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# Interpolating Bézier spline curves with local control

A. P. Pobegailo

**Abstract.** The paper presents a technique for construction of interpolating spline curves in linear spaces by means of blending parametric curves. A class of polynomials which satisfy special boundary conditions is used for blending. Properties of the polynomials are stated. An application of the technique to construction of interpolating Bézier spline curves with local control is considered. The presented interpolating Bézier spline curves can be used in on-line geometric applications or for fast sketching and prototyping of spline curves in geometric design.

**Mathematics subject classification:** 65D05, 65D07, 65D17.
**Keywords and phrases:** Blending parametric curves, interpolating curves, spline curves, Bezier curves.

## 1 Introduction

Blending curves is an important technique for smoothing corners of curves in computer-aided geometric design. Besides the technique can be applied to the design of parametric spline curves which have local shape control. Firstly the construction of spline curves by linear blending of parabolic arcs was proposed by Overhauser [5] and considered by Rogers and Adams [8]. The construction of spline curves by linear blending of circular arcs was considered by Zavjalov, Leus, Skorospelov [15], Wenz [10] and Liska, Shashkov, Swartz [3]. The construction of spline curves by trigonometric blending of circular arcs was considered by Szilvási-Nagy, Vendel [12], Séquin, Kiha Lee, Jane Yen [11]. Using linear blending of conics for the construction of spline curves was considered by Chuan Sun, Huanxi Zhao [1]. The paper presents an approach to the construction of interpolating spline curves by means of blending quadric Bezier curves using a class of polynomials which ensure a necessary continuity of the designed curves. The properties of the polynomials are stated. The presented approach can be considered as a generalization of the linear blending.

## 2 Polynomials approximating a jump function

The purpose is to determine polynomials which can be used for smooth deformation of parametric curves in linear spaces. To solve the problem define polynomials which smoothly approximate the jump function

$$\delta(u)=\begin{cases}0, & 0\le u<1/2\\ 1/2, & u=1/2\\ 1, & 1/2<u\le 1\end{cases}.$$

It can be seen that the jump function $\delta(u)$ is infinitely smooth at the boundaries but has a discontinuity at the middle of the domain. In order to avoid the discontinuity approximate the jump function $\delta(u)$ by means of Bernstein polynomials

$$b_{n,m}(u)=\frac{n!}{m!(n-m)!}(1-u)^{n-m}u^m,\ u\in[0,1].$$

For this purpose introduce the following knot sequences:

$$(\underbrace{0,0,\dots,0}_{n},\underbrace{1,1,\dots,1}_{n}),\ n\in\mathbb{N}$$

and define the polynomials

$$w_n(u)=\sum_{i=0}^{n-1}0\cdot b_{2n-1,i}(u)+\sum_{i=n}^{2n-1}1\cdot b_{2n-1,i}(u)=\sum_{i=n}^{2n-1}b_{2n-1,i}(u)$$

for $n\in\mathbb{N}$ . It follows from this definition that the polynomials $w_n(u)$ have the following boundary values:

$$w_n(0)=0,\ w_n(1)=1 \tag{1}$$

and their derivatives satisfy the following boundary conditions:

$$w_n^{(m)}(0)=w_n^{(m)}(1)=0 \tag{2}$$

for $m\in\{1,2,...,n-1\}$ . The following polynomials:

$$w_1(u)=u,\ w_2(u)=3(1-u)u^2+u^3,\ w_3(u)=10(1-u)^2u^3+5(1-u)u^4+u^5$$

are usually used in geometric applications. The polynomials $w_n(u)$ have the following properties.

*Property 1.* The polynomials $w_n(u)$ satisfy the equation

$$w_n(u)+w_n(1-u)=1.$$

*Proof.* This property follows from the property of Bernstein polynomials

$$\sum_{m=0}^{n}b_{n,m}(u)=1,\ \forall n\in\mathbb{N}.$$

*Property 2.* The polynomials $w_n(u)$ are symmetric with respect to the point $u = 1/2$.

*Proof.* It follows from Property 1 that

$$w_n(1/2 + v) + w_n(1/2 - v) = 1, \ \forall v \in [-1/2, 1/2].$$

This means that polynomials $w_n(u)$ are symmetric with respect to the point $u = 1/2$.

