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On π-quasigroups of type T_1

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Abstract. Quasigroups satisfying the identity $x(x \cdot xy) = y$ are called π-quasigroups of type T_1. The spectrum of the defining identity is precisely $q = 0$ or $1 (\mathrm{mod}\ 3)$, except for $q = 6$. Necessary conditions when a finite π-quasigroup of type T_1 has the order $q = 0\ (\mathrm{mod}\ 3)$, are given. In particular, it is proved that a finite π-quasigroup of type T_1 such that the order of its inner mapping group is not divisible by three has a left unit. Necessary and sufficient conditions when the identity $x(x \cdot xy) = y$ is invariant under the isotopy of quasigroups (loops) are found.

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Let $\Sigma(Q)$ be the set of all binary quasigroup operations defined on a nonempty set Q. V. Belousov proved in [1] that the minimal length of nontrivial identities $w_1 = w_2$ in $\Sigma(Q)$, of two free elements, is five and that any such minimal identity can be represented in the form: $A(x, B(x, C(x, y))) = y$. Moreover, every identity of the given above form implies the orthogonality of some pairs of parastrophes of the quasigroup operations A, B and C.

A binary quasigroup (Q, A) is called a π-quasigroup of type $[\alpha, \beta, \gamma]$, where $\alpha, \beta, \gamma \in S_3$, if it satisfies the identity:

$$^{\alpha}A(x, {}^{\beta}A(x, {}^{\gamma}A(x, y))) = y \tag{1}$$

(where $^{\sigma}A$ denotes the σ-parastrophe of A).

V. Belousov (1983) and, independently, F. Bennett (1989) gave a classification of all identities (1), consisting of seven classes. Denoting A by "$\cdot$", the representatives of these classes are the following (their types are given according to [1]): $x(x \cdot xy) = y$ (of type $T_1 = [\varepsilon, \varepsilon, \varepsilon]$); $x(y \cdot yx) = y$ (of type $T_2 = [\varepsilon, \varepsilon, l]$); $x \cdot xy = yx$ (of type $T_4 = [\varepsilon, \varepsilon, lr]$); $xy \cdot x = y \cdot xy$ (of type $T_6 = [\varepsilon, l, lr]$); $xy \cdot y = x \cdot xy$ (of type $T_8 = [\varepsilon, rl, lr]$); $xy \cdot yx = y$ (of type $T_{10} = [\varepsilon, lr, l]$); $yx \cdot xy = y$ (of type $T_{11} = [\varepsilon, lr, rl]$), where $l = (13)$, $r = (23)$. Quasigroups satisfying identities from this classification have been studied by many authors (see, for example, [1, 4–6, 10]). An open problem is to describe groups isotopic to π-quasigroups of different types.

π-Quasigroups of type T_1, i.e. binary quasigroups $(Q, \cdot)$ satisfying the identity:

$$x \cdot (x \cdot xy) = y, \tag{2}$$

are studied in the present work. It is known that the spectrum of the defining identity is precisely $q = 0$ or $1 (\mathrm{mod}\ 3)$, except for $q = 6$ ([5]). Necessary conditions when a finite π-quasigroup of type T_1 has the order $q = 0 \pmod 3$ are given. In particular, we prove that a π-quasigroup of type T_1 for which the order of inner mapping group is not divisible by three always has a left unit. Necessary and sufficient conditions when the identity (2) is invariant under the isotopy of quasigroups (loops) are proved. Also, π-quasigroups of type T_1 isotopic to groups, in particular π-T-quasigroups of type T_1, are considered.

Proposition 1. *A quasigroup $(Q, \cdot)$ is a π-quasigroup of type T_1 if and only if its parastrophe $(Q, \backslash)$, where "$\backslash$" is the right division in $(Q, \cdot)$, is a π-quasigroup of type T_1.*

Proof. If $(Q, \cdot)$ is a quasigroup, then the identity (2) is equivalent to the identity $x\backslash[x\backslash(x\backslash y)] = y$. □

Remark that the identity (2) in a quasigroup $(Q, \cdot)$ is equivalent to the condition $L_x^3 = \varepsilon$, $\forall x \in Q$, where $L_x : Q \mapsto Q, L_x(a) = x \cdot a, \forall a \in Q$, is the left translation in $(Q, \cdot)$. We will denote by $M(Q, \cdot)$ (respectively, $LM(Q, \cdot)$, $RM(Q, \cdot)$) the multiplication group (respectively, left multiplication group, right multiplication group) of a quasigroup $(Q, \cdot)$.

