BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 3–8
ISSN 1024–7696

On a class of weighted composition operators on Fock space

Namita Das

Abstract. Let T_ϕ be the Toeplitz operator defined on the Fock space $L^2_a(\mathbb{C})$ with symbol $\phi \in L^\infty(\mathbb{C})$. Let for $\lambda \in \mathbb{C}$, $k_\lambda(z) = e^{\frac{\bar{\lambda}z}{2} - \frac{|\lambda|^2}{4}}$, the normalized reproducing kernel at λ for the Fock space $L^2_a(\mathbb{C})$ and $t_\alpha(z) = z - \alpha, z, \alpha \in \mathbb{C}$. Define the weighted composition operator W_α on $L^2_a(\mathbb{C})$ as $(W_\alpha f)(z) = k_\alpha(z)(f \circ t_\alpha)(z)$. In this paper we have shown that if M and H are two bounded linear operators from $L^2_a(\mathbb{C})$ into itself such that $MT_\psi H = T_{\psi \circ t_\alpha}$ for all $\psi \in L^\infty(\mathbb{C})$, then M and H must be constant multiples of the weighted composition operator W_α and its adjoint respectively.

Mathematics subject classification: 47B35, 32M15.
Keywords and phrases: Fock space; Toeplitz operators; weighted composition operators.

1 Introduction

For $x, y \in \mathbb{C}^N$(for some integer $N \geq 1$), we write $\bar{x}y = \sum_{n=1}^N \bar{x}_n y_n$ and $|x| = (\bar{x}x)^{\frac{1}{2}}$. Thus, $|x-y|$ is the usual Euclidean distance between x and y. The symbol dz denotes the Lebesgue measure in $\mathbb{C}^N$ for all $N \geq 1$. The Gaussian measure on $\mathbb{C}^N$ is, by definition, $d\mu(z) = (2\pi)^{-N} e^{-\frac{|z|^2}{2}} dz$. Denote $L^p(\mathbb{C}^N, d\mu)$ the usual Lebesgue spaces on $\mathbb{C}^N$ with respect to the measure μ; $L^\infty(\mathbb{C}^N, d\mu)$ shall be occasionally abbreviated to $L^\infty(\mathbb{C}^N) = L^\infty(\mathbb{C}^N, dz)$, since they happen to coincide [5]. Set, for $1 \leq p \leq \infty$,

$$L^p_a(\mathbb{C}^N) = \{f \in L^p(\mathbb{C}^N, d\mu) : f \text{ is an entire function on } \mathbb{C}^N\}.$$

The space $L^p_a(\mathbb{C}^N)$ is a closed subspace of $L^p(\mathbb{C}^N, d\mu)$, $L^\infty_a(\mathbb{C}^N) = H^\infty(\mathbb{C}^N)$. For $p = 2$, $L^2_a(\mathbb{C}^N)$ is a Hilbert space, called the Fock or Siegal-Bargmann space.

For a multiindex $n = (n_1, n_2, \cdots, n_N) \in \mathbb{N}^N$, the following abbreviations will be employed:

$$a_n = a_{n_1, n_2, \cdots, n_N},$$

$$z^n = z_1^{n_1} z_2^{n_2} \cdots z_N^{n_N} (\text{ for } z \in \mathbb{C}^N),$$

$$n! = n_1! n_2! \cdots n_N!,$$

$$2^n = 2^{n_1 + n_2 + \cdots + n_N}.$$

If f is an entire function, $f(z) = \sum_{n\in\mathbb{N}^N} f_n z^n$, then

$$\int_{\mathbb{C}^N} |f(z)|^2 d\mu(z) = \sum_{n\in\mathbb{N}^N} n!2^n |f_n|^2.$$

Consequently, $f \in L^2_a(\mathbb{C}^N)$ if and only if the last expression is finite. The inner product of f and $g(z) = \sum_{n\in\mathbb{N}^N} g_n z^n$, $f, g \in L^2_a(\mathbb{C}^N)$, is given by