*Property 3.*

$$\lim_{n \to \infty} \int_0^{1/2} w_n(u) du = 0.$$

*Proof.* It is obvious that the polynomials $w_n(u)$ can be represented by linear combinations of polynomials from the power polynomial basis $u^n$, $u^{n+1}$, $\ldots$, $u^{2n-1}$ with coefficients linearly depending on $n$. Then the indefinite integral of the polynomial $w_n(u)$ is a linear combination of the polynomials $u^{n+1}$, $u^{n+2}$, $\ldots$, $u^{2n}$ whose coefficients also linearly depend on $n$. Therefore the limit of the definite integrals equals zero.

It follows from Properties 2 and 3 that the polynomial $w_n(u)$ indefinitely close approaches the jump function $\delta(u)$ while its degree is rising.

*Property 4.* The polynomial $w_n(u)$ is a minimum of the functional

$$J_n(f) = \int_0^1 |f^{(n)}(u)|^2 du, \ \forall n \in \mathbb{N}$$

where the function $f(u)$, $u \in [0, 1]$, satisfies the following boundary conditions:

$$f(0) = 0, \ f(1) = 1, \ f^{(m)}(0) = f^{(m)}(1) = 0 \tag{3}$$

for $m \in \{1, 2, ..., n - 1\}$ .

*Proof.* Assume that a function $g(u)$ is a minimum of the functional $J_n(f)$. Consider the function

$$(g - w_n)(u) = g(u) - w_n(u).$$

Then

$$|(g - w_n)^{(n)}|^2 = |g^{(n)} - w_n^{(n)}|^2 = (g^{(n)})^2 - 2g^{(n)} w_n^{(n)} + (w_n^{(n)})^2.$$

or equivalently

$$|(g - w_n)^{(n)}|^2 = (g^{(n)})^2 - (w_n^{(n)})^2 - 2(g^{(n)} - w_n^{(n)}) w_n^{(n)}.$$

It follows from the last equation that

$$J_n(g - w_n) = J_n(g) - J_n(w_n) - 2 \int_0^1 (g^{(n)}(u) - w_n^{(n)}(u)) w_n^{(n)}(u) du.$$

The last integral can be computed by parts as follows:

$$\int_0^1 (g^{(n)}(u) - w_n^{(n)}(u))w_n^{(n)}(u)du = \int_0^1 w_n^{(n)}(u)d(g^{(n-1)}(u) - w_n^{(n-1)}(u)) =$$
$$= w_n^{(n)}(u)(g^{(n-1)}(u) - w_n^{(n-1)}(u))\Big|_0^1 - \int_0^1 (g^{(n-1)}(u) - w_n^{(n-1)}(u))w_n^{(n+1)}(u)du =$$
$$= -\int_0^1 (g^{(n-1)}(u) - w_n^{(n-1)}(u))w_n^{(n+1)}(u)du$$

taking into account that

$$g^{(n-1)}(0) = w_n^{(n-1)}(0) = 0,\ g^{(n-1)}(1) = w_n^{(n-1)}(1) = 0.$$

Recurrently computing the obtained integrals by parts and taking into account that the function $w_n^{(2n-1)}(u)$ is a constant it is obtained that

$$\int_0^1 (g^{(n-1)}(u) - w_n^{(n-1)}(u))w_n^{(n+1)}(u)du =$$
$$= (-1)^n \int_0^1 (g^{'}(u) - w_n^{'}(u))w_n^{(2n-1)}(u)du =$$
$$= (-1)^n (g(u) - w_n(u))w_n^{(2n-1)}(u)\Big|_0^1 = 0$$

because

$$g(0) = w_n(0) = 0,\ g(1) = w_n(1) = 1.$$

Thus it is proven that

$$J_n(g - w_n) = J_n(g) - J_n(w_n).$$

The last equation can be rewritten as follows:

$$J_n(g) = J_n(w_n) + J_n(g - w_n).$$

It follows from the definition of the functional $J_n(f)$ that

$$J_n(g - w_n) \geq 0.$$

Therefore

$$J_n(w_n) \leq J_n(g).$$

But the function $g(u)$ is a minimum of the functional $J_n(f)$ by assumption, therefore

$$g(u) = w_n(u).$$

Thus it is proven that the polynomial $w_n(u)$ is a minimum of the functional $J_n(f)$.