A mapping $\alpha \in M(Q, \cdot)$ is called an inner mapping of a quasigroup $(Q, \cdot)$ with respect to an element $h \in Q$ if $\alpha(h) = h$. The group of inner mappings of the quasigroup $(Q, \cdot)$ with respect to h will be denoted by I_h [2, 9].

Proposition 2. *If $(Q, \cdot)$ is a finite π-quasigroup of type T_1 without a left unit, then $\mid I_h \mid \equiv 0 \ (mod\ 3)$, for every $h \in Q$.*

Proof. Let $(Q, \cdot)$ be a π-quasigroup of type T_1, $h \in Q$ and let f_h be the local left unit of the element h: $f_h \cdot h = h$. Then $L_{f_h}(h) = h$, so $L_{f_h} \in I_h$. If $(Q, \cdot)$ has not a left unit, then $L_{f_h} \neq \varepsilon$ and, using the equality $L_{f_h}^3 = \varepsilon$, we get $\mid I_h \mid \equiv 0 \pmod 3$. □

Proposition 3. *Let $(Q, \cdot)$ be a π-quasigroup of type T_1 and $h \in Q$. If $\mid I_h \mid \equiv 1$ or 2 $(mod\ 3)$, then $(Q, \cdot)$ has a left unit.*

Proof. As it was remarked above, $L_{f_h} \in I_h$ and $L_{f_h}^3 = \varepsilon$, where f_h is the local left unit of the element h. If $\mid I_h \mid \equiv 1$ or $2 \pmod 3$, then every element of I_h has the order not divisible by three, so the order of the mapping L_{f_h} has the form $2k + 1$ or $2k + 2$. In both cases we get $L_{f_h} = \varepsilon$, which means that f_h is a left unit in $(Q, \cdot)$. □

Proposition 4. *If a π-quasigroup $(Q, \cdot)$ of type T_1 has a left unit and is isotopic to an abelian group, then its left multiplication group $LM(Q, \cdot)$ is abelian.*

Proof. Let $(Q, \cdot)$ be a π-quasigroup of type T_1, with the left unit f. If $(Q, \cdot)$ is isotopic to an abelian group then, according to [3], the corresponding e-quasigroup $(Q, \cdot, \backslash, /)$ satisfies the identity

$$x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v)).$$

On the other hand, the equality $x\backslash y = x\cdot xy$ follows from (2), so the previous identity is equivalent to:

$$x\cdot x(y(u\cdot uv)) = u\cdot u(y(x\cdot xv)),$$

which, for $x = f$, implies

$$y(u\cdot uv) = u\cdot (u\cdot yv),$$

i.e. $L_yL_u^2 = L_u^2L_y$. So as $L_u^2 = L_u^{-1}$, $\forall u\in Q$, we get $L_yL_u = L_uL_y$, for every $y,u\in Q$. □

Let $(Q,\cdot)$ be a quasigroup. Following [2,9], the left (resp. middle) nucleus of $(Q,\cdot)$ is the set $N_l = \{a\in Q \mid a\cdot xy = ax\cdot y, \forall x,y\in Q\}$ (resp. $N_m = \{a\in Q \mid xa\cdot y = x\cdot ay, \forall x,y\in Q\}$). A mapping $\lambda : Q\mapsto Q$ is called a left regular mapping of the quasigroup $(Q,\cdot)$ if $\lambda(x\cdot y) = \lambda(x)\cdot y$, $\forall x,y\in Q$.

Proposition 5. *Let $(Q,\cdot)$ be a π-quasigroup of type T_1. The following statements hold:*

1) if $(Q,\cdot)$ has a left unit, then $\lambda^3 = \varepsilon$, for every left regular mapping λ of $(Q,\cdot)$;

2) if $(Q,\cdot)$ is finite and its left nucleus N_l contains at least two elements, then $|Q|\equiv 0(mod\ 3)$;

3) if $(Q,\cdot)$ is a finite π-loop of type T_1 and its middle nucleus contains at least two elements, then $|Q|\equiv 0(mod\ 3)$.