$$\langle f, g\rangle = \sum_{n\in\mathbb{N}^N} n!2^n f_n \bar{g}_n.$$

The set $\{(n!2^n)^{-\frac{1}{2}} z^n\}_{n\in\mathbb{N}^N}$ is an orthonormal basis of $L^2_a(\mathbb{C}^N)$. The polynomials are dense in $L^2_a(\mathbb{C}^N)$. The space $L^2_a(\mathbb{C}^N)$ is a reproducing kernel space; the reproducing kernel at $\lambda \in \mathbb{C}^N$ is given by $g_\lambda(z) = e^{\frac{\bar{\lambda}z}{2}}$, and $\|g_\lambda\|_2 = e^{\frac{|\lambda|^2}{4}}$. For $\phi \in L^\infty(\mathbb{C}^N, d\mu) = L^\infty(\mathbb{C}^N)$, the Toeplitz operator T_ϕ is defined from $L^2_a(\mathbb{C}^N)$ into itself as $T_\phi f = P(\phi f)$ where P is the orthogonal projection from $L^2(\mathbb{C}^N, d\mu)$ onto $L^2_a(\mathbb{C}^N)$. Further, for $\phi \in L^\infty(\mathbb{C}^N)$, define the Hankel operator H_ϕ from $L^2_a(\mathbb{C}^N)$ into $(L^2_a(\mathbb{C}^N))^\perp$ by $H_\phi f = (I-P)(\phi f)$. Here $(L^2_a(\mathbb{C}^N))^\perp$ denotes the orthogonal complement of $L^2_a(\mathbb{C}^N)$. Define for $\lambda \in \mathbb{C}^N$, $k_\lambda(z) = \frac{g_\lambda(z)}{\|g_\lambda\|} = e^{\frac{\bar{\lambda}z}{2} - \frac{|\lambda|^2}{4}}$, the normalized reproducing kernel at λ for the Fock space $L^2_a(\mathbb{C}^N)$. In this paper we shall only concentrate our attention on the Fock space $L^2_a(\mathbb{C})$. Notice that it has an orthonormal basis $\{e_n\}_{n=0}^\infty$ where

$$e_n(z) = (n!2^n)^{-\frac{1}{2}} z^n.$$

For $\alpha \in \mathbb{C}$, define W_α from $L^2_a(\mathbb{C})$ into itself by $(W_\alpha f)(z) = k_\alpha(z) f(z-\alpha)$. Note for $f \in L^2_a(\mathbb{C}), W^*_\alpha f = (f \circ t_{-\alpha}) k_{-\alpha} = W_{-\alpha} f$ and therefore the operator W_α is a unitary operator on $L^2_a(\mathbb{C})$ for each $\alpha \in \mathbb{C}$ and the operator can be defined on $L^2(\mathbb{C})$.

2 The forward shift operator and Toeplitz algebra on Fock space

Let Z be the forward shift operator with respect to the basis $\{e_n\}_{n=0}^\infty$, and let $\Phi(z) = \frac{z}{|z|} = e^{i \arg z}$. Let $\mathcal{L}(L^2_a(\mathbb{C}))$ be the space of all bounded linear operators from $L^2_a(\mathbb{C})$ into itself and $\mathcal{LC}(L^2_a(\mathbb{C}))$ be the space of all compact operators in $\mathcal{L}(L^2_a(\mathbb{C}))$. For $M, T \in \mathcal{L}(L^2_a(\mathbb{C}))$, let $[M, T] = MT - TM$. Let

$$\mathcal{A}(T_\Phi) = \{T \in \mathcal{L}(L^2_a(\mathbb{C})) : [T, T_\Phi] \in \mathcal{LC}(L^2_a(\mathbb{C}))\}$$

and

$$\mathcal{A}(Z) = \{T \in \mathcal{L}(L^2_a(\mathbb{C})) : [T, Z] \in \mathcal{LC}(L^2_a(\mathbb{C}))\}.$$

Lemma 2.1. *The following hold.*

(i) *The operator T_Φ is a compact perturbation of Z and $\mathcal{A}(T_\Phi) = \mathcal{A}(Z)$.*

(ii) *The Toeplitz operator $T_\Psi \in \mathcal{A}(T_\Phi)$ for every $\Psi \in L^\infty(\mathbb{C})$.*

Proof. (i) Notice that

$$\begin{aligned}\langle T_\Phi z^n, z^m\rangle &= \int_{\mathbb{C}} \frac{z}{|z|} z^n \bar{z}^m d\mu(z) \\ &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} r^{n+m} e^{i(n-m+1)t} e^{-\frac{r^2}{2}} r dt dr.\end{aligned}$$

