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Closure operators in the categories of modules. Part III (Operations in $\mathbb{CO}$ and their properties)

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Abstract. This article is a continuation of the works [1] and [2] (Part I and Part II) and contains some results on the family $\mathbb{CO}$ of all closure operators of a module category R-Mod. The principal operations in $\mathbb{CO}$ (meet, join, product, coproduct) are studied and their properties are elucidated. Also the question on the preservation of types of closure operators with respect to these operations is investigated.

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1 Introduction. Preliminary notions

Continuing the investigation of closure operators of a module category [1, 2], in this part of the work the principal operations are analyzed, which are defined in the family of all closure operators $\mathbb{CO}$ of a module category R-Mod: the meet, join, product and coproduct [1–5]. The properties of these operations will be studied, as well as the relations between them. Moreover, the types of operators are indicated (weakly hereditary, idempotent, hereditary, maximal, minimal, cohereditary) which are preserved by the application of these operations.

The main definitions and some preliminary results can be found in [1–6]. For convenience we would remind some necessary definitions and facts.

Let R be a ring with unit and R-Mod be the category of unitary left R-modules. For the module $M \in R$-Mod we denote by $\mathbb{L}(M)$ the lattice of all submodules of M. *A closure operator* in R-Mod is a function C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule of M denoted by $C_M(N)$ with the conditions: (c_1) $N \subseteq C_M(N)$; (c_2) if $N, P \in \mathbb{L}(M)$ and $N \subseteq P$, then $C_M(N) \subseteq C_M(P)$ (monotony); (c_3) if $f : M \to M'$ is an R-morphism and $N \subseteq M$, then $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ (continuity). We denote by $\mathbb{CO}$ the family of all closure operators of a category R-Mod.

The *principal operations* in $\mathbb{CO}$ are defined as follows, where $N \in \mathbb{L}(M)$:
1. *The meet* $\bigwedge\limits_{\alpha \in \mathfrak{A}} C_\alpha$ of a family $\{C_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$:

$$\Big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\Big)_M(N) = \bigcap_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_M(N)\big]; \tag{1.1}$$

2. *The join* $\bigvee_{\alpha\in\mathfrak{A}} C_\alpha$ of a family $\{C_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M(N) = \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_M(N)\big]; \tag{1.2}$$

3. *The product* $C \cdot D$ of two closure operators $C, D \in \mathbb{CO}$:

$$\big(C \cdot D\big)_M(N) = C_M\big(D_M(N)\big); \tag{1.3}$$

4. *The coproduct* $C \# D$ of two closure operators $C, D \in \mathbb{CO}$:

$$\big(C \# D\big)_M(N) = C_{D_M(N)}(N). \tag{1.4}$$

It is easy to observe that by the rules (1.1)–(1.4) we obtain the closure operators and the family $\mathbb{CO}$ of all closure operators of R-Mod is a complete "big lattice" with respect to the meet and join (which will be named the lattice operations). As to the other two operations we can remark that they are associative and for every $C, D \in \mathbb{CO}$ we have: $C \cdot D \geq C \vee D$, $C \# D \leq C \wedge D$ [3].

We remind in continuation the most important *types of closure operators* [1–3]. The operator $C \in \mathbb{CO}$ is called:

1) *weakly hereditary* if for every $N \subseteq M$ is true the relation:

$$C_{C_M(N)}(N) = C_M(N); \tag{1.5}$$

2) *idempotent* if for every $N \subseteq M$ we have:

$$C_M\big(C_M(N)\big) = C_M(N); \tag{1.6}$$

3) *hereditary* if for every submodules $L \subseteq N \subseteq M$ the relation holds:

$$C_N(L) = C_M(L) \cap N; \tag{1.7}$$

4) *cohereditary* if for every submodules $K, N \in \mathbb{L}(M)$ we have:

$$\big(C_M(N) + K\big)/K = C_{M/K}\big((N+K)/K\big); \tag{1.8}$$

5) *maximal* if for every $N \subseteq M$ is true the relation:

$$C_M(N)/N = C_{M/N}(\bar{0}); \tag{1.9}$$

or: for every submodules $K \subseteq N \subseteq M$ we have:

$$C_M(N)/K = C_{M/K}(N/K); \tag{1.9$'$}$$

6) *minimal* if for every $N \subseteq M$ is true the relation:

$$C_M(N) = C_M(0) + N; \tag{1.10}$$

or: for every submodules $L \subseteq N \subseteq M$ we have:

$$C_M(N) = C_M(L) + N. \tag{1.10$'$}$$

We remark the following known facts:

a) every hereditary closure operator is weakly hereditary;

b) every cohereditary closure operator is idempotent;

c) the operator $C \in \mathbb{CO}$ is cohereditary if and only if it is maximal and minimal ([2], Lemma 6.2).

2 Operations in $\mathbb{CO}$: distributivity

In this section we will study the interaction between the lattice operations $(\wedge)$, $(\vee)$ of $\mathbb{CO}$ and the operations $(\cdot)$, $(\#)$ of product and coproduct. We begin with the following *relations of distributivity*.

Proposition 2.1. *For every family of closure operators* $\{C_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ *and for every operator* $D \in \mathbb{CO}$ *the following relations hold:*

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D); \tag{2.1}$$

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D); \tag{2.2}$$

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \# D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \# D); \tag{2.3}$$

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big) \# D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \# D). \tag{2.4}$$

Proof. (2.1). For every $N \subseteq M$ from the definitions of operations it follows:

$$\Big[\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D\Big]_M(N) = \Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M\big(D_M(N)\big) = \bigcap_{\alpha\in\mathfrak{A}} \Big[\big(C_\alpha\big)_M\big(D_M(N)\big)\Big] =$$

$$= \bigcap_{\alpha\in\mathfrak{A}} \Big[\big(C_\alpha \cdot D\big)_M(N)\Big] = \Big[\bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D)\Big]_M(N),$$

therefore the relation (2.1) is true.

(2.2). By definition for every $N \subseteq M$ we have:

$$\Big[\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D\Big]_M(N) = \Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M\big(D_M(N)\big) = \sum_{\alpha\in\mathfrak{A}} \Big[\big(C_\alpha\big)_M\big(D_M(N)\big)\Big] =$$

$$= \sum_{\alpha\in\mathfrak{A}} \Big[\big(C_\alpha \cdot D\big)_M(N)\Big] = \Big[\bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D)\Big]_M(N),$$

hence the relation (2.2) holds.

(2.3). For $N \subseteq M$ by definition we obtain:

$$\Big[\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \# D\Big]_M(N) = \Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_{D_M(N)}(N) = \bigcap_{\alpha\in\mathfrak{A}} \Big[\big(C_\alpha\big)_{D_M(N)}(N)\Big] =$$

$$= \bigcap_{\alpha\in\mathfrak{A}} \Big[\big(C_\alpha \# D\big)_M(N)\Big] = \Big[\bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \# D)\Big]_M(N),$$

which proves (2.3).

(2.4). Similarly, for every $N \subseteq M$ we have:

$$\big[\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big) \,\#\, D\big]_M(N) = \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_{D_M(N)}(N) = \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_{D_M(N)}(N)\big] =$$

$$= \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha \,\#\, D)_M(N)\big] = \big[\bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \,\#\, D)\big]_M(N),$$

hence (2.4) is true. □

The other relations of distributivity of the indicated types can be obtained by some *supplementary conditions* on the operators. To concretize this idea we need the following two auxiliary statements.