Proof. 1. Let $(Q,\cdot)$ be a π-quasigroup of type T_1 with the left unit f and let λ be a left regular mapping of $(Q,\cdot)$. Taking $x\mapsto\lambda(x)$ in (2), get: $y = \lambda(x)\cdot(\lambda(x)\cdot(\lambda(x)\cdot y)) = \lambda(x\cdot\lambda(x\cdot\lambda(x\cdot y)))$, $\forall x,y\in Q$. Now, for $x = f$, from the last equalities follows: $\lambda^3(y) = y, \forall y\in Q$, i.e. $\lambda^3 = \varepsilon$.

2. Let $|N_l|\geq 2$ and $a\in N_l^{(\cdot)}$. Then $a\cdot xy = ax\cdot y, \forall x,y\in Q$. Using the identity (2), have: $a\cdot(a\cdot ay) = y\Rightarrow a\cdot(a^2\cdot y) = y\Rightarrow (a\cdot a^2)\cdot y = y$, so $a\cdot a^2 = f$, where f is the left unit of $(Q,\cdot)$. So as $(N_l,\cdot)$ is a group, we get that $a^3 = e, \forall a\in N_l$. From $|N_l|\equiv 0(mod\ 3)$ and the fact that $|N_l|$ divides $|Q|$ follows $|Q|\equiv 0(mod\ 3)$.

3. Let $(Q,\cdot)$ be a finite π-loop of type T_1 with the unit f. If the middle nucleus N_m contains at least two elements, then there exists $a\in N_m\backslash\{f\}$ which satisfies the equality $x\cdot ay = xa\cdot y$, $\forall x,y\in Q$, hence $y = a(a\cdot ay) = a^2\cdot ay = (a^2\cdot a)y = a^3\cdot y, \forall y\in Q\Rightarrow a^3 = f, \forall a\in N_m$ (N_m is a group), which implies $|N_m|\equiv 0(mod\ 3)$ and $|Q|\equiv 0(mod\ 3)$. □

Corollary. *If the group of left regular mappings of a finite π-quasigroup $(Q,\cdot)$ of type T_1 with a left unit has at least two elements, then $|Q|\equiv 0(mod\ 3)$.*

Proof. In this case the group of left regular mappings will contain at least one element of order three, so its order will be a multiple of three and then $|Q|\equiv 0(mod\ 3)$. □

A loop $(Q,\cdot)$ is called an LPA-loop (or a left power alternative loop) if, for $\forall m,n\in Z$ and $\forall x,y\in Q$, the following equality holds:

$$x^m\cdot x^ny = x^{m+n}y.$$

It is known that LPA-loops are power associative, i.e. each element of an LPA-loop generates an associative subloop [8]. For example, left Bol loops are LPA-loops.

Proposition 6. *An LPA-loop $(Q,\cdot)$ is a π-loop of type T_1 if and only if $x^3 = e, \forall x \in Q$, where e is the unit of $(Q,\cdot)$.*

Proof. Let $(Q,\cdot)$ be an LPA-loop with the unit e. If $(Q,\cdot)$ is a π-loop of type T_1, then $y = x\cdot(x\cdot xy) = x^3\cdot y, \forall x,y \in Q$, so $x^3 = e$. Conversely, if $x^3 = e, \forall x \in Q$, then $x\cdot(x\cdot xy) = x^3\cdot y = e\cdot y = y, \forall x,y\in Q$, i.e. $(Q,\cdot)$ is a π-loop of type T_1. □

Corollary 1. *A left Bol loop $(Q,\cdot)$ is a π-loop of type T_1 if and only if $x^3 = e, \forall x \in Q$, where e is the unit of $(Q,\cdot)$.*

Corollary 2. A group $(Q,\cdot)$ is a π-group of type T_1 if and only if $x^3 = e$, $\forall x \in Q$, where e is the unit of the group $(Q,\cdot)$.

