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Irreducible triangulations of the Möbius band *

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Abstract. A complete list of irreducible triangulations is identified on the Möbius band.

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1 Introduction

Let $S \in \{S_g, N_k\}$ be the closed orientable surface S_g of genus g or the closed non-orientable surface N_k of non-orientable genus k. In particular, S_0 is the sphere and N_1 is the projective plane. Let D be an open disk in S and let $S - D$ denote S with D removed; therefore, the boundary $\partial(S - D)$ ($=\partial D$) is homeomorphic to a circle. In particular, $S_0 - D$ is the disk and $N_1 - D$ is the Möbius band. We use the notation Σ whenever we assume the general case: $\Sigma \in \{S, S - D\}$.

If a graph G is 2-cell embedded in Σ, the components of $\Sigma - G$ are called *faces*. A *triangulation* of Σ with a simple graph G (without loops or multiple edges) is a 2-cell embedding $T : G \to \Sigma$ in which each face is bounded by a 3-cycle (that is, a cycle of length 3) of G and any two faces are either disjoint, share a single vertex, or share a single edge. We denote by $V = V(T)$, $E = E(T)$, and $F = F(T)$ the sets of vertices, edges, and faces of T, respectively. The cardinality $|V(T)|$ is called the *order* of T. By $G(T)$ we denote the graph $(V(T), E(T))$ of triangulation T. Two triangulations T_1 and T_2 are called *isomorphic* if there is a bijection, called an *isomorphism*, $\varphi : V(T_1) \to V(T_2)$ such that $uvw \in F(T_1)$ if and only if $\varphi(u)\varphi(v)\varphi(w) \in F(T_2)$. Throughout this paper we distinguish triangulations only up to isomorphism. For $\Sigma = S - D$, let ∂T ($= \partial D$) denote the boundary cycle of T. The vertices and edges of ∂T are called *boundary vertices* and *boundary edges* of T.

A triangulation is called *irreducible* if no edge can be shrunk without producing multiple edges or changing the topological type of the underlying surface. The term "irreducible triangulation" is more accurately introduced in Section 2. The irreducible triangulations of Σ form a basis for the family of all triangulations of Σ, in the sense that any triangulation of Σ can be obtained from a member of

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the basis by repeatedly applying the *splitting* operation (introduced in Section 2) a finite number of times. Barnette and Edelson [2] and independently Negami [9] have proved that for every closed surface S the basis of irreducible triangulations is finite. At present such bases are known for seven closed surfaces: the sphere (Steinitz and Rademacher [10]), projective plane (Barnette [1]), torus (Lawrencenko [6]), Klein bottle (25 Lawrencenko and Negami's [8] triangulations plus 4 more irreducible triangulations found later by Sulanke [12]) as well as S_2, N_3, and N_4 (Sulanke [13, 14]). Boulch, Colin de Verdière, and Nakamoto [3] have established upper bounds on the order of an irreducible triangulation of $S - D$. In this paper we obtain a complete list of irreducible triangulations of $N_1 - D$.

2 Preliminaries

Let T be a triangulation of Σ. An unordered pair of distinct adjacent edges vu and vw of T is called a *corner* of T at vertex v, denoted by $\langle u, v, w\rangle$. The *splitting* of a corner $\langle u, v, w\rangle$, denoted by $\mathrm{sp}\langle u, v, w\rangle$, is the operation which consists in cutting T open along the edges vu and vw and then closing the resulting hole with two new triangular faces, $v'v''u$ and $v'v''w$, where v' and v'' denote the two images of v appearing as a result of cutting. Under this operation, vertex v is extended to the edge $v'v''$ and the two faces having this edge in common are inserted into the triangulation. Especially in the case $\{\Sigma = S - D \wedge uv \in E(T) \wedge v \in V(\partial T)\}$, the operation $\mathrm{sp}\langle u, v]$ of *splitting a truncated corner* $\langle u, v]$ produces a single triangular face $uv'v''$, where $v'v'' \in E(\partial(\mathrm{sp}\langle u, v](T)))$.

