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A semi-isometric isomorphism on a ring of matrices

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Abstract. Let (R, ξ) be a pseudonormed ring and R_n be a ring of matrices over the ring R. We prove that if $1 \leq \gamma, \sigma \leq \infty$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$, then the function $\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n. Let now $\varphi : (R, \xi) \to (\bar{R}, \overline{\xi})$ be a semi-isometric isomorphism of pseudonormed rings. We prove that $\Phi : (R_n, \eta_{\xi,\gamma,\sigma}) \to (\bar{R}_n, \eta_{\bar{\xi},\gamma,\sigma})$ is a semi-isometric isomorphism too for all $1 \leq \gamma, \sigma \leq \infty$ such that $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$.

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The following theorem on isomorphism is often applied in algebra and, in particular, in the ring theory:

Theorem 1. *If A is a subring of a ring R and I is an ideal of the ring R, then the quotient rings $A/(A \bigcap I)$ and $(A+I)/I$ are isomorphic rings.*

In particular, if $A \bigcap I = 0$, then the ring A is isomorphic to the ring $(A+I)/I$, i.e. the rings A and $(A+I)/I$ possess identical algebraic properties.

Since it is necessary to take into account properties of pseudonorms when studying the pseudonormed rings then one needs to consider isomorphisms which keep pseudonorms. Such isomorphisms are called isometric isomorphisms.

Theorem 1 does not always take place for pseudonormed rings. As is shown in Theorem 2.1 from [1] it is impossible to tell anything more than the validity of the inequality $\bar{\xi}(\varphi(r)) \leq \xi(r)$ in case $A \bigcap I = 0$.

The case when A is an ideal of a pseudonormed ring (R, ξ) was studied in [1], the case when A is a one-sided ideal of a pseudonormed ring (R, ξ) was studied in [2].

The following definition was introduced in [1]:

Definition 1. Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings. An isomorphism $\varphi : R \to \bar{R}$ is called a semi-isometric isomorphism if there exists a pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$ such that the following conditions are valid:

1) the ring R is an ideal in the ring $\hat{R}$;

2) $\hat{\xi}(r) = \xi(r)$ for any $r \in R$;

3) the isomorphism φ can be extended up to an isometric homomorphism $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$ of the pseudonormed rings, i.e. $\bar{\xi}(\hat{\varphi}(\hat{r})) = \inf\left\{\hat{\xi}(\hat{r}+a) \,|\, a \in \ker\hat{\varphi}\right\}$ for all $\hat{r} \in \hat{R}$.

The following theorem was proved in [1]:

Theorem 2. *Let* (R,ξ) *and* $\left(\bar{R},\bar{\xi}\right)$ *be pseudonormed rings and* $\varphi : R \to \bar{R}$ *be a ring isomorphism. Then the isomorphism* $\varphi : (R,\xi) \to \left(\bar{R},\bar{\xi}\right)$ *is a semi-isometric isomorphism of the pseudonormed rings iff the inequalities* $\xi(a\cdot b) \le \bar{\xi}(\varphi(a))\cdot\xi(b)$, $\xi(b\cdot a) \le \bar{\xi}(\varphi(a))\cdot\xi(b)$ *and* $\bar{\xi}(\varphi(a)) \le \xi(a)$ *are true for any* $a,b \in R$.

This paper is a continuation of [1] and [2] and it is devoted to the study of pseudonorms on a ring of matrices which keep a semi-isometric isomorphism.

We will use the following propositions. The proof of Propositions 1 – 3 can be found in [3]; the proof of Propositions 4, 5 can be found in [4].

Proposition 1. *Let* λ *and* λ^* *be positive real numbers such that* $\lambda > 1$, $\lambda^* > 1$ *and* $\frac{1}{\lambda} + \frac{1}{\lambda^*} = 1$. *Then the inequality*

$$\sum_{k=1}^{n} a_k b_k \le \left(\sum_{k=1}^{n} a_k^{\lambda}\right)^{\frac{1}{\lambda}} \cdot \left(\sum_{k=1}^{n} b_k^{\lambda^*}\right)^{\frac{1}{\lambda^*}}$$

is true for all $a_k \ge 0$ *and* $b_k \ge 0$.

