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On 2-primal Ore extensions over Noetherian Weak σ-rigid rings

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Abstract. Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. In this article, we discuss skew polynomial rings over 2-primal weak σ-rigid rings. We show that if R is a 2-primal Noetherian weak σ-rigid ring, then $R[x;\sigma,\delta]$ is a 2-primal Noetherian weak $\overline{\sigma}$-rigid ring.

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1 Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The fields of complex numbers and rational numbers are denoted by $\mathbb{C}$ and $\mathbb{Q}$ respectively. The set of prime ideals of R is denoted by $Spec(R)$. The set of minimal prime ideals of R is denoted by $Min.Spec(R)$. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$, respectively.

Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R, i. e. $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$. Recall that the skew polynomial ring $R[x;\sigma,\delta]$ is the set of polynomials

$$\{\sum_{i=0}^{n} x^i a_i \colon a_i \in R, n \in \mathbb{N}\}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i. e. $\sigma(I) = I$) and is also δ-invariant (i. e. $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$. We note that $O(I) = I(O(R))$. This article concerns the study of skew polynomial rings (Ore extensions) in terms of 2-primal rings.

2-Primal Rings

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i. e. if the prime radical is a completely semiprime. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$. We note that a reduced ring (a ring with no non zero nilpotent elements) is 2-primal and so is a commutative ring. Also let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then R is 2-primal.

2-Primal rings have been studied in recent years and are being treated by authors for different structures. In [10], Greg Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ an automorphism of R and δ a σ-derivation of R. In Greg Marks [10], it has been shown that for a local ring R with a nilpotent maximal ideal, the Ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $R[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition. Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [7].

$\sigma(*)$-rings

Let R be a ring and σ an endomorphism of R. Then σ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in R$, and R is said to be a σ-rigid ring (Krempa [8]).

For example let $R = \mathbb{C}$, and $\sigma : \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a+ib) = a - ib$, $a,\, b \in \mathbb{R}$. Then it can be seen that σ is a rigid endomorphism of R.

In Theorem 3.3 of [8], Krempa has proved the following:

Let R be a ring, σ *an endomorphism of* R and δ a σ-derivation of R. If σ is a monomorphism, then the skew polynomial ring $R[x;\sigma,\delta]$ is reduced if and only if R is reduced and σ is rigid. Under these conditions any minimal prime ideal (annihilator) of $R[x;\sigma;\delta]$ is of the form $P[x;\sigma;\delta]$ where P is a minimal prime ideal (annihilator) in R.

Definition 1 (see [9], Kwak)**.** Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

Remark 1. A $\sigma(*)$-ring need not be a σ-rigid. For let $0 \neq a \in F$ in above example (Example 1). Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Kwak in [9] establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew polynomial ring $R[x;\sigma]$. It has been proved in Theorem 5 of [9] that if R is a 2-primal ring and σ is an automorphism of R, then

R is a $\sigma(*)$-ring if and only if $\sigma(P) = P$ for all $P \in Min.Spec(R)$. In Theorem 12 of [9] it has been proved that if R is a $\sigma(*)$-ring with $\sigma(P(R)) = P(R)$, then $R[x;\sigma]$ is 2-primal if and only if $P(R)[x;\sigma] = P(R[x;\sigma])$.

2 Preliminaries

We have the following:

Proposition 1. *Let R be a Noetherian ring and σ an automorphism of R. If R is a $\sigma(*)$-ring, then R is 2-primal.*

Proof. Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) = a\sigma(a^2)\sigma^2(a) \in \sigma(P(R))$. Now R is Noetherian, so $\sigma(P(R)) = P(R)$. Therefore $a\sigma(a)\sigma(a\sigma(a)) \in P(R)$ which implies that $a\sigma(a) \in P(R)$ and so $a \in P(R)$. Hence R is 2-primal. $\square$

The following example shows that a 2-primal ring need not be a $\sigma(*)$-ring:

Let $R = F[x]$ be the polynomial ring over a field F. Then R is an integral domain and so is 2-primal with $P(R) = 0$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$ for $f(x) \in F[x]$. Let $f(x) = xa$, $a \in F$. Then $f(x)\sigma(f(x)) = 0 \in P(R)$, but $f(x) \notin P(R)$.