Now prove that this minimum is unique. Suppose the opposite. Let there exist such a function $g(u)$ which satisfies the condition

$$J_n(g) = J_n(w_n).$$

It follows from this equation that

$$J_n(g - w_n) = 0,$$

which is equivalent to

$$g^{(n)}(u) = w_n^{(n)}(u),\ \forall u \in [0,1].$$

It follows from the last equation that

$$g(u) = w_n(u) + \sum_{i=0}^{n-1} a_i u^i.$$

But the coefficients $a_i$ are equal to zero $\forall i \in \{0, 1, ..., n-1\}$ taking into account boundary conditions which must be satisfied by the function $g(u)$. Therefore

$$g(u) = w_n(u).$$

Thus the property is proven.

The functional $J_n(f)$ can be considered as energy of $n-th$ derivative of the function which satisfies boundary conditions (3). Property 4 shows that the polynomial $w_n(u)$ is a minimum of the functional $J_n(f)$.

The polynomials $w_n(u)$ were firstly introduced by the author [7, 8] for the construction of spline curves by blending of circular arcs and linear segments. Then the polynomials were used by the author for the construction of spline curves on smooth manifolds [9]. The polynomials were also used by Jakubiak, Leite and Rodrigues [3] for smooth spline generation on Riemannian manifolds and by Hartmann [2] for parametric $G^n$ blending of curves and surfaces. Wiltsche [14] proposed Bézier representation of the considered polynomials.

## 3 Polynomial blending of parametric curves

Consider two parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$, $u \in [0,1]$, which have the same boundary points, that is

$$\mathbf{p}_1(0) = \mathbf{p}_2(0),\ \mathbf{p}_1(1) = \mathbf{p}_2(1). \tag{4}$$

The problem is to construct a parametric curve $\mathbf{p}(u)$, $u \in [0,1]$, which has the boundaries

$$\mathbf{p}(0) = \mathbf{p}_1(0),\ \mathbf{p}(1) = \mathbf{p}_2(1) \tag{5}$$

and derivatives of the parametric curve $\mathbf{p}(u)$ satisfy the following boundary conditions:

$$\mathbf{p}^{(m)}(0) = \mathbf{p}_1^{(m)}(0),\ \mathbf{p}^{(m)}(1) = \mathbf{p}_2^{(m)}(1),\ \forall m \in \{1, 2, ..., n\}, \tag{6}$$

where $n \in \mathbb{N}$. In topology a parametric curve $\mathbf{p}(u)$ which satisfies conditions (5) is called a deformation of the parametric curve $\mathbf{p}_1(u)$ into the parametric curve $\mathbf{p}_2(u)$.

In this case the parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$ are called homotopic. In geometric design the parametric curve $\mathbf{p}(u)$ must satisfy additional boundary conditions (6) and in this case $\mathbf{p}(u)$ is called a parametric curve blending the parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$.

Using the polynomials $w_n(u)$ define a blending parametric curve $\mathbf{p}(u)$ as follows:

$$\mathbf{p}(u) = (1 - w_n(u))\mathbf{p}_1(u) + w_n(u)\mathbf{p}_2(u),\ u \in [0, 1]. \tag{7}$$

It follows form the definition of the polynomials $w_n(u)$ that the parametric curve $\mathbf{p}(u)$ satisfies conditions (5) because

$$\mathbf{p}(0) = (1 - w_n(0))\mathbf{p}_1(0) + w_n(0)\mathbf{p}_2(0) = \mathbf{p}_1(0) \tag{8}$$

and

$$\mathbf{p}(1) = (1 - w_n(1))\mathbf{p}_1(1) + w_n(1)\mathbf{p}_2(1) = \mathbf{p}_2(1). \tag{9}$$