Proposition 2. *Let* $a_{ik} \ge 0$ *for* $1 \le i \le m$, $1 \le k \le n$. *Then the inequality*

$$\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{m} a_{ik}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \le \sum_{i=1}^{m}\left(\sum_{k=1}^{n} a_{ik}^{\lambda}\right)^{\frac{1}{\lambda}}$$

is true for any $\lambda > 1$.

Proposition 3. *Let* $a_k \ge 0$ *for all* $1 \le k \le n$ *and* $G_\lambda(a_1, a_2, \ldots, a_n)$ *be a real function such that*

$$G_\lambda(a_1, a_2, \ldots, a_n) = \left(\sum_{k=1}^{n} a_k^{\lambda}\right)^{\frac{1}{\lambda}} \text{ for } 1 \le \lambda < \infty,$$

$$G_\infty(a_1, a_2, \ldots, a_n) = \max_{1\le k\le n} a_k \text{ for } \lambda = \infty.$$

Then the family of functions $\{G_\lambda | 1 \le \lambda \le \infty\}$ *has the following properties:*

1) *if* $\lambda_1 \le \lambda_2$, *then* $G_{\lambda_1}(a_1, a_2, \ldots, a_n) \ge G_{\lambda_2}(a_1, a_2, \ldots, a_n)$ *for all* $a_k \ge 0$;
2) $\lim\limits_{\lambda\to+\infty} G_\lambda(a_1, a_2, \ldots, a_n) = G_\infty(a_1, a_2, \ldots, a_n)$ *for all* $a_k \ge 0$;
3) $\sup\limits_{\lambda>1} G_\lambda(a_1, a_2, \ldots, a_n) = G_1(a_1, a_2, \ldots, a_n)$ *for all* $a_k \ge 0$.

Definition 2. A direction is a partially ordered set $(\Gamma, \le)$ that satisfies the following condition: for any two elements $\gamma_1, \gamma_2 \in \Gamma$ there exists the third element $\gamma_3 \in \Gamma$ such that $\gamma_1 \le \gamma_3$ and $\gamma_2 \le \gamma_3$.

Proposition 4. *Let Γ be some set and R be a ring. If $\{\xi_\gamma \,|\gamma \in \Gamma\}$ is a family of pseudonorms on the ring R, then the following statements are valid:*

1. If Γ is a direction and for every $r \in R$ there exists $\lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$ such that $\lim\limits_{\gamma\in\Gamma} \xi_\gamma(r) \neq 0$ for every $r \neq 0$, then the function $\xi(r) = \lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$ is a pseudonorm on the ring R;

2. If the set $\{\xi_\gamma(r)\,|\gamma \in \Gamma\}$ is bounded from above for every $r \in R$, then the function $\xi(r) = \sup\limits_{\gamma\in\Gamma} \xi_\gamma(r)$ is a pseudonorm on the ring R.

Proposition 5. *Let R and $\bar{R}$ be rings and let $\varphi : R \to \bar{R}$ be a ring isomorphism. If $\{\xi_\gamma \,|\gamma \in \Gamma\}$ and $\left\{\bar{\xi}_\gamma \,|\gamma \in \Gamma\right\}$ are families of pseudonorms such that $\varphi : (R, \xi_\gamma) \to \left(\bar{R}, \bar{\xi}_\gamma\right)$ is a semi-isometric isomorphism for any $\gamma \in \Gamma$, then the following statements are true:*

1. If Γ is a direction and there exist $\lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$, $\lim\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r})$ for every $r \in R$, $\bar{r} \in \bar{R}$ such that $\lim\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r}) \neq 0$ for every $\bar{r} \neq 0$ and $\xi(r) = \lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$, $\bar{\xi}(\bar{r}) = \lim\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r})$ for every $r \in R$, $\bar{r} \in \bar{R}$, then $\varphi : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism;

2. If the set $\{\xi_\gamma(r)\,|\gamma \in \Gamma\}$ is bounded from above for every $r \in R$ and $\xi(r) = \sup\limits_{\gamma\in\Gamma} \xi_\gamma(r)$, $\bar{\xi}(\bar{r}) = \sup\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r})$ for every $r \in R$, $\bar{r} \in \bar{R}$, then $\varphi : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism.