Weak σ-rigid rings:

Definition 2 (see Ouyang [12])**.** Let R be a ring and σ an endomorphism of R. Then R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.

Example 2 (see Example 2.1 of Ouyang [12])**.** Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \text{ a, b, c, d} \in R \right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. Then it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

Ouyang has proved in [12] that if σ is an endomorphism of a ring R, then R is σ-rigid if and only if R is weak σ-rigid and reduced.

Let R be a Noetherian ring and σ an automorphism of R. We now give a characterization for R to be a weak σ-rigid ring.

Theorem 1. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring if and only if $N(R)$ is a completely semiprime ideal of R.*

Proof. R is commutative implies that $N(R)$ is an ideal of R. We show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of R. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So

$$I = \sigma^{-1}(N(R)) = \{a \in R \text{ such that } \sigma(a) = n \in N(R)\}$$

is an ideal of R. Now I is nilpotent, so $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$.

Now let R be a weak σ-rigid ring. Let $a \in R$ be such that $a^2 \in N(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R).$$

Therefore, $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.

Conversely let $N(R)$ be completely semiprime. Let $a \in R$ be such that $a\sigma(a) \in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a)) \in N(R)$ implies that $a^2 \in N(R)$, and so $a \in N(R)$. Hence R is a weak σ-rigid ring. □

Completely prime ideals

Let R be a ring. Recall that an ideal $P \neq R$ is completely prime if R/P is a domain or equivalently if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$ (McCoy [11]). In commutative rings completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

We note that in a 2-primal ring R, for example a reduced ring, all minimal prime ideals are completely prime.

Regarding the relation between the completely prime ideals of a ring R and those of $O(R)$, the following result has been proved in Bhat [1]:

Theorem 2.4 of [1]. Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then:

1. For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P)$ is a completely prime ideal of $O(R)$.

2. For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.

The following result gives a characterization of a Notherian $\sigma(*)$-ring R, where σ is an automorphism of R.

Theorem 2 (see [2]). *Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. To make the paper self contained, we give a sketch of the proof.

Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. For each i, $a \in U_i$ or $\sigma(a) \in U_i$ and U_i is completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring and let $U = U_1$ be a minimal prime ideal of R. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now suppose that $U = U_1$ is not completely prime. Then there exist $a, b \in R \setminus U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. Now $c \in P(R)$ by Proposition 1 and, thus $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime. □

From now onwards, we deal with σ-derivation δ and its higher orders, therefore, the ring R is also taken as an algebra over $\mathbb{Q}$.

Proposition 2. *Let R be a Noetherian $\sigma(*)$-ring which is also an algebra over $\mathbb{Q}$ and δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$. Then $\delta(U) \subseteq U$ for all $U \in MinSpec(R)$.*

Proof. Let $U \in MinSpec(R)$. Then $\sigma(U) = U$ by Theorem 2. Consider the set

$$T = \{a \in U \mid \delta^k(a) \in U \text{ for all integers } k \geq 1\}.$$

First of all, we will show that T is an ideal of R. Let a, $b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1$. Therefore $a - b \in T$. Now let $a \in T$ and $r \in R$. We see that $\delta^k(ar) \in U$ and $\delta^k(ra) \in U$ for some $k \geq 1$ as both are sums of terms involving $\delta^j(a)$ for some $j \geq 1$. So T is a δ-invariant ideal of R.

We will now show that $T \in Spec(R)$. Suppose the contrary. Let $a \notin T$, $b \notin T$ be such that $aRb \subseteq T$. Let t, s be least positive integers such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that

$$\delta^t(a)c\sigma^t(\delta^s(b)) \notin U \tag{1}$$

as otherwise $\delta^t(a) \in U$ or $\delta^s(b) \in U$. Let $d = \sigma^{-t}(c)$. Now $aRb \subseteq T$ implies that $acb \subseteq T$. Therefore $\delta^{t+s}(adb) \in U$. This implies on simplification that

$$\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U \tag{2}$$

where u is a sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$, i.e.