Derivatives of the parametric curve $\mathbf{p}(u)$ can be computed as follows:

$$\mathbf{p}^{(m)}(u) = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!}((1 - w_n(u))^{(i)}\mathbf{p}_1^{(m-i)}(u) + w_n^{(i)}(u)\mathbf{p}_2^{(m-i)}(u)),\ \forall m \in \mathbb{N}.$$

Substitution of equations (2) into the last equation yields that

$$\mathbf{p}^{(m)}(0) = (1 - w_n(0))\mathbf{p}_1^{(m)}(0) + w_n(0)\mathbf{p}_2^{(m)}(0) = \mathbf{p}_1^{(m)}(0),\ \forall m \in \{1, 2, ..., n-1\}$$

and

$$\mathbf{p}^{(m)}(1) = (1 - w_n(1))\mathbf{p}_1^{(m)}(1) + w_n(1)\mathbf{p}_2^{(m)}(1) = \mathbf{p}_2^{(m)}(1),\ \forall m \in \{1, 2, ..., n-1\}.$$

Besides taking into account Equations (4) it is obtained that

$$\begin{gathered}\mathbf{p}^{(n)}(0) = -w_n^{(n)}(0)\mathbf{p}_1(0) + (1 - w_n(0))\mathbf{p}_1^{(n)}(0) + \\ + w_n^{(n)}(0)\mathbf{p}_2(0) + w_n(0)\mathbf{p}_2^{(n)}(0) = \mathbf{p}_1^{(n)}(0)\end{gathered}$$

and

$$\begin{gathered}\mathbf{p}^{(n)}(1) = -w_n^{(n)}(1)\mathbf{p}_1(1) + (1 - w_n(1))\mathbf{p}_1^{(n)}(1) + \\ + w_n^{(n)}(1)\mathbf{p}_2(1) + w_n(1)\mathbf{p}_2^{(n)}(1) = \mathbf{p}_2^{(n)}(1).\end{gathered}$$

Therefore the boundary conditions described by Equations (6) are also fulfilled.

The polynomials $w_n(u)$ can be considered as a generalization of the polynomial $w_1(u)$, which is widely used in geometric applications for blending. Blending by means of the polynomials $w_1(u)$ and $(1 - w_1(u))$ is called linear. It can be seen that the proposed approach for blending parametric curves ensures $C^n$ parametric continuity of a blending curve with the blended curves at its boundaries. Polynomial blending which ensures $G^n$ geometric continuity is considered in other articles by Hartmann [2], Meek and Walton [5].

## 4 Construction of spline curves by curve blending

The considered approach to blending of parametric curves can be used for the construction of spline curves in a linear space. Suppose that it is necessary to construct a spline curve $\mathbf{p}(u) \in C^n$, $n \in \mathbb{N}$, interpolating a sequence of knot points $\mathbf{p}_i$, $i \in \{1, 2, ..., l\}$, which belong to a linear space. In this case segments $\mathbf{p}_i(u)$, $0 < i < l$, of the spline curve are constructed by blending two predefined parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ using Equation (7) as follows:

$$\mathbf{p}_i(u) = (1 - w_n(u))\mathbf{p}_{i,1}(u) + w_n(u)\mathbf{p}_{i,2}(u) \tag{10}$$

where the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ must satisfy the following conditions:

$$\mathbf{p}_{i,1}(0) = \mathbf{p}_{i,2}(0) = \mathbf{p}_i, \ \mathbf{p}_{i,1}(1) = \mathbf{p}_{i,2}(1) = \mathbf{p}_{i+1}. \tag{11}$$

Besides in order to ensure $C^n$ continuity of the spline curve $\mathbf{p}(u)$ the parametric curves $\mathbf{p}_{i-1,2}(u)$ and $\mathbf{p}_{i,1}(u)$ must be smoothly joined at the point $\mathbf{p}_i$ that is

$$\mathbf{p}_{i-1,2}(1) = \mathbf{p}_{i,1}(0) = \mathbf{p}_i \tag{12}$$

and

$$\mathbf{p}_{i-1,2}^{(m)}(1) = \mathbf{p}_{i,1}^{(m)}(0), \ \forall m \in \{1, 2, ..., n\}. \tag{13}$$