We will consider a ring of matrices R_n over a pseudonormed ring (R, ξ) and a family of functions $\{\eta_{\xi,\gamma,\sigma} | 1 \leq \gamma, \sigma \leq \infty\}$ on R_n such that

$$\eta_{\xi,\gamma,\sigma}(A) = \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(\xi\left(a_{ij}\right)\right)^\sigma\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}},$$

$$\eta_{\xi,\gamma,\infty}(A) = \left(\sum_{i=1}^{n} \left(\max_{1\leq j\leq n} \xi\left(a_{ij}\right)\right)^\gamma\right)^{\frac{1}{\gamma}},$$

$$\eta_{\xi,\infty,\sigma}(A) = \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} \left(\xi\left(a_{ij}\right)\right)^\sigma\right)^{\frac{1}{\sigma}},$$

$$\eta_{\xi,\infty,\infty}(A) = \max_{1\leq i,j\leq n} \xi\left(a_{ij}\right)$$

for any $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in R_n$.

The following theorem gives conditions for γ and σ such that the functions $\eta_{\xi,\gamma,\sigma}$ define pseudonorms on the ring R_n.

Theorem 3. *Let (R, ξ) be a pseudonormed ring and let R_n be a ring of matrices over the ring R with the natural operations of addition and multiplication. Then the function $\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n for all γ and σ such that $1 \leq \gamma, \sigma \leq \infty$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$.*

Proof. Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$ and $\eta = \eta_{\xi,\gamma,\sigma}$, $1 < \gamma, \sigma < \infty$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$.

It is obvious that:

1) $\eta(A) = 0 \Leftrightarrow A = 0$;

2) $\eta(-A) = \eta(A)$ for any $A \in R_n$.

Since ξ is a pseudonorm on the ring R then $\xi(a_{ij} + b_{ij}) \leq \xi(a_{ij}) + \xi(b_{ij})$ for all $1 \leq i, j \leq n$, and hence

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} (\xi(a_{ij} + b_{ij}))^{\sigma} \right)^{\frac{\gamma}{\sigma}} \right)^{\frac{1}{\gamma}} \leq \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} (\xi(a_{ij}) + \xi(b_{ij}))^{\sigma} \right)^{\frac{\gamma}{\sigma}} \right)^{\frac{1}{\gamma}}.$$

It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n} (\xi(a_{ij}) + \xi(b_{ij}))^{\sigma} \right)^{\frac{1}{\sigma}} \leq \left(\sum_{j=1}^{n} (\xi(a_{ij}))^{\sigma} \right)^{\frac{1}{\sigma}} + \left(\sum_{j=1}^{n} (\xi(b_{ij}))^{\sigma} \right)^{\frac{1}{\sigma}}.$$

Then

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} (\xi(a_{ij}) + \xi(b_{ij}))^{\sigma} \right)^{\frac{\gamma}{\sigma}} \right)^{\frac{1}{\gamma}} \leq$$

$$\left(\sum_{i=1}^{n} \left(\left(\sum_{j=1}^{n} (\xi(a_{ij}))^{\sigma} \right)^{\frac{1}{\sigma}} + \left(\sum_{j=1}^{n} (\xi(b_{ij}))^{\sigma} \right)^{\frac{1}{\sigma}} \right)^{\gamma} \right)^{\frac{1}{\gamma}}.$$