$\delta^t(a)c\sigma^t(\delta^s(b)) \in U$. This is a contradiction to 1. Therefore $T \in Spec(R)$. Now $T \subseteq U$, so $T = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$. □

Remark 2. In above proposition the condition that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$ is necessary. For example if $s = t = 1$, then $a \in U$, $b \in U$ and therefore, $\sigma^i(a) \in U$, $\sigma^i(b) \in U$ for all integers $i \geq 1$ as $\sigma(U) = U$. Now $\delta^2(adb) \in U$ implies that

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) + u \in U.$$

where u is a sum of terms involving a or b, or $\sigma^i(b)$. Therefore by assumption $u \in U$. This implies that

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) \in U.$$

If $\delta(\sigma(a)) \neq \sigma(\delta(a))$, for all $a \in R$, then we get nothing out of it and if $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$, we get $\delta(a)\sigma(d)\sigma(\delta(b)) \in U$ which gives a contradiction.

We now give a relation between a $\sigma(*)$-ring and a weak σ-rigid ring:

Proposition 3. *Let R be a Noetherian ring and σ an automorphism of R. Then*

1. *R is a $\sigma(*)$-ring implies that R is a weak σ-rigid ring.*
2. *R is a 2-primal weak σ-rigid ring implies that R is a $\sigma(*)$-ring.*

Proof. **1**. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Now Proposition 1 implies that R is 2-primal, i.e. $N(R) = P(R)$. Thus $a\sigma(a) \in N(R) = P(R)$ implies that $a \in P(R) = N(R)$. Hence R is a weak σ-rigid ring.

2. Let R be 2-primal weak σ-rigid ring. Then $N(R) = P(R)$ and $a\sigma(a) \in N(R)$ implies that $a \in N(R)$. Therefore, $a\sigma(a) \in P(R)$ implies that $a \in P(R)$. Hence R is a $\sigma(*)$-ring. □

Corollary 1. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a 2-primal weak σ-rigid ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. Combine Theorem 2 and Proposition 3. □

3 Skew polynomial rings over 2-primal weak σ-rigid rings

Proposition 4. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R is a $\sigma(*)$-ring. Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. If $U \in Min.Spec(R)$, then $U(O(R)) = U[x;\sigma,\delta]$ is a completely prime ideal of $O(R) = R[x;\sigma,\delta]$.*

Proof. Let $U \in Min.Spec(R)$. Then $\sigma(U) = U$ by Theorem 2 and $\delta(U) \subseteq U$ by Proposition 2. Now R is 2-primal by Proposition 1 and furthermore U is completely prime by Theorem 2. Now consider canonical maps $\overline{\sigma}$ and $\overline{\delta}$ between R/U associated to σ and δ. It is well known that $O(R)/U(O(R)) \simeq (R/U)[x;\overline{\sigma},\overline{\delta}]$ and hence $U(O(R))$ is a completely prime ideal of $O(R)$. □

Theorem 3. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R is a $\sigma(*)$-ring. Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. If $P_1 \in Min.Spec(R)$, then $O(P_1) \in Min.Spec(O(R))$.*

Proof. Let $P_1 \in Min.Spec(R)$. Now by Theorem 2 $\sigma(P_1) = P_1$, and by Proposition 2 $\delta(P_1) \subseteq P_1$. Now Proposition 3.3 of [5] implies that $O(P_1) \in Spec(O(R))$. Suppose $O(P_1) \notin Min.Spec(O(R))$ and $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then

$$P_2 = O(P_2 \cap R) \subset O(P_1) \in Min.Spec(O(R)).$$

Therefore $P_2 \cap R \subset P_1$ which is a contradiction, as $P_2 \cap R \in Spec(R)$. Hence $O(P_1) \in Min.Spec(O(R))$. □

Theorem 4 (see [3])**.** *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R is a $\sigma(*)$-ring. Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $R[x;\sigma,\delta]$ is 2-primal if and only if $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.*

Proof. Let $R[x;\sigma,\delta]$ be 2-primal. Now Theorem 3 implies that $P(R[x;\sigma,\delta]) \subseteq P(R)[x;\sigma,\delta]$. Let

$$f(x) = \sum_{j=0}^{n} x^j a_j \in P(R)[x;\sigma,\delta].$$

Now R is a 2-primal subring of $R[x;\sigma,\delta]$ by Proposition 1, which implies that a_j is nilpotent and thus

$$a_j \in N(R[x;\sigma,\delta]) = P(R[x;\sigma,\delta]).$$

So we have $x^j a_j \in P(R[x;\sigma,\delta])$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(R[x;\sigma,\delta])$. Hence $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.