This is zero unless $m = n+1$, and in that case it equals

$$\int_0^\infty r^{2n+1} e^{-\frac{r^2}{2}} r dr = \int_0^\infty 2^{n+\frac{1}{2}} t^{n+\frac{1}{2}} e^{-t} dt = 2^{n+\frac{1}{2}}\Gamma\left(n+\frac{3}{2}\right),$$

where Γ is Euler's gamma function. Thus

$$\langle T_\Phi e_n, e_m\rangle = \begin{cases} 0 & \text{if} \quad m \neq n+1; \\ (n!2^n)^{-\frac{1}{2}}(m!2^m)^{-\frac{1}{2}} 2^{n+\frac{1}{2}}\Gamma(n+\frac{3}{2}) & \text{if} \quad m = n+1. \end{cases}$$

Consequently, $T_\Phi e_n = c_n e_{n+1}$, where $c_n = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)^{\frac{1}{2}}\Gamma(n+2)^{\frac{1}{2}}}$. Let $\operatorname{diag}(1-c_n)$ be the diagonal matrix whose nth diagonal entry is $1-c_n$. Now it follows that $Z - T_\Phi = Z\cdot \operatorname{diag}(1-c_n)$, and in order to verify our claim it suffices to show that $c_n \to 1$ as $n \to +\infty$. According to Stirling's formula [1],

$$\Gamma(x+1) \sim \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x},$$

where "$\sim$" means that the ratio of the right-hand to the left-hand side approaches 1 as $x \to +\infty$. Substituting this into the expression for c_n produces

$$c_n \sim \frac{\left(n+\frac{1}{2}\right)^{n+1} e^{-n-\frac{1}{2}}\sqrt{2\pi}}{n^{\frac{n}{2}+\frac{1}{4}} e^{-\frac{n}{2}} (2\pi)^{\frac{1}{4}} (n+1)^{\frac{n}{2}+\frac{3}{4}} e^{-\frac{n}{2}-\frac{1}{2}} (2\pi)^{\frac{1}{4}}}.$$

The terms containing π cancel, as well as those containing e, and what remains is the product of

$$\left(\frac{n+\frac{1}{2}}{n}\right)^{\frac{n}{2}}, \left(\frac{n+\frac{1}{2}}{n+1}\right)^{\frac{n+1}{2}} \quad \text{and} \quad \frac{\left(n+\frac{1}{2}\right)^{\frac{1}{2}}}{n^{\frac{1}{4}}(n+1)^{\frac{1}{4}}},$$

which tend to $e^{\frac{1}{4}}, e^{-\frac{1}{4}}$ and 1, respectively. So, $c_n \to 1$ and the assertion (i) follows.

Now we shall prove (ii). The formulas

$$T_{\psi\theta} - T_\psi T_\theta = H^*_{\overline{\psi}} H_\theta, \tag{1}$$

$$T_\psi T_\theta - T_\theta T_\psi = H^*_{\overline{\theta}} H_\psi - H^*_{\overline{\psi}} H_\theta, \tag{2}$$

hold for arbitrary $\psi, \theta \in L^\infty(\mathbb{C})$. Owing to (2),

$$T_\psi T_\Phi - T_\Phi T_\psi = H^*_{\overline{\Phi}} H_\psi - H^*_{\overline{\psi}} H_\Phi$$

will be compact for arbitrary $\psi \in L^\infty(\mathbb{C})$ if $H_\Phi, H_{\overline{\Phi}}$ are compact. The latter is equivalent to $H_\Phi^* H_\Phi, H_{\overline{\Phi}}^* H_{\overline{\Phi}}$ are compact, respectively, and from (1) it follows that this is equivalent to $I - T_\Phi^* T_\Phi$ and $I - T_\Phi T_\Phi^*$ are compact, respectively. Owing to (i), the last two operators are compact perturbations of $I - Z^*Z = 0$ and $I - ZZ^* = \langle ., e_0 \rangle e_0$, respectively and the result follows. □