Using Proposition 2 we obtain

$$\left(\sum_{i=1}^{n} \left(\left(\sum_{j=1}^{n} (\xi(a_{ij}))^{\sigma} \right)^{\frac{1}{\sigma}} + \left(\sum_{j=1}^{n} (\xi(b_{ij}))^{\sigma} \right)^{\frac{1}{\sigma}} \right)^{\gamma} \right)^{\frac{1}{\gamma}} \leq$$

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} (\xi(a_{ij}))^{\sigma} \right)^{\frac{\gamma}{\sigma}} \right)^{\frac{1}{\gamma}} + \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} (\xi(b_{ij}))^{\sigma} \right)^{\frac{\gamma}{\sigma}} \right)^{\frac{1}{\gamma}}.$$

Therefore, $\eta(A + B) \leq \eta(A) + \eta(B)$ for any $A, B \in R_n$.

Verify the inequality $\eta(A \cdot B) \leq \eta(A) \cdot \eta(B)$ for any $A, B \in R_n$. We consider

$$\eta(A\cdot B)=\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

Since ξ is a pseudonorm then

$$\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(a_{ik}\cdot b_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right).$$

Hence

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

Let σ^* be a positive real number such that $\frac{1}{\sigma}+\frac{1}{\sigma^*}=1$. It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}=$$

$$\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right).$$

Using Proposition 1 we obtain

$$\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right)\leq$$

$$\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Then

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}$$

and

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

We have the inequality

$$\eta(A\cdot B)=\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot$$

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}=\eta(A)\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Since $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$ and $\frac{1}{\sigma}+\frac{1}{\sigma^*}=1$ then $\frac{1}{\gamma}\geq\frac{1}{\sigma^*}$ and so $\gamma\leq\sigma^*$. Hence it follows from Proposition 3 that

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

Therefore $\eta(A\cdot B)\leq\eta(A)\cdot\eta(B)$ for any $A,B\in R_n$.

Thus, the function $\eta=\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n for all γ and σ such that $1<\gamma,\sigma<\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$.

It follows from Proposition 3 that

$$\eta_{\xi,1,\sigma}=\sup_{1<\gamma\leq\frac{\sigma}{\sigma-1}}\eta_{\xi,\gamma,\sigma}=\sup_{\gamma>1}\eta_{\xi,\gamma,\sigma},$$

$$\eta_{\xi,\gamma,1}=\sup_{1<\sigma\leq\frac{\gamma}{\gamma-1}}\eta_{\xi,\gamma,\sigma}=\sup_{\sigma>1}\eta_{\xi,\gamma,\sigma},$$

$$\eta_{\xi,1,1}=\sup_{\gamma>1}\eta_{\xi,\gamma,1}=\sup_{\sigma>1}\eta_{\xi,1,\sigma},$$

$$\eta_{\xi,1,\infty}=\lim_{\sigma\to+\infty}\eta_{\xi,1,\sigma},$$

$$\eta_{\xi,\infty,1}=\lim_{\gamma\to+\infty}\eta_{\xi,\gamma,1}.$$

Therefore by Proposition 4 the functions $\eta_{\xi,1,\sigma}$, $\eta_{\xi,\gamma,1}$, $\eta_{\xi,1,1}$, $\eta_{\xi,1,\infty}$, $\eta_{\xi,\infty,1}$ are pseudonorms on the ring R_n too for all $1<\gamma,\sigma<\infty$.

Thus, the function $\eta=\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n for any γ and σ such that $1\leq\gamma,\sigma\leq\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$. □

Remark 1. The conditions $\gamma,\sigma\geq 1$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$ are essential. Consider examples which show that if these conditions are not satisfied, then the function $\eta_{\xi,p,q}$ is not a pseudonorm on the ring R_n .

Let R be the ring of real numbers and $\xi(r)=|r|$ be a norm on the ring R ; let R_2 be the ring of real matrices $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$, $\gamma,\sigma>0$ and $\eta=\eta_{\xi,\gamma,\sigma}$ be a pseudonorm on the ring R_2.