Proposition 7. *Let $(Q,\cdot)$ be a π-quasigroup of type T_1 and let $(Q,\circ)$ be its isotope with the isotopy (α,β,γ). Then $(Q,\circ)$ is a π-quasigroup of type T_1 if and only if, for every $x,y\in Q$, the following equality holds:*

$$\gamma\beta^{-1}[x\cdot(x\cdot y)] = x\cdot\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}y). \tag{3}$$

Proof. The isotope $(Q,\circ)$ is a π-quasigroup of type T_1 if and only if it satisfies the identity

$$x\circ(x\circ(x\circ y)) = y. \tag{4}$$

Using the isotopy $x\circ y = \gamma^{-1}(\alpha(x)\cdot\beta(y))$, the identity (4) gets the form:

$$\gamma^{-1}(\alpha(x)\cdot\beta\gamma^{-1}(\alpha(x)\cdot\beta\gamma^{-1}(\alpha(x)\cdot\beta(y))) = y.$$

Taking $x\mapsto\alpha^{-1}x$ and $y\mapsto\beta^{-1}y$ in the last identity, we have the equality:

$$\gamma^{-1}(x\cdot\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(x\cdot y))) = \beta^{-1}(y),$$

which is equivalent to

$$x\cdot\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(x\cdot y)) = \gamma\beta^{-1}(y).$$

So as $(Q,\cdot)$ satisfies (2), the last equality implies

$$\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(x\cdot y)) = x\cdot(x\cdot\gamma\beta^{-1}(y)),$$

hence

$$x\cdot\beta\gamma^{-1}(x\cdot y) = \gamma\beta^{-1}(x\cdot(x\cdot\gamma\beta^{-1}(y))), \tag{5}$$

for $\forall x,y\in Q$. Denoting $x\cdot y = z$ and using (2), have $y = x\cdot xz$, so (5) takes the form:

$$x\cdot\beta\gamma^{-1}(z) = \gamma\beta^{-1}(x\cdot(x\cdot\gamma\beta^{-1}(x\cdot xz))),$$

which implies $\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(z)) = x\cdot(x\cdot\gamma\beta^{-1}(x\cdot xz))$, hence $x\cdot(\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(z))) = \gamma\beta^{-1}(x\cdot xz)$, for $\forall x,z\in Q$.

Conversely, if the π-quasigroup $(Q,\cdot)$ of type T_1 satisfies (3) then, taking $y\mapsto \gamma\beta^{-1}(y)$, we have:

$$x\cdot\beta\gamma^{-1}(x\cdot y) = \gamma\beta^{-1}(x\cdot(x\cdot\gamma\beta^{-1}(y))),$$

which, for $y=\beta\gamma^{-1}(x\cdot y)$, implies

$$x\cdot\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(x\cdot y)) = \gamma\beta^{-1}(x\cdot(x\cdot xy)) = \gamma\beta^{-1}(y).$$

Using the isotopy $x\cdot y=\gamma(\alpha^{-1}(x)\circ\beta^{-1}(y))$, from the last equalities we get:

$$\gamma(\alpha^{-1}(x)\circ(\alpha^{-1}(x)\circ(\alpha^{-1}(x)\circ\beta^{-1}(y)))) = \gamma\beta^{-1}(y),$$

or, replacing $x\mapsto\alpha(x)$, $y\mapsto\beta(y)$ and using the fact that γ is a bijection, we obtain: $x\circ(x\circ(x\circ y)) = y$, i.e. $(Q,\circ)$ is a π-quasigroup of type T_1. □

Corollary 1. *Let $(Q,\cdot)$ be a π-loop of type T_1. If the isotope $(Q,\circ)$, where $(\circ) = (\cdot)^{(\alpha,\beta,\varepsilon)}$, is a π-quasigroup of type T_1, then $\beta^3=\varepsilon$.*

Proof. If $(Q,\circ)$ is a π-loop of type T_1, then $(Q,\cdot)$ satisfies the equality (3). Taking $x=e$ in (3), where e is the unit of the loop $(Q,\cdot)$, we get $\beta(y)=\beta^{-2}(y), \forall y\in Q$, so $\beta^3=\varepsilon$. □

Corollary 2. *The identity (2) is invariant under quasigroup isotopies with equal second and third components.*

Proof. Let $(Q,\cdot)$ be a quasigroup satisfying the identity (2) and let consider the isotope $(\circ) = (\cdot)^{(\alpha,\beta,\beta)}$. Using the equality $x\cdot y = \beta(\alpha^{-1}(x)\cdot\beta^{-1}(y))$, from (2) follows

$$\beta(\alpha^{-1}x\circ(\alpha^{-1}x\circ(\alpha^{-1}x\circ\beta^{-1}y))) = y,$$