Lemma 2.2. *If $C \in \mathbb{CO}$ is a* ***hereditary*** *closure operator, then it preserves the intersection in the superior term, i.e. for every family of submodules $\{N_\alpha \in \mathbb{L}(M) \,|\, \alpha \in \mathfrak{A}\}$ and every submodule $K \subseteq N_\alpha$ $(\alpha \in \mathfrak{A})$ the following relation holds:*

$$C_{\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha}(K) = \bigcap_{\alpha\in\mathfrak{A}} \big[C_{N_\alpha}(K)\big]. \tag{2.5}$$

Proof. From the heredity of $C \in \mathbb{CO}$ $\big($see (1.7)$\big)$ in the situation $K \subseteq N_\alpha \subseteq M$ we obtain $C_{N_\alpha}(K) = C_M(K) \cap N_\alpha$ for every $\alpha \in \mathfrak{A}$, therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} \big[C_{N_\alpha}(K)\big] = C_M(K) \cap \big(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha\big)$.

On the other hand, by the hereditary of C in the situation $K \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq M$ we have $C_{\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha}(K) = C_M(K) \cap \big(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha\big)$ and comparing with the previous relation we obtain (2.5). □

Lemma 2.3. *If $C \in \mathbb{CO}$ is a* ***minimal*** *closure operator, then it preserves the sum in the inferior term, i.e. for every family of submodules $\{N_\alpha \in \mathbb{L}(M) \mid \alpha \in \mathfrak{A}\}$ the relation is true:*

$$C_M\Big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\Big) = \sum_{\alpha\in\mathfrak{A}} \big[C_M(N_\alpha)\big]. \tag{2.6}$$

Proof. Let $L \subseteq N_\alpha \subseteq M$. From the minimality of C $\big($see (1.10′)$\big)$ it follows that

$$\sum_{\alpha\in\mathfrak{A}} \big[C_M(N_\alpha)\big] = \sum_{\alpha\in\mathfrak{A}} \big[C_M(L) + N_\alpha\big] = C_M(L) + \big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\big).$$

By the minimality of C in the situation $L \subseteq \sum\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq M$ we have

$$C_M\big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\big) = C_M(L) + \big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\big),$$

hence (2.6) is true. □

Using the Lemmas 2.2 and 2.3 we obtain the following relations of distributivity.

Proposition 2.4. a) *If the closure operator $C \in \mathbb{CO}$ is **hereditary**, then for every family of closure operators $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ the following relation holds:*

$$C \,\#\, \big(\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha \big) = \bigwedge_{\alpha \in \mathfrak{A}} (C \,\#\, D_\alpha). \tag{2.7}$$

b) *If the closure operator $C \in \mathbb{CO}$ is **minimal**, then for every family of closure operators $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ the relation is true:*

$$C \cdot \big(\bigvee_{\alpha \in \mathfrak{A}} D_\alpha \big) = \bigvee_{\alpha \in \mathfrak{A}} (C \cdot D_\alpha). \tag{2.8}$$

Proof. a) For every $N \subseteq M$ from the definitions it follows that:

$$\big[C \,\#\, \big(\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha \big)\big]_M(N) = C_{(\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha)_M(N)}(N) = C_{\bigcap_{\alpha \in \mathfrak{A}} [(D_\alpha)_M(N)]}(N);$$

$$\big[\bigwedge_{\alpha \in \mathfrak{A}} \big(C \,\#\, D_\alpha\big)\big]_M(N) = \bigcap_{\alpha \in \mathfrak{A}} [(C \,\#\, D_\alpha)_M(N)] = \bigcap_{\alpha \in \mathfrak{A}} [C_{(D_\alpha)_M(N)}(N)].$$

By assumption the operator C is hereditary, therefore it preserves the intersection in superior term (Lemma 2.2). The application of (2.5) in our case shows that the right sides of the previous relations coincide, therefore (2.7) is true.