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded measurable functions on $\mathbb{T}$ with the essential supremum norm. Let H^2 be the Hardy space on the unit circle $\mathbb{T}$. For $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator B_ϕ with symbol ϕ is the operator on H^2 sending $f \in H^2$ to $P_+(\phi f)$, where P_+ is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. It is easy to check that $B_z^* B_\phi B_z = B_\phi$ for any $\phi \in L^\infty(\mathbb{T})$. According to a classical result [3], the converse holds: if an operator $T \in \mathcal{L}(H^2)$ satisfies $B_z^* T B_z = T$, then $T = B_\phi$ for some $\phi \in L^\infty(\mathbb{T})$. This result serves as a starting point for the theory of symbols of operators. It is also shown in [3], that the only compact Toeplitz operator is the zero Toeplitz operator. If $\phi \in H^\infty(\mathbb{T})$ then $B_\phi \in \mathcal{L}(H^2)$ is called an analytic Toeplitz operator and $B_\phi^* = B_{\bar{\phi}}$ is called a coanalytic Toeplitz operator. Let

$$\begin{aligned} \mathcal{A}(B_z) &= \{T \in \mathcal{L}(H^2) : T - B_z^* T B_z \in \mathcal{LC}(H^2)\} \\ &= \{T \in \mathcal{L}(H^2) : [T, B_z] \in \mathcal{LC}(H^2)\}, \end{aligned}$$

the essential commutant of the forward shift operator B_z on H^2. It is known [2] that $\mathcal{A}(B_z)$ is a C^*−subalgebra of $\mathcal{L}(H^2)$ and $B_\phi \in \mathcal{A}(B_z)$ for all $\phi \in L^\infty(\mathbb{T})$.

Lemma 2.2. *There exists a unitary operator $U : H^2 \to L_a^2(\mathbb{C})$ such that the transformation $T \mapsto U^* T U$ is a C^*-isomorphism of $\mathcal{A}(Z)$ onto $\mathcal{A}(B_z)$.*

Proof. Define $U : H^2 \to L_a^2(\mathbb{C})$ by mapping the standard basis of H^2 onto the basis $\{e_n\}_{n \in \mathbb{N}}$ of $L_a^2(\mathbb{C})$,

$$U : z^n \in H^2 \mapsto \frac{z^n}{\sqrt{n! 2^n}} \in L_a^2(\mathbb{C}).$$

This operator is unitary and the transformation $T \to U^* T U$ maps Z to B_z; hence,

$$\begin{aligned} T \in \mathcal{A}(Z) &\Leftrightarrow [T, Z] \in \mathcal{LC}(L_a^2(\mathbb{C})) \\ &\Leftrightarrow U^* T Z U - U^* Z T U \in \mathcal{LC}(H^2) \\ &\Leftrightarrow (U^* T U)(U^* Z U) - (U^* Z U)(U^* T U) \in \mathcal{LC}(H^2) \\ &\Leftrightarrow (U^* T U) B_z - B_z (U^* T U) \in \mathcal{LC}(H^2) \\ &\Leftrightarrow U^* T U \in \mathcal{A}(B_z). \end{aligned}$$

The proof is complete. □

3 Main result

We now prove the main result of the work.

Theorem 3.1. *Let $\alpha \in \mathbb{C}$ and define the translation operator on $\mathbb{C}$ as $t_\alpha(z) = z - \alpha$. Suppose M and H are two linear bounded operators from $L_a^2(\mathbb{C})$ into itself such that*

$MT_\psi H = T_{\psi\circ t_\alpha}$ for all $\psi \in L^\infty(\mathbb{C}, dz)$. Then $M = cW_\alpha$ and $H = \frac{1}{c}W_\alpha^*$ and $MH = I$, the identity operator on $L_a^2(\mathbb{C})$.

Proof. Notice that the Fock space $L_a^2(\mathbb{C})$ is an invariant subspace for W_α and $W_\alpha^* = W_{-\alpha}$ and therefore $PW_\alpha = W_\alpha P$. For $f \in L_a^2(\mathbb{C})$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} T_\psi W_\alpha f &= T_\psi\left[(f\circ t_\alpha)\,k_\alpha\right] \\ &= P\left(\psi\,(f\circ t_\alpha)\,k_\alpha\right) \\ &= P\left((\psi\circ t_{-\alpha}\circ t_\alpha)\,(f\circ t_\alpha)k_\alpha\right) \\ &= P\left[(((\psi\circ t_{-\alpha})f)\circ t_\alpha)k_\alpha\right] \\ &= PW_\alpha\left[(\psi\circ t_{-\alpha})f\right] \\ &= W_\alpha P\left[(\psi\circ t_{-\alpha})f\right] \\ &= W_\alpha T_{\psi\circ t_{-\alpha}} f. \end{aligned}$$