Under the inverse operation, *shrinking* the edge $v'v''$, denoted by $\mathrm{sh}\rangle v'v''\langle$, this edge collapses to a single vertex v, the faces $v'v''u$ and $v'v''w$ collapse to the edges vu and vw, respectively. Therefore $\mathrm{sh}\rangle v'v''\langle \circ\, \mathrm{sp}\langle u, v, w\rangle(T) = T$. It should be noticed that in the case $\{\Sigma = S - D \wedge v'v'' \in E(\partial T)\}$, there is only one face incident with $v'v''$, and only that single face collapses to an edge under $\mathrm{sh}\rangle v'v''\langle$. Clearly, the operation of splitting doesn't change the topological type of Σ. We demand that the shrinking operation must preserve the topological type of Σ as well; moreover, multiple edges must not be created in a triangulation. A 3-cycle of T is called *nonfacial* if it doesn't bound a face of T. In the case in which an edge $e \in E(T)$ occurs in some nonfacial 3-cycle, if we still insist on shrinking e, multiple edges would be produced, which would expel $\mathrm{sh}\rangle e\langle(T)$ from the class of triangulations. An edge e is called *shrinkable*, or a *cable* if $\mathrm{sh}\rangle e\langle(T)$ is still a triangulation of Σ; otherwise the edge is called *unshrinkable*, or a *rod*. The subgraph of $G(T)$ made up of all cables is called the *cable-subgraph* of $G(T)$.

The only impediment to edge shrinkability in a triangulation T of a closed surface S is identified in [1, 2, 6]: an edge $e \in E(T)$ is a rod if and only if e satisfies the following condition:

(2.1) e is in a nonfacial 3-cycle of $G(T)$.

The impediments to edge shrinkability in a triangulation T of a punctured surface $S - D$ are identified in [3]: an edge $e \in E(T)$ is a rod if and only if e satisfies either condition (2.1) or the following condition:

(2.2) e is a *chord* of D — that is, the end vertices of e are in $V(\partial D)$ but $e \notin E(\partial D)$.

A triangulation is said to be *irreducible* if it is free of cables or in other words, each edge is a rod. For instance, a single triangle is the only irreducible triangulation of the disk $S_0 - D$ although its edges don't meet either of conditions (2.1) and (2.2). Thus, we have yet one more impediment to edge shrinkability:

(2.3) e is a boundary edge in the case the boundary cycle is a 3-cycle.

Although condition (2.3) is a specific case of condition (2.1) (unless $S = S_0$) and is not explicitly stated in [3], it deserves especial mention.

3 The structure of irreducible punctured triangulations

In the remainder of this paper we assume that $S \neq S_0$. Let T be an irreducible triangulation of $S - D$. Let us restore the disk D in T, add a vertex p in D and join p to the vertices in ∂D. We thus obtain a triangulation, T^*, of the closed surface S. In this setting we call D the *patch*, call p the *central vertex of the patch*, and say that T is obtained from the corresponding triangulation T^* of S by the *patch removal*. Notice that T^* may turn out to be an irreducible triangulation of S, but not necessarily.

A vertex of a triangulation R of S is called a *pylonic vertex* if that vertex is incident with all cables of R. A triangulation that has at least one cable and at least one pylonic vertex is called a *pylonic triangulation*. It should be noticed that there exist triangulations of the torus with exactly one cable, and thereby with two different pylonic vertices; however, if a pylonic triangulation R has at least two cables, R has a unique pylonic vertex.

Lemma 1. *If T^* has at least two cables, then the central vertex p of the patch is the only pylonic vertex of T^*.*

Proof. Using the assumption that T is irreducible and the fact that each cable of T^* fails to satisfy condition (2.1), it can be easily seen that in the case T^* is not irreducible, all cables of T^* have to be entirely in $D \cup \partial D$ and, moreover, there is no cable that is entirely in ∂D. In particular, we observe that any chord of D is a rod in T because it meets condition (2.2), and is also a rod in T^* because it meets condition (2.1). ☐