b) For every $N \subseteq M$ we have:

$$\big[C \cdot \big(\bigvee_{\alpha \in \mathfrak{A}} D_\alpha \big)\big]_M(N) = C_M\big[\big(\bigvee_{\alpha \in \mathfrak{A}} D_\alpha \big)_M(N)\big] = C_M\big[\sum_{\alpha \in \mathfrak{A}} \big((D_\alpha)_M(N)\big)\big];$$

$$\big[\bigvee_{\alpha \in \mathfrak{A}} (C \cdot D_\alpha)\big]_M(N) = \sum_{\alpha \in \mathfrak{A}} \big[\big(C \cdot D_\alpha\big)_M(N)\big] = \sum_{\alpha \in \mathfrak{A}} \big[C_M\big((D_\alpha)_M(N)\big)\big].$$

The operator C is minimal, hence it preserves the sum in the inferior term (Lemma 2.3). By the relation (2.6) we obtain that the right sides of the previous equalities coincide. This proves (2.8). ☐

To give a complete picture we can mention also the last two possible cases of distributivity of considered operations, which are obtained by some supplementary assumptions on the closure operators.

Proposition 2.5. a) *If the closure operator $C \in \mathbb{CO}$ preserves the intersection in the inferior term, i.e.*

$$C_M\big(\bigcap_{\alpha \in \mathfrak{A}} N_\alpha \big) = \bigcap_{\alpha \in \mathfrak{A}} \big[C_M(N_\alpha)\big], \tag{2.9}$$

where $\{N_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{L}(M)$, then for every family of closure operators $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ the relation holds:

$$C \cdot \big(\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha \big) = \bigwedge_{\alpha \in \mathfrak{A}} \big(C \cdot D_\alpha\big). \tag{2.10}$$

b) *If the closure operator* $C \in \mathbb{CO}$ *preserves the sum in the superior term, i.e.*

$$C_{\sum\limits_{\alpha\in\mathfrak{A}}(N_\alpha)}(N) = \sum_{\alpha\in\mathfrak{A}} \big[C_{N_\alpha}(N)\big], \tag{2.11}$$

where $\{N_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{L}(M)$, $N \subseteq N_\alpha$ $(\alpha \in \mathfrak{A})$, *then for every family of closure operators* $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ *the relation is true:*

$$C \,\#\, \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big) = \bigvee_{\alpha\in\mathfrak{A}} \big(C \,\#\, D_\alpha\big). \tag{2.12}$$

Proof. a) For every $N \subseteq M$ we have:

$$\big[C \cdot \big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big)\big]_M(N) = C_M\big[\big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big)_M(N)\big] = C_M\big[\bigcap_{\alpha\in\mathfrak{A}} \big((D_\alpha)_M(N)\big)\big];$$

$$\big[\bigwedge_{\alpha\in\mathfrak{A}} (C \cdot D_\alpha)\big]_M(N) = \bigcap_{\alpha\in\mathfrak{A}} [(C \cdot D_\alpha)_M(N)] = \bigcap_{\alpha\in\mathfrak{A}} \big[C_M\big((D_\alpha)_M(N)\big)\big].$$

By assumption the operator C preserves the intersection in the inferior term, and so applying the relation (2.9) we see that the right sides of the previous equalities coincide, therefore (2.10) is true.

b) Similarly, for every $N \subseteq M$ we have:

$$\big[C \,\#\, \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big)\big]_M(N) = C_{(\bigvee\limits_{\alpha\in\mathfrak{A}} D_\alpha)_M(N)}(N) = C_{\sum\limits_{\alpha\in\mathfrak{A}}[(D_\alpha)_M(N)]}(N);$$

$$\big[\bigvee_{\alpha\in\mathfrak{A}} (C \,\#\, D_\alpha)\big]_M(N) = \sum_{\alpha\in\mathfrak{A}} [(C \,\#\, D_\alpha)_M(N)] = \sum_{\alpha\in\mathfrak{A}} \big[C_{(D_\alpha)_M(N)}(N)\big].$$

By hypothesis C preserves the sum in the superior term, and applying (2.11) now we obtain (2.12). □

3 Principal operations and preservation of types of operators

Now we will study the question on the behaviour of closure operators when the principal operations are applied. For that we consider consecutively all principal operations of $\mathbb{CO}$ and show the types of closure operators which are preserved by the application of given operation. Some similar facts are mentioned in [3].