$\forall x,y\in Q$, which, for $x\mapsto\alpha(x)$ and $y\mapsto\beta(y)$, implies:

$$x\circ(x\circ(x\circ y)) = y,$$

$\forall x,y\in Q$, so $(Q,\circ)$ is a π-quasigroup of type T_1. □

Proposition 8. *The identity (2) is universal in a loop $(Q,\cdot)$ if and only if $(Q,\cdot)$ satisfies the identity:*

$$x\cdot b(b\cdot x(b(b\cdot xy))) = by. \tag{6}$$

Proof. Let $(Q,\cdot)$ be a loop with universal identity (2). Then $(Q,\cdot)$ and every its loop isotope satisfy (2). Let $a,b\in Q$ and $(\circ) = (\cdot)^{(R_a^{-1},L_b^{-1},\varepsilon)}$. According to Proposition 5, the loop $(Q,\cdot)$ satisfies (3):

$$L_b(x\cdot xy) = x\cdot L_b^{-1}(x\cdot L_b^{-1}(y)),$$

$\forall x, y \in Q$. Taking $y \mapsto x \cdot y$ and using (2), from the last identity follows

$$L_b(y) = x \cdot L_b^{-1}(x \cdot L_b^{-1}(x \cdot y)), \tag{7}$$

$\forall x, y \in Q$. So as the loop $(Q, \cdot)$ satisfies the equality $L_b^3 = \varepsilon$, for $\forall b \in Q$, (7) is equivalent to

$$L_b(y) = x \cdot L_b^2(x \cdot L_b^2(x \cdot y)),$$

$\forall x, y \in Q$. Conversely, if a loop $(Q, \cdot)$ satisfies the identity (6) then, taking $b = e$, where e is the unit of $(Q, \cdot)$, we get the identity (2), i.e. $(Q, \cdot)$ is a π-loop of type T_1. So as every loop isotope of a loop is isomorphic to an LP-isotope, we may consider only LP-isotopes of $(Q, \cdot)$. Let $a, b \in Q$ and $(\circ) = (\cdot)^{(R_a^{-1}, L_b^{-1}, \varepsilon)}$. Using (6) and the equality $L_b^3 = \varepsilon$, have: $x \cdot L_b^{-1}(x \cdot L_b^{-1}(x \cdot y)) = b \cdot y \Rightarrow R_a^{-1}x \cdot L_b^{-1}(R_a^{-1}x \cdot L_b^{-1}(R_a^{-1}x \cdot L_b^{-1}y)) = b \cdot L_b^{-1}y = y \Rightarrow x \circ (x \circ (x \circ y)) = y$, i. e. $(Q, \circ)$ is a π-loop of type T_1. $\square$

Example 1. The couple $(Z_3^3, \circ)$, where Z_3 is the field of residues modulo 3 and the operation $(\circ)$ is defined as follows:

$$(i, j, k) \circ (p, q, r) = (i + p, j + q, k + r + ijp),$$

$\forall (i, j, k), (p, q, r) \in Z_3^3$, is a non-associative loop for which the identity (2) is universal. Remark that the left nucleus of $(Z_3^3, \circ)$ is $N_l = \{(0, 0, 0), (0, 0, 1), (0, 0, 2)\}$.

Example 2. The loop $(Q, *)$, where $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and the operation " $*$ " is given by the table below, is a π-loop of type T_1, for which the identity (2) is not universal.

$*$	0	1	2	3	4	5	6	7	8
0	2	0	1	4	6	8	3	5	7
1	0	1	2	3	4	5	6	7	8
2	3	2	4	5	1	0	7	8	6
3	4	3	5	6	8	7	1	2	0
4	8	4	6	0	7	2	5	1	3
5	7	5	8	1	2	3	0	6	4
6	5	6	7	2	0	4	8	3	1
7	1	7	3	8	5	6	4	0	2
8	6	8	0	7	3	1	2	4	5

T-Quasigroups are defined and partially studied in [7]. A quasigroup $(Q, \cdot)$ is called a T-quasigroup if there exists an abelian group $(Q, +)$, its automorphisms $\varphi, \psi \in Aut(Q, +)$, and an element $g \in Q$ such that, for every $x, y \in Q$, the following equality holds:

$$x \cdot y = \varphi(x) + \psi(y) + g.$$

The tuple $((Q, +), \varphi, \psi, g)$ is called a T-form and the group $(Q, +)$ is called a T-group of the T-quasigroup $(Q, \cdot)$.