Therefore in order to apply the proposed technique to the construction of spline curves the following problem must be solved: how to choose the parametric curves $\mathbf{p}_{i-1,2}(u)$ and $\mathbf{p}_{i,1}(u)$ which satisfy Equations (12) and (13). A solution of this problem depends on the application which uses the technique or more precisely on the modeled physical process. For example, circular arcs have been using for blending curves in the paper [7] because the application was intended for robot trajectory planning. At that time most robots supported only techniques for the interpolation of circular arcs and therefore it was not difficult to use deformation of circular arcs for the construction of spline trajectories. Generally, since spline curves constructed by the proposed technique have local shape control it is reasonable to suppose that the proposed technique will be very suitable to solve problems for on-line point interpolation.

## 5 Interpolating Bézier spline curves with local control

In geometric design a problem of choosing the model curve is motivated by two reasons: shape and smoothness control of modeled curves. Nowadays it is accepted that Bézier and B-spline curves are most suitable for this purpose. Taking into account these considerations and since the polynomials $w_n(u)$ are represented by means of Bernstein polynomials, Bézier curves are chosen for representation of the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ used for blending.

In order to reduce degree of the designed Bézier spline curve it is reasonable to use Bézier curves of the most low degree for blending. Therefore quadric Bézier curves are chosen for parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$. In order to ensure unique choice of the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ it is supposed that the parametric curves $\mathbf{p}_{i-1,2}(u)$ and $\mathbf{p}_{i,1}(u)$ are smoothly joined at the knot point $\mathbf{p}_i$. Analytical representation of such quadric Bézier curves $\mathbf{p}_{i-1,1}(u)$ and $\mathbf{p}_{i,2}(u)$ can be obtained from the following conditions:

$$\mathbf{p}_{i-1,2}(1) = \mathbf{p}_{i,1}(0),\ \mathbf{p}_{i-1,2}^{'}(1) = \mathbf{p}_{i,1}^{'}(0),\ \mathbf{p}_{i-1,2}^{''}(1) = \mathbf{p}_{i,1}^{''}(0). \tag{14}$$

In order to simplify index notations consider two quadric Bézier curves

$$\mathbf{p}_j(u) = (1-u)^2\mathbf{p}_{j,0} + 2(1-u)u\mathbf{p}_{j,1} + u^2\mathbf{p}_{j,2},\ j \in \{1,2\},$$

which have the following boundary points:

$$\mathbf{p}_1(0) = \mathbf{p}_0,\ \mathbf{p}_1(1) = \mathbf{p}_2(0) = \mathbf{p}_1,\ \mathbf{p}_2(1) = \mathbf{p}_2 \tag{15}$$

and are smoothly joined at the point $\mathbf{p}_1$, that is

$$\mathbf{p}_1^{'}(1) = \mathbf{p}_2^{'}(0),\ \mathbf{p}_1^{''}(1) = \mathbf{p}_2^{''}(0) \tag{16}$$

Resolution of these equations yields the following values of unknown knot and control points of the quadric Bézier curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$:

$$\mathbf{p}_{1,0} = \mathbf{p}_0,\ \mathbf{p}_{1,2} = \mathbf{p}_{2,0} = \mathbf{p}_1,\ \mathbf{p}_{2,2} = \mathbf{p}_2, \tag{17}$$

$$\mathbf{p}_{1,1} = \mathbf{p}_1 - \frac{\mathbf{p}_2 - \mathbf{p}_0}{4},\ \mathbf{p}_{2,1} = \mathbf{p}_1 + \frac{\mathbf{p}_2 - \mathbf{p}_0}{4}. \tag{18}$$

It follows from these constructions that the quadric Bézier curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$ are smoothly joined at the knot point $\mathbf{p}_1$ and therefore can be used for the construction of spline curves with any degree of continuity. Actually the parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$ are two segments of the same parabola. Besides the segments are joined at such the point $\mathbf{p}_1$ that the distance from the point to the line connecting the points $\mathbf{p}_0$ and $\mathbf{p}_2$ is maximal for all points belonging to the parabola.