1. If $A = B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\frac{1}{\gamma} + \frac{1}{\sigma} < 1$, then

$$\eta(A \cdot B) = 2^{\frac{1}{\gamma}+\frac{1}{\sigma}+1} > 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} \cdot 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} = \eta(A) \cdot \eta(B).$$

2. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \infty$, $\sigma > 1$, then

$$\eta(A \cdot B) = 2^{\frac{1}{\sigma}+1} > 2^{\frac{1}{\sigma}} \cdot 2^{\frac{1}{\sigma}} = \eta(A) \cdot \eta(B).$$

3. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\sigma = \infty$, $\gamma > 1$, then

$$\eta(A \cdot B) = 2^{\frac{1}{\gamma}+1} > 2^{\frac{1}{\gamma}} \cdot 2^{\frac{1}{\gamma}} = \eta(A) \cdot \eta(B).$$

4. If $A = B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\gamma = \sigma = \infty$, then

$$\eta(B \cdot A) = 2 > 1 \cdot 1 = \eta(B) \cdot \eta(A).$$

5. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\sigma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} > 2^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}} = \eta(A) + \eta(B).$$

6. If $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\gamma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} > 2^{\frac{1}{\sigma}} + 2^{\frac{1}{\sigma}} = \eta(A) + \eta(B).$$

7. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\gamma = \infty$, $\sigma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\sigma}} > 1 + 1 = \eta(A) + \eta(B).$$

8. If $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma = \infty$, $\gamma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\gamma}} > 1 + 1 = \eta(A) + \eta(B).$$

So $\eta_{\xi,\gamma,\sigma}$ is not a pseudonorm on the ring R_2 if the conditions $\gamma, \sigma \geq 1$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$ are violated.

Theorem 4. *Let* (R, ξ), $(\bar{R}, \bar{\xi})$ *be pseudonormed rings,* $\varphi : (R, \xi) \to (\bar{R}, \overline{\xi})$ *be a semi-isometric isomorphism,* R_n *and* $\bar{R}_n$ *be rings of matrices over the rings* R *and* $\bar{R}$ *with the pseudonorms* $\eta_{\xi,\gamma,\sigma}$ *and* $\eta_{\bar{\xi},\gamma,\sigma}$*, respectively, where* $1 \leq \gamma, \sigma \leq \infty$ *and* $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$ *. Then the mapping* $\Phi : (R_n, \eta_{\xi,\gamma,\sigma}) \to (\bar{R}_n, \eta_{\bar{\xi},\gamma,\sigma})$ *given by*

$$\Phi\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \varphi(a_{11}) & \dots & \varphi(a_{1n}) \\ \dots & \dots & \dots \\ \varphi(a_{n1}) & \dots & \varphi(a_{nn}) \end{pmatrix}$$

is a semi-isometric isomorphism too.

Proof. Let $1 < \gamma, \sigma < \infty$, $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$ and let $\eta = \eta_{\xi,\gamma,\sigma}$ and $\overline{\eta} = \eta_{\overline{\xi},\gamma,\sigma}$ be pseudonorms on the rings R_n and $\bar{R}_n$. We verify the conditions of Theorem 2 for the mapping $\Phi : (R_n, \eta) \to (\bar{R}_n, \overline{\eta})$.

Let us show that the inequality $\eta(A \cdot B) \leq \overline{\eta}(\Phi(A)) \cdot \eta(B)$ is valid for any

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \in R_n.$$

Since $\xi(a+b) \leq \xi(a) + \xi(b)$ and $\xi(a \cdot b) \leq \overline{\xi}(\varphi(a)) \cdot \xi(b)$ by Theorem 2 then

$$\xi\left(\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right) \leq \sum_{k=1}^{n} \xi\left(a_{ik} \cdot b_{kj}\right) \leq \sum_{k=1}^{n} \overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \xi\left(b_{kj}\right),$$

and hence

$$\eta(A \cdot B) = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}} \leq$$
$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n} \overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n} \overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} \leq \sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right).$$