Proposition 9. *A T-quasigroup $(Q,\cdot)$, with a T-form $T=((Q,+),\varphi,\psi,g)$, is a π-quasigroup of type T_1 if and only if $\psi^2+\psi+\varepsilon=\omega$, where $\omega: Q\to Q$, $\omega(x)=0$, $\forall x\in Q$, 0 is the neutral element of the group $(Q,+)$.*

Proof. Let $(Q,\cdot)$ be a T-quasigroup with a T-form $T=((Q,+),\varphi,\psi,g)$. Then

$$x\cdot y=\varphi(x)+\psi(y)+g, \tag{8}$$

$\forall x,y\in Q$. If $(Q,\cdot)$ is a π-quasigroup of type T_1, then if satisfies the identity (2). Using (8), the identity (2) takes the form:

$$\varphi(x)+\psi\varphi(x)+\psi^2\varphi(x)+\psi^3(y)+\psi^2(g)+\psi(g)+g=y, \tag{9}$$

$\forall x,y\in Q$. Taking $x=y=0$ in (9), where 0 is the neutral element of the group $(Q,+)$, we get:

$$\psi^2(g)+\psi(g)+g=0. \tag{10}$$

Also, taking $x=0$ in (9), we have $\psi^3(y)=y, \forall y\in Q$, i. e.

$$\psi^3=\varepsilon, \tag{11}$$

where $\varepsilon: Q\mapsto Q, \varepsilon(x)=x, \forall x\in Q$. Now, using (10) and (11), the equality (9) implies: $\varphi(x)+\psi\varphi(x)+\psi^2\varphi(x)=0$, hence $(\varepsilon+\psi+\psi^2)\varphi(x)=0$, $\forall x\in Q$. So as φ is a bijection, the last equality implies

$$\varepsilon+\psi+\psi^2=\omega. \tag{12}$$

Conversely, if the equality (12) holds, then

$$\psi^3-\varepsilon=(\psi-\varepsilon)(\varepsilon+\psi+\psi^2)=\omega,$$

hence (11) holds. Using (11) and (12), we get: $y=\omega(x)+\psi^3(y)+\omega(g)=(\varepsilon+\psi+\psi^2)\varphi(x)+\psi^3(y)+\psi^2(g)+\psi(g)+g=x\cdot(x\cdot xy)=y$, $\forall x,y\in Q$, so $(Q,\cdot)$ is a π-quasigroup of type T_1. □

The following example shows that the class of π-T-quasigroups of type T_1 is not empty.

Example 3. The quasigroup $(Z_7,\cdot)$, where

$$x\cdot y=\bar{5}x+\bar{2}y+\bar{3},$$

$\forall x,y\in Q$, is a π-T-quasigroup of type T_1 with the T-form $((Z_7,+),\varphi,\psi,\bar{3})$, where $\varphi(x)=\bar{5}x$, $\psi(x)=\bar{2}x$, $\forall x\in Z_7$.

Proposition 10. *If $(Q,\cdot)$ is a finite π -T-quasigroup of type T_1 with a left unit, then $|Q|\equiv 0(mod\ 3)$.*

Proof. Let $(Q,\cdot)$ be a finite π-T-quasigroup of type T_1 with a T-form $T=((Q,+),\varphi,\psi,g)$ and with the left unit f. Then,

$$x = f \cdot x = \varphi(f) + \psi(x) + g, \tag{13}$$

so, taking $x = 0$ where 0 is the neutral element of the T-group $(Q,+)$, we get $\varphi(f) = -g$. From the last equality and (13), we obtain $\psi = \varepsilon$, where ε is the identical mapping on Q. According to Proposition 9, if $(Q,\cdot)$ is a π-quasigroup of type T_1, then $\psi^2 + \psi + \varepsilon = \omega$, hence $3\varepsilon = \omega$, i.e. $x + x + x = 0$, $\forall x \in Q$, which implies $|Q| \equiv 0(\mathrm{mod}\ 3)$. □

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