4 Irreducible triangulations of the Möbius band

Barnette's theorem [1] states that there exist two irreducible triangulations of N_1; those are presented in Figure 1: P_1 and P_2. (For each hexagon identify each antipodal pair of points in the boundary to obtain an actual triangulation of N_1.) By repeatedly applying the splitting operation to P_1 and P_2, we can generate all triangulations of N_1. Sulanke [11] has generated by computer all triangulations of N_1 with up to 19 vertices; in particular, among them there are 20 triangulations with up to 8 vertices. Independently, the authors of the present paper have identified the same list of 20 triangulations by hand (Figure 1), using the automorphisms of P_1 and P_2. An *automorphism* of a triangulation P is an isomorphism of P with itself. The set of all automorphisms of P forms a group, called the *automorphism group* of P (denoted $\mathrm{Aut}(P)$).

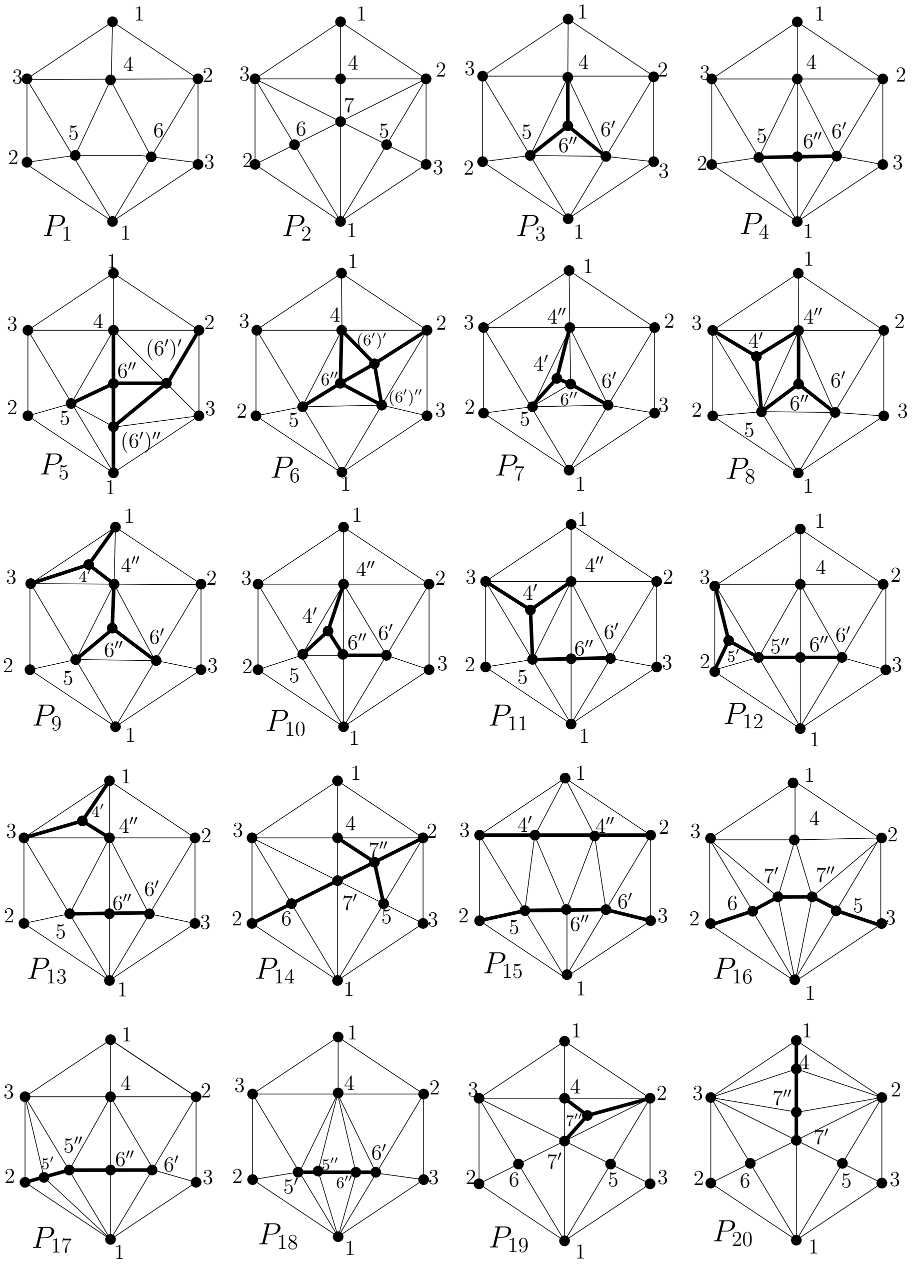

Figure 1. All projective plane triangulations with up to 8 vertices

Lemma 2 (see [11])**.** *There are precisely one (up to isomorphism) triangulation of N_1 with 6 vertices, three with 7 vertices, and sixteen with 8 vertices. They are shown in Figure 1, in which the bold edges indicate the cable-subgraphs of the triangulations.*

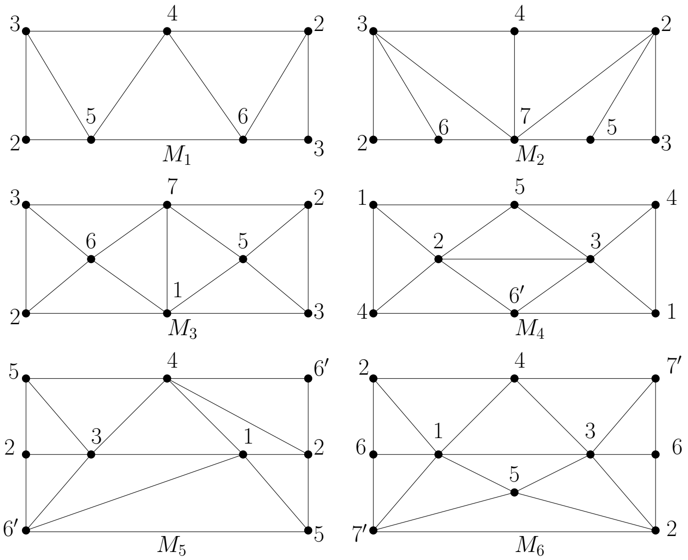

Figure 2. Irreducible triangulations of the Möbius band

Theorem 1. *There are precisely six non-isomorphic irreducible triangulations of the Möbius band, namely M_1 to M_6, shown in Figure 2 in which the left and right sides of each rectangle are identified with opposite orientation to obtain an actual triangulation of the Möbius band.*