Thus we get $W_\alpha^* T_\psi W_\alpha f = T_{\psi\circ t_{-\alpha}} f$, for $\alpha \in \mathbb{C}$. Now let $R_\alpha = W_\alpha^* M$ and $S_\alpha = HW_\alpha$. Since $MT_\psi H = T_{\psi\circ t_\alpha}$ it follows that $R_\alpha T_\psi S_\alpha = W_\alpha^* M T_\psi H W_\alpha = W_\alpha^* T_{\psi\circ t_\alpha} W_\alpha = T_\psi$ for all $\psi \in L^\infty(\mathbb{C})$. It is known [4] that the norm closure of the set of all Toeplitz operators in $\mathcal{L}(L_a^2(\mathbb{C}))$ contains $\mathcal{LC}(L_a^2(\mathbb{C}))$. In fact, if $\mathcal{T}_1 = \{T_\phi : \phi \in \mathcal{D}(\mathbb{C})\}$ then $\text{clos}\mathcal{T}_1 = \mathcal{LC}(L_a^2(\mathbb{C}))$ where $\mathcal{D}(\mathbb{C})$ is the set of all infinitely differentiable functions on $\mathbb{C}$ whose supports are compact subsets of $\mathbb{C}$. Thus

$$\begin{aligned} R_\alpha T_\psi S_\alpha T_\Phi &= T_\psi T_\Phi = T_{\psi\Phi} + G \quad (\text{for some } G \in \mathcal{LC}(L_a^2(\mathbb{C}))) \\ &= R_\alpha T_{\psi\Phi} S_\alpha + G \\ &= R_\alpha (T_\psi T_\Phi - G) S_\alpha + G \\ &= R_\alpha (T_\psi T_\Phi - \lim_{n\to\infty} T_{\phi_n}) S_\alpha + G \quad (\text{where } G = \lim_{n\to\infty} T_{\phi_n}) \\ &= R_\alpha T_\psi T_\Phi S_\alpha - \lim_{n\to\infty} R_\alpha T_{\phi_n} S_\alpha + G \\ &= R_\alpha T_\psi T_\Phi S_\alpha - \lim_{n\to\infty} T_{\phi_n} + G \\ &= R_\alpha T_\psi T_\Phi S_\alpha - G + G \\ &= R_\alpha T_\psi T_\Phi S_\alpha. \end{aligned}$$

It follows therefore that $R_\alpha T_\psi (S_\alpha T_\Phi - T_\Phi S_\alpha) = 0$. We shall now show that $S_\alpha T_\Phi - T_\Phi S_\alpha = 0$. Suppose on the contrary that there is some $x \neq 0$ in $\text{Ran}(S_\alpha T_\Phi - T_\Phi S_\alpha)$. Then, by the last relation, $R_\alpha T_\psi x = 0$ for all $\psi \in L^\infty(\mathbb{C})$, so the kernel of R_α contains the set $\{T_\psi x : \psi \in L^\infty(\mathbb{C})\}$. Consider some $y \in L_a^2(\mathbb{C})$ orthogonal to this set. Then $0 = \langle y, T_\psi x\rangle = \langle y, P(\psi x)\rangle = \int_{\mathbb{C}} y(z)\overline{\psi(z)x(z)}d\mu(z)$ for all $\psi \in L^\infty(\mathbb{C})$; because $\bar{x}y \in L^1(\mathbb{C}, d\mu)$, we conclude that $\bar{x}y = 0$, and this is only possible if at least one of the analytic functions x, y is identically zero. But $x \neq 0$ by assumption, so y must be zero, which means that our set is dense in $L_a^2(\mathbb{C})$. Because this set is contained in $\ker R_\alpha$, we have $R_\alpha = 0$, so $T_\psi = R_\alpha T_\psi S_\alpha = 0$ for all ψ – a contradiction. This proves that $S_\alpha T_\Phi - T_\Phi S_\alpha = 0$. Hence $S_\alpha T_\Phi^n = T_\Phi^n S_\alpha$ for all $n \in \mathbb{N}$. Therefore $S_\alpha (Z + \widetilde{K})^n = (Z + \widetilde{K})^n S_\alpha$ as $T_\Phi = Z + \widetilde{K}$ for some $\widetilde{K} \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Hence, it follows that $S_\alpha Z^n - Z^n S_\alpha = K_n$ for some $K_n \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Thus

$$(U^* S_\alpha U)(U^* Z^n U) - (U^* Z^n U)(U^* S_\alpha U) = C_n$$

for some $C_n \in \mathcal{LC}(H^2)$ for all $n \in \mathbb{N}$. Hence $U^*S_\alpha U$ lies in the essential commutant of all analytic Toeplitz operators in $\mathcal{L}(H^2)$. Thus $U^*S_\alpha U = B_\phi + K$ for some $\phi \in H^\infty(\mathbb{T})$ and $K \in \mathcal{LC}(H^2)$.

