which give

Corollary 3.11. *The mappings* (Φ, Ψ_1) $\big($*or* $(\Phi, \Psi_2)\big)$ *define a monotone bijection between the* ***cotorsions*** *of* R*-Mod and weakly hereditary, cohereditary closure operators of* $\mathbb{CO}$.

The description of cotorsions of R-Mod by idempotent ideals of R $\big($Proposition 1.1, 4)$\big)$ now implies

Corollary 3.12. *There exists a bijection between the* ***idempotent ideals*** *of the ring* R *and the weakly hereditary, cohereditary closure operators of* $\mathbb{CO}$.

Finally, we consider the case of *hereditary and cohereditary preradicals* of R-Mod (see Propositions 2.7 or 3.5, and 3.9).

Corollary 3.13. *The mappings* (Φ, Ψ_1) $\big($*or* $(\Phi, \Psi_2)\big)$ *define a monotone bijection between the hereditary and cohereditary preradicals of R-Mod and the hereditary, cohereditary closure operators of* $\mathbb{CO}$.

From Proposition 1.1, 5) now follows

Corollary 3.14. *There exists a bijection between the* ***still ideals*** *of the ring R and the hereditary, cohereditary closure operators of* $\mathbb{CO}$.

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