a) *The join in* $\mathbb{CO}$

Proposition 3.1. *If the closure operators* C_α $(\alpha \in \mathfrak{A})$ *of* $\mathbb{CO}$ *are* ***weakly hereditary***, *then the operator* $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ *also is weakly hereditary.*

Proof. By the monotony and weak heredity of C_α $\big($see (1.5)$\big)$, for every $N \subseteq M$ and $\alpha \in \mathfrak{A}$ we have:

$$(C_\alpha)_{\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)]}(N) \supseteq (C_\alpha)_{(C_\alpha)_M(N)}(N) = (C_\alpha)_M(N),$$

and from the relation $\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)] \subseteq M$ the inverse inclusion follows. Therefore

$$(C_\alpha)_{\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)]}(N) = (C_\alpha)_M(N)$$

for every $\alpha \in \mathfrak{A}$, consequently

$$\sum_{\alpha\in\mathfrak{A}}[(C_\alpha)_{\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)]}(N)] = \sum_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)].$$

From the definition of join in $\mathbb{CO}$ now we have:

$$\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_{(\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha)_M(N)}(N) = \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N),$$

i.e. the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is weakly hereditary. □

Proposition 3.2. *If the closure operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **maximal**, then the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is also maximal.*

Proof. By definition $\big($see (1.9$'$)$\big)$ for every submodules $K \subseteq N \subseteq M$ and every $\alpha \in \mathfrak{A}$ we have $[(C_\alpha)_M(N)]/K = (C_\alpha)_{M/K}(N/K)$. Using this relation we obtain:

$$\big[\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N)\big]/K = \big[\sum_{\alpha\in\mathfrak{A}}\big((C_\alpha)_M(N)\big)\big]/K = \sum_{\alpha\in\mathfrak{A}}\big[\big((C_\alpha)_M(N)\big)/K\big] =$$

$$= \sum_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_{M/K}(N/K)\big] = \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_{M/K}(N/K),$$

which means that the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is maximal. □

Proposition 3.3. *If the closure operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **minimal**, then the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is also minimal.*

Proof. We consider the situation: $L \subseteq N \subseteq M$. The minimality of C_α $\big($see (1.10$'$)$\big)$ implies $(C_\alpha)_M(N) = (C_\alpha)_M(L) + N$. Using this relation we obtain:

$$\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \sum_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_M(N)\big] = \sum_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_M(L) + N\big] =$$

$$= \big[\sum_{\alpha\in\mathfrak{A}}\big((C_\alpha)_M(L)\big)\big] + N = \big[\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(L)\big] + N,$$

therefore the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is minimal. □

Taking into account that $C \in \mathbb{CO}$ is cohereditary if and only if it is maximal and minimal (see Section 1), from Propositions 3.2 and 3.3 follows

Corollary 3.4. *If the operators C_α $(\alpha \in \mathfrak{A})$ are **cohereditary**, then the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is also cohereditary.* □

b) *The meet in* $\mathbb{CO}$

Proposition 3.5. *If the operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **hereditary**, then the operator $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is also hereditary.*

Proof. By definition (see (1.7)) the heredity of C_α means that for every submodules $L \subseteq N \subseteq M$ we have $(C_\alpha)_N(L) = (C_\alpha)_M(L) \cap N$. Therefore:

$$\big[\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(L)\big] \cap N = \big[\bigcap_{\alpha\in\mathfrak{A}} \big((C_\alpha)_M(N)\big)\big] \cap N = \bigcap_{\alpha\in\mathfrak{A}} \big[\big((C_\alpha)_M(L)\big) \cap N\big] =$$

$$= \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_N(L)\big] = \big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)_N(L),$$

so $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is hereditary. □

Proposition 3.6. *If the operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **maximal**, then the operator $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is also maximal.*