Similarly one can show that $U^*R_\alpha U = B_{\bar\theta} + K'$, for some $\theta \in H^\infty(\mathbb{T})$ and $K' \in \mathcal{LC}(H^2)$. This is because $R_\alpha T_\psi S_\alpha = T_\psi$ for all $\psi \in L^\infty(\mathbb{C})$ implies $S_\alpha^* T_\psi R_\alpha^* = T_\psi$ for all $\psi \in L^\infty(\mathbb{C})$. Now $(U^*R_\alpha U)(U^*S_\alpha U) = B_{\bar\theta\phi} + C$, for some $C \in \mathcal{LC}(H^2)$. Hence $I = (U^*R_\alpha S_\alpha U) = B_{\bar\theta\phi} + C$ and therefore $B_{1-\bar\theta\phi} = C$. This implies $1 - \bar\theta\phi = 0$ as the only compact Toeplitz operator in $\mathcal{L}(H^2)$ is the zero Toeplitz operator. Thus $C = 0$ and $\bar\theta = \frac{1}{\phi}$. This implies $\theta \in H^\infty(\mathbb{T})$ and $\bar\theta \in H^\infty(\mathbb{T})$. Thus $\bar\theta = d$ and $\phi = \frac{1}{d}$ for some constant d. Hence it follows that $U^*R_\alpha U = B_d + K' = dI + K'$ and $U^*S_\alpha U = B_{\frac{1}{d}} + K = \frac{1}{d}I + K$. Thus $I = (dI + K')(\frac{1}{d}I + K)$ and therefore

$$dK + \frac{K'}{d} + K'K = 0. \tag{3}$$

On the other hand, $U^*S_\alpha U = \frac{1}{d}I + K$ implies $S_\alpha = \frac{1}{d} + UKU^* = \frac{1}{d} + E$ where $E = UKU^* \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Hence

$$Z^{*n}S_\alpha Z^n \to \frac{1}{d} \tag{4}$$

as $Z^{*n}EZ^n \to 0$ (see [2] for the proof) strongly. Further, since $S_\alpha Z^n - Z^n S_\alpha = K_n$ for some $K_n \in \mathcal{LC}(L_a^2(\mathbb{C}))$, hence

$$Z^{*n}S_\alpha Z^n - S_\alpha = J_n \tag{5}$$

for some $J_n = Z^{*n}K_n \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Since $\{J_n\}$ converges strongly to 0, we obtain from (4) and (5) that $S_\alpha = \frac{1}{d}$. Hence $E = 0$ and therefore $K = 0$. It follows hence from (3) that $K' = 0$. Thus $U^*S_\alpha U = \frac{1}{d}$ and $U^*R_\alpha U = d$. Hence $S_\alpha = \frac{1}{d}$ and $R_\alpha = d$. Thus $M = W_\alpha R_\alpha = dW_\alpha$ and $H = S_\alpha W_\alpha^* = \frac{1}{d}W_\alpha^*$ and the theorem follows. □

References

[1] Ahlfors L. V. *Complex Analysis*. McGraw-Hill, 1979.

[2] Barria J., Halmos P. R. *Asymptotic Toeplitz operators*. Trans. Amer. Math. Soc., 1982, **273**, 621–630.

[3] Brown A., Halmos P. R. *Algebraic properties of Toeplitz operators*. J. Reine Angew. Math., 1964, **213**, 89–102.

[4] Englis M. *Density of algebras generated by Toeplitz operators on Bergman spaces*. Arkiv for Matematik, 1992, **30**, 227–243.

[5] Rudin W. *Real and complex Analysis*. 3rd Edition, McGraw-Hill, New York, 1987.

Namita Das
P. G. Department of Mathematics
Utkal University, Vani Vihar
Bhubaneswar, Orissa, India 751004
E-mail: *namitadas440@yahoo.co.in*

Received November 1, 2011