Let σ^* be a positive real number such that $\frac{1}{\sigma} + \frac{1}{\sigma^*} = 1$. Using Proposition 1 we obtain the ineguality

$$\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right) \leq$$
$$\left(\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} \cdot \left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Then

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Since $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$ and $\frac{1}{\sigma}+\frac{1}{\sigma^*}=1$ then $\gamma\leq\sigma^*$. Hence it follows from Proposition 3 that

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}=\eta(B).$$

We have the inequality

$$\eta(A\cdot B)\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\overline{\eta}(\Phi(A))\cdot\eta(B).$$

Let us show that the inequality $\eta(B\cdot A)\leq\overline{\eta}(\Phi(A))\cdot\eta(B)$ is true for any $A,B\in R_n$. Since $\xi\left(\sum_{k=1}^{n}b_{ik}\cdot a_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(b_{ik}\cdot a_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)$, then

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}b_{ik}\cdot a_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}$$

. It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right).$$

Using Proposition 1 we have

$$\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right)\leq$$

$$\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Then

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq$$

$$\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}},$$

and hence

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Since $\gamma\leq\sigma^*$ then it follows from Proposition 3 that

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}=\overline{\eta}(\Phi(A)).$$

We obtain the inequality

$$\eta(B\cdot A)\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\eta\left(B\right)\cdot\overline{\eta}(\Phi(A)).$$

The inequality $\overline{\eta}(\Phi(A))\leq\eta(A)$ follows from the inequality $\overline{\xi}(\varphi(a_{ij}))\leq\xi(a_{ij})$.

All conditions of Theorem 2 are valid. Therefore the mapping $\Phi:(R_n,\eta_{\xi,\gamma,\sigma})\to(\bar{R}_n,\eta_{\bar{\xi},\gamma,\sigma})$ is a semi-isometric isomorphism when $1<\gamma,\sigma<\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$.

Since $\eta_{\xi,1,\sigma}=\sup\limits_{1<\gamma\leq\frac{\sigma}{\sigma-1}}\eta_{\xi,\gamma,\sigma}$, $\eta_{\xi,\gamma,1}=\sup\limits_{1<\sigma\leq\frac{\gamma}{\gamma-1}}\eta_{\xi,\gamma,\sigma}$, $\eta_{\xi,1,1}=\sup\limits_{\gamma>1}\eta_{\xi,\gamma,1}=\sup\limits_{\sigma>1}\eta_{\xi,1,\sigma}$, $\eta_{\xi,1,\infty}=\lim\limits_{\sigma\to+\infty}\eta_{\xi,1,\sigma}$ and $\eta_{\xi,\infty,1}=\lim\limits_{\gamma\to+\infty}\eta_{\xi,\gamma,1}$ then it follows from Proposition 5 that $\Phi:(R_n,\eta_{\xi,1,\sigma})\to(\bar{R}_n,\eta_{\bar{\xi},1,\sigma})$ for any $1<\sigma<\infty$, $\Phi:(R_n,\eta_{\xi,\gamma,1})\to(\bar{R}_n,\eta_{\bar{\xi},\gamma,1})$ for any $1<\gamma<\infty$, $\Phi:(R_n,\eta_{\xi,1,\infty})\to(\bar{R}_n,\eta_{\bar{\xi},1,\infty})$, $\Phi:(R_n,\eta_{\xi,\infty,1})\to(\bar{R}_n,\eta_{\bar{\xi},\infty,1})$ and $\Phi:(R_n,\eta_{\xi,1,1})\to(\bar{R}_n,\eta_{\bar{\xi},1,1})$ are semi-isometric isomorphisms too.

Thus the mapping $\Phi:(R_n,\eta_{\xi,\gamma,\sigma})\to(\bar{R}_n,\eta_{\bar{\xi},\gamma,\sigma})$ is a semi-isometric isomorphism for any γ and σ such that $1\leq\gamma,\sigma\leq\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$. $\square$

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