Proof. In the situation $K \subseteq N \subseteq M$ the maximality of C_α (see (1.9$'$)) implies the relation $[(C_\alpha)_M(N)]\,/\,K = (C_\alpha)_{M/K}(N/K)$. Therefore $\bigcap\limits_{\alpha\in\mathfrak{A}}\big[\big((C_\alpha)_M(N)\big)/K\big] = \bigcap\limits_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_{M/K}(N/K)\big]$, and so $\big[\bigcap\limits_{\alpha\in\mathfrak{A}}\big((C_\alpha)_M(N)\big)\big]/K = \bigcap\limits_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_{M/K}(N/K)\big]$. Now by the definition of the meet in $\mathbb{CO}$ it is clear that $\big[\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N)\big]/K = \big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_{M/K}(N/K)$, i.e. the operator $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is maximal. □

c) *The product in* $\mathbb{CO}$

Proposition 3.7. *If the closure operators $C, D \in \mathbb{CO}$ are **maximal**, then the operator $C \cdot D$ is also maximal.*

Proof. Let $K \subseteq N \subseteq M$. The maximality of C and D implies the relations $C_M(N)/K = C_{M/K}(N/K)$ and $D_M(N)/K = D_{M/K}(N/K)$, which permit to obtain:

$$\big[(C \cdot D)_M(N)\big]/K = \big[C_M\big(D_M(N)\big)\big]/K = C_{M/K}\big[D_M(N)/K\big] =$$

$$= C_{M/K}\big[D_{M/K}(N/K)\big] = (C \cdot D)_{M/K}(N/K).$$

This shows that the operator $C \cdot D$ is maximal. □

Proposition 3.8. *If the closure operators $C, D \in \mathbb{CO}$ are **minimal**, then the operator $C \cdot D$ is also minimal.*

Proof. Let $L \subseteq N \subseteq M$. By the minimality of C and D we have $C_M(N) = C_M(L) + N$ and $D_M(N) = D_M(L) + N$. From the second relation we obtain $(C \cdot D)_M(N) = C_M\big(D_M(N)\big) = C_M\big(D_M(L) + N\big)$, and from the first relation in the situation $D_M(L) \subseteq D_M(L) + N \subseteq N$ we have:

$$C_M\big(D_M(L) + N\big) = C_M\big(D_M(L) + (D_M(L) + N)\big) = \big[C_M\big(D_M(L)\big)\big] + N.$$

Since $\big[\big(C \cdot D\big)_M(L)\big] + N = \big[C_M\big(D_M(L)\big)\big] + N$, now it is clear that $\big(C \cdot D\big)_M(N) = \big(C \cdot D\big)_M(L) + N$, i.e. the operator $C \cdot D$ is minimal. $\square$

From Propositions 3.7 and 3.8 follows

Corollary 3.9. *If the closure operators $C, D \in \mathbb{CO}$ are **cohereditary**, then the operator $C \cdot D$ is also cohereditary.* $\square$

The preservation of some properties of closure operators under the application of the operation of product can be obtained by some *addititional conditions* on the operators. We show in continuation two examples of such situations.

Example 1. Let $C, D \in \mathbb{CO}$ and $C \cdot D = D \cdot C$. If the operators C and D are *idempotent*, then the operator $C \cdot D$ is also idempotent.

Example 2. Let $C \in \mathbb{CO}$ preserves the intersection in the inferior term: $C_M(N_1 \cap N_2) = C_M(N_1) \cap C_M(N_2)$, where $N_1, N_2 \in \mathbb{L}(M)$. If the operators $C, D \in \mathbb{CO}$ are *hereditary*, then the operator $C \cdot D$ is also hereditary. Indeed, if $L \subseteq N \subseteq M$, then by hypotheses $C_N(L) = C_M(L) \cap N$ and $D_N(L) = D_M(L) \cap N$. Since C preserves the intersections, we have $C_M\big[D_M(L) \cap N\big] = \big[C_M\big(D_M(L)\big)\big] \cap C_M(N)$. This relation together with the heredity of C and D implies:

$$\big(C \cdot D\big)_N(L) = C_N\big(D_N(L)\big) = C_N\big[D_M(L) \cap N\big] =$$

$$= \big[C_M\big(D_M(L) \cap N\big)\big] \cap N = \big[\big(C_M(D_M(L))\big) \cap C_M(N)\big] \cap N =$$

$$= \big[C_M\big(D_M(L)\big)\big] \cap N = \big[\big(C \cdot D\big)_M(L)\big] \cap N,$$

i.e. $C \cdot D$ is hereditary.

d) *The coproduct in* $\mathbb{CO}$

Proposition 3.10. *If the closure operators $C, D \in \mathbb{CO}$ are **hereditary**, then the operator $C \# D$ is also hereditary.*

Proof. Let $L \subseteq N \subseteq M$. By the definition of coproduct and heredity of C in the situation $L \subseteq D_N(L) \subseteq M$ we obtain:

$$(C \# D)_N(L) = C_{D_N(L)}(L) = C_M(L) \cap D_N(L).$$

On the other hand, by definition we have:

$$\big[(C \# D)_M(L)\big] \cap N = \big[C_{D_M(L)}(L)\big] \cap N,$$

and applying the heredity of C in the situation $L \subseteq D_M(L) \subseteq M$, we obtain $C_{D_M(L)}(L) = C_M(L) \cap D_M(L)$. These facts together with the heredity of D $\big($i.e. $D_M(L) \cap N = D_N(L)\big)$ show that

$$\big[(C \# D)_M(L)\big] \cap N = \big[C_{D_M(L)}(L)\big] \cap N =$$

$$= \big[C_M(L) \cap D_M(L)\big] \cap N = C_M(L) \cap D_N(L).$$

Comparing with the foregoing, we conclude that $(C \# D)_N(L) = \big[(C \# D)_M(L)\big] \cap N$, i.e. $C \# D$ is hereditary. □

Proposition 3.11. *If the operators $C, D \in \mathbb{CO}$ are **maximal**, then the operator $C \# D$ is also maximal.*

Proof. Let $K \subseteq N \subseteq M$. By the maximality of C and D we have $C_M(N)/K = C_{M/K}(N/K)$ and $D_M(N)/K = D_{M/K}(N/K)$. These relations and the definition of coproduct imply:

$$\big[(C \# D)_M(N)\big]/K = \big[C_{D_M(N)}(N)\big]/K = C_{D_M(N)/K}(N/K) =$$

$$= C_{D_{M/K}(N/K)}(N/K) = (C \# D)_{M/K}(N/K),$$

therefore $C \# D$ is maximal. □

Proposition 3.12. *If the closure operators $C, D \in \mathbb{CO}$ are **cohereditary**, then the operator $C \# D$ is also cohereditary.*

Proof. Let $K, N \in \mathbb{L}(M)$. Since C and D are cohereditary we have:

$$\big[C_M(N) + K\big]/K = C_{M/K}\big[(N + K)/K\big];$$

$$\big[D_M(N) + K\big]/K = D_{M/K}\big[(N + K)/K\big].$$

From these relations and the definition of coproduct we obtain:

$$\big[\big((C \# D)_M(N)\big) + K\big]/K = \big[\big(C_{D_M(N)}(N)\big) + K\big]/K =$$

$$= C_{(D_M(N)+K)/K}\big((N + K)/K\big) = C_{D_{M/K}((N+K)/K)}\big((N + K)/K\big) =$$

$$= (C \# D)_{M/K}\big((N + K)/K\big),$$

hence the operator $C \# D$ is cohereditary. □

Similarly to the case of product (see Example 1) the commutativity $C \# D = D \# C$ implies the preservation of weak heredity, i.e. if C, D are *weakly hereditary*, then the operator $C \# D$ is also weakly hereditary, which can be proved by the direct verification.

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