Conversely suppose that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$. We will show that $R[x;\sigma,\delta]$ is 2-primal. Let

$$g(x) = \sum_{i=0}^{n} x^i b_i \in R[x;\sigma,\delta],\ b_n \neq 0$$

be such that

$$(g(x))^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta].$$

We will show that $g(x) \in P(R[x;\sigma,\delta])$. Now the leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in Min.Spec(R)$. Also $\sigma(P) = P$ and P is completely prime by Theorem 3. Therefore we have $b_n \in P$, for all $P \in Min.Spec(R)$, i. e. $b_n \in P(R)$. Since $\delta(P) \subseteq P$ for all $P \in Min.Spec(R)$ by Proposition 2, we get

$$(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta]$$

and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $g(x) \in P(R)[x;\sigma,\delta]$, i.e. $g(x) \in P(R[x;\sigma,\delta])$. Therefore, $P(R[x;\sigma,\delta])$ is completely semiprime. Hence $R[x;\sigma,\delta]$ is 2-primal. □

Proposition 5. *Let R be a 2-primal Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R be a $\sigma(*)$-ring. Let δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $O(N(R)) = N(O(R))$.*

Proof. The proof is on the same lines as in Proposition 5 of [2]. We take R to be 2-primal in place of commutative.

It is easy to see that $O(N(R)) \subseteq N(O(R))$. We will show that $N(O(R)) \subseteq O(N(R))$. Let

$$f = \sum_{i=0}^{m} x^i a_i \in N(O(R)).$$

Then $(f)(O(R)) \subseteq N(O(R))$, and $(f)(R) \subseteq N(O(R))$. Let $((f)(R))^k = 0$, $k > 0$. Then equating the leading term to zero, we get

$$(x^m a_m R)^k = 0.$$

After simplification and equating the leading term to zero, we get

$$x^{km}\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)...a_m R = 0.$$

Therefore,

$$\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)...a_m R = 0 \subseteq P,$$

for all $P \in Min.Spec(R)$. This implies that $\sigma^{(k-j)m}(a_m R) \subseteq P$, for some j, $1 \leq j \leq k$. Therefore, $a_m R \subseteq \sigma^{-(k-j)m}(P)$. But $\sigma^{-(k-j)m}(P) = P$ by Theorem 2, so we have $a_m R \subseteq P$, for all $P \in Min.Spec(R)$. Therefore, $a_m \in P(R)$, and R being 2-primal implies that $a_m \in N(R)$. Now $x^m a_m \in O(N(R)) \subseteq N(O(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(O(R))$, and with the same process, in a finite number of steps, it can be seen that $a_i \in P(R) = N(R)$, $0 \leq i \leq m-1$. Therefore, $f \in O(N(R))$. Hence $N(O(R)) \subseteq O(N(R))$ and the result follows. □

Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma,\delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x;\sigma,\delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.

We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$ for all $a \in R$, then the above does not hold. For example let $f(x) = xa$ and $g(x) = xb$, $a, b \in R$. Then

$$\overline{\delta}(f(x)g(x)) = x^2\{\delta(\sigma(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\},$$

but

$$\overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x)) = x^2\{\sigma(\delta(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\}.$$

Theorem 5. *Let R be a 2-primal Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $O(R) = R[x;\sigma,\delta]$ is a 2-primal Noetherian weak $\overline{\sigma}$-rigid ring.*

Proof. $O(R)$ is Noetherian by the Hilbert Basis Theorem (see for example, Theorem 1.12 of Goodearl and Warfield [6]). Now R being 2-primal weak σ-rigid ring implies that R is a $\sigma(*)$-ring by Proposition 3. Now by Theorem 1.3 of [4] $P \in Min.Spec(O(R))$ implies that $P \cap R \in Min.Spec(R)$. Now use Theorem 3 to get that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$. Therefore, Theorem 4 implies that $O(R)$ is 2-primal. Also Theorem 7 of [2] implies that $O(R)$ is a weak $\overline{\sigma}$-rigid ring. Hence $O(R)$ is a 2-primal Noetherian weak $\overline{\sigma}$-rigid ring. □

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