Using Equations (17) and (18) Bézier curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ which are used for the construction of a Bézier spline curve can be determined as follows:

$$\mathbf{p}_{i,k}(u) = (1-u)^2\mathbf{p}_i + 2(1-u)u\mathbf{p}_{i,k} + u^2\mathbf{p}_{i+1},\ k \in \{1,2\}, \tag{19}$$

where

$$\mathbf{p}_{i,1} = \mathbf{p}_i + \frac{\mathbf{p}_{i+1} - \mathbf{p}_{i-1}}{4},\ \mathbf{p}_{i,2} = \mathbf{p}_{i+1} - \frac{\mathbf{p}_{i+2} - \mathbf{p}_i}{4}. \tag{20}$$

That is the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ are blended in Equation (10) for the construction of the spline curve segment $\mathbf{p}_i(u)$.

Find another representation of the spline curve segment $\mathbf{p}_i(u)$ which clarifies its geometric construction. For this purpose substitute the obtained expressions for parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ into Equation (10). It is obtained that

$$\mathbf{p}_i(u) = (1-u)^2\mathbf{p}_i + 2(1-u)u((1-w_n(u))\mathbf{p}_{i,1} + w_n(u)\mathbf{p}_{i,2}) + u^2\mathbf{p}_{i+1}$$

where the control points $\mathbf{p}_{i,1}$ and $\mathbf{p}_{i,2}$ are defined by Equations (20). This representation shows that the spline curve segment $\mathbf{p}_i(u)$ can be considered as a quadric Bézier curve with a smoothly modified control point.

A spline curve segment $\mathbf{p}_i(u)$ can be also represented as a Bézier curve. To obtain this representation modify Equation (10) using Equations (19) and taking into account Property 1 of the polynomials $w_n(u)$ as follows:

$$
\begin{aligned}
\mathbf{p}_i(u) &= (1 - w_n(u))\mathbf{p}_{i,1}(u) + w_n(u)\mathbf{p}_{i,2}(u) = \\
&= w_n(1-u)\mathbf{p}_{i,1}(u) + w_n(u)\mathbf{p}_{i,2}(u) = \\
&= \sum_{k=0}^{n-1} b_{2n-1,k}(u)\mathbf{p}_{i,1}(u) + \sum_{k=n}^{2n-1} b_{2n-1,k}(u)\mathbf{p}_{i,2}(u) = \\
&= \sum_{k=0}^{n-1} b_{2n-1,k}(u)(b_{2,0}(u)\mathbf{p}_i + b_{2,1}(u)\mathbf{p}_{i,1} + b_{2,2}(u)\mathbf{p}_{i+1}) + \\
&+ \sum_{k=n}^{2n-1} b_{2n-1,k}(u)(b_{2,0}(u)\mathbf{p}_i + b_{2,1}(u)\mathbf{p}_{i,2} + b_{2,2}(u)\mathbf{p}_{i+1}) = \\
&= \sum_{k=0}^{2n-1} b_{2n-1,k}(u)b_{2,0}(u)\mathbf{p}_i + \sum_{k=0}^{n-1} b_{2n-1,k}(u)b_{2,1}(u)\mathbf{p}_{i,1} + \\
&+ \sum_{k=n}^{2n-1} b_{2n-1,k}(u)b_{2,1}(u)\mathbf{p}_{i,2} + \sum_{k=0}^{2n-1} b_{2n-1,k}(u)b_{2,2}(u)\mathbf{p}_{i+1} = \\
&= \sum_{k=0}^{2n-1} b_{2n+1,k}(u)c_{0,k}\mathbf{p}_i + \sum_{k=1}^{n} b_{2n+1,k}(u)c_{1,k}\mathbf{p}_{i,1} + \\
&+ \sum_{k=n+1}^{2n} b_{2n+1,k}(u)c_{1,k}\mathbf{p}_{i,2} + \sum_{k=2}^{2n+1} b_{2n+1,k}(u)c_{2,k}\mathbf{p}_{i+1}
\end{aligned}
$$

where

$$
\begin{aligned}
c_{0,k} &= \frac{(2n-k)(2n-k+1)}{2n(2n+1)}, \ 0 \le k \le 2n-1, \\
c_{1,k} &= \frac{k(2n-k+1)}{n(2n+1)}, \ 1 \le k \le 2n, \\
c_{2,k} &= \frac{(k-1)k}{2n(2n+1)}, \ 2 \le k \le 2n+1.
\end{aligned}
$$