Proof. Observe that in Figure 1 only the following three non-irreducible members have a pylonic vertex: P_3 and P_4 with pylonic vertex $6''$, and P_{19} with pylonic vertex $7''$. It can be easily proved that if a triangulation of N_1 has at least two cables but has no pylonic vertex, then no pylonic vertex can be created under further splitting of the triangulation. On the other hand, it can be easily seen that any one splitting applied to the pylonic triangulations P_3, P_4, or P_{19} destroys their pylonicity. Therefore, by Lemma 1, each irreducible triangulation of $N_1 - D$ is obtainable either by removing a vertex from an irreducible triangulation in $\{P_1, P_2\}$, or by removing the pylonic vertex from a pylonic triangulation in $\{P_3, P_4, P_{19}\}$. It is known [4,5,7] that $\mathrm{Aut}(P_1)$ acts transitively on the vertex set $V(P_1)$, while under the action of $\mathrm{Aut}(P_2)$ the set $V(P_2)$ breaks into two orbits as follows: $\mathrm{orbit}_1 = \{1, 2, 3, 7\}$, $\mathrm{orbit}_2 = \{4, 5, 6\}$. Therefore, all irreducible triangulations of $N_1 - D$ are covered by the followings: $M_1 = P_1$ minus vertex 1 (subtracted with the incident edges and faces), $M_2 = P_2$ minus vertex 1, $M_3 = P_2$ minus vertex 4, $M_4 = P_4$ minus vertex $6''$, $M_5 = P_3$ minus vertex $6''$, $M_6 = P_{19}$ minus vertex $7''$. To see that these triangulations are pairwise non-isomorphic, observe that they have different vertex degree sequences except for the pair $\{M_3, M_4\}$; however, all boundary vertices have degree 5 in M_3 but not all in M_4. $\square$

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