It follows from the obtained equations that the segment $\mathbf{p}_i(u)$ has the following Bézier representation:

$$
\begin{aligned}
\mathbf{p}_i(u) &= b_{2n+1,0}(u)\mathbf{p}_i + b_{2n+1,1}(u)(c_{0,1}\mathbf{p}_i + c_{1,1}\mathbf{p}_{i,1}) + \\
&+ \sum_{k=2}^{n} b_{2n+1,k}(u)(c_{0,k}\mathbf{p}_i + c_{1,k}\mathbf{p}_{i,1} + c_{2,k}\mathbf{p}_{i+1}) +
\end{aligned}
$$

$$+\sum_{k=n+1}^{2n-1} b_{2n+1,k}(u)(c_{0,k}\mathbf{p}_i + c_{1,k}\mathbf{p}_{i,2} + c_{2,k}\mathbf{p}_{i+1}) +$$
$$+ b_{2n+1,2n}(u)(c_{1,2n}\mathbf{p}_{i,2} + c_{2,2n}\mathbf{p}_{i+1}) + b_{2n+1,2n+1}(u)\mathbf{p}_{i+1}.$$

For example, segments of $C^1$ and $C^2$ continuous spline curves have the following Bézier representations:

$$\mathbf{p}_i(u) = b_{3,0}(u)\mathbf{p}_i + b_{3,1}(u)\frac{1}{3}(\mathbf{p}_i + 2\mathbf{p}_{i,2}) + +b_{3,2}(u)\frac{1}{3}(2\mathbf{p}_{i,1} + \mathbf{p}_{i+1}) + b_{3,3}(u)\mathbf{p}_{i+1},$$
$$\mathbf{p}_i(u) = b_{5,0}(u)\mathbf{p}_i + b_{5,1}(u)\frac{1}{5}(3\mathbf{p}_i + 2\mathbf{p}_{i,2}) + b_{5,2}(u)\frac{1}{10}(3\mathbf{p}_i + 6\mathbf{p}_{i,2} + \mathbf{p}_{i+1}) +$$
$$+ b_{5,3}(u)\frac{1}{10}(\mathbf{p}_i + 6\mathbf{p}_{i,1} + 3\mathbf{p}_{i+1}) + b_{5,4}(u)\frac{1}{5}(2\mathbf{p}_{i,1} + 3\mathbf{p}_{i+1}) + b_{5,5}(u)\mathbf{p}_{i+1},$$

respectively.

It can be seen that a segment $\mathbf{p}_i(u)$ of a $C^n$ continuous spline curve is a Bézier curve of degree $2n+1$. Let $\mathbf{p}_{i,k},\ k \in \{1, 2, \ldots, 2n\}$, denote control points of the spline curve segment $\mathbf{p}_i(u)$. It is known that Bézier curves have convex hull property, that is a Bézier curve lies completely in the convex hull of its control points. It follows from this property that deviation of a spline curve segment $\mathbf{p}_i(u)$ from the line segment $\overline{P_iP_{i+1}}$ can be estimated as follows:

$$\varepsilon < max(dist(\overline{P_iP_{i+1}}, \mathbf{p}_{i,k})),\ \forall k \in \{1, 2, \ldots, 2n\}.$$

## 6 Conclusions

The approach to the construction of $C^n$ continuous interpolating spline curves by means of blending quadric Bézier curves is introduced. Properties of the polynomials which are used for blending are considered. The considered spline curves are constructed locally, that ensures local shape control of the constructed spline curves. Bézier representations of the introduced spline curves is presented. The considered interpolating spline curves can be used in on-line geometric applications or for fast sketching and prototyping of spline curves in geometric design. It also can be noted that the proposed approach enables drawing of Bézier spline curves of $C^1$ continuity by means of any software packages which support drawing of cubic Bézier curves.

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A. P. Pobegailo
Department of Applied Mathematics
and Computer Science
Belarussian State University
pr. Nezavisimosty 4, Minsk
Belarus

E-mail: *pobegailo@bsu.by*

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