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On stability of multicriteria investment Boolean problem with Wald's efficiency criteria

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Abstract. Based on Markowitz's portfolio theory we construct the multicriteria Boolean problem with Wald's maximin efficiency criteria and the Pareto-optimality principle. We obtained lower and upper attainable bounds for the stability radius of the problem in the cases of linear metric l_1 in the portfolio and the market state spaces and of the Chebyshev metric l_∞ in the criteria space.

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Back in the early XX century J. Hadamard included the stability in the concept of the correct mathematical problem as a necessary condition that reflects some physical reality. Subsequently it was found that many mathematical problems are unstable to small changes in input data (parameters). In 1960 this led to the creation of the theory of ill-posed problems, basics of which were laid by A. N. Tikhonov, M. M. Lavrentiev, V. K. Ivanov and others (see e. g. [1–3]).

Usually, the stability of the optimization problem (both scalar and vector) is understood as one of the classical properties of continuity or semi-continuity optimal mapping [4–7]. In the case of the discrete problem the definition of the stability rephrases easily in terms of the existence of 'the stability ball', i. e. a surroundings of the initial data in the problem parameter space, that any 'perturbed' problem with the parameters from this surroundings has some property of invariance to the original problem.

The widespread occurrence of discrete optimization models has given a start to the interest of many experts to studying various types of stability aspects, parametric and post-optimal analysis of both scalar (single criterion) and vector (multicriteria) discrete optimization problems (e. g. monographs [7–9], surveys [10–12], and annotated bibliographies [13, 14]).

One of the well-known approaches to the stability analysis of multicriteria discrete optimization problems is focused on obtaining quantitative characteristics of the stability and consists in finding an ultimate level of perturbations of the initial data of the problem that do not result in new Pareto-optimal solutions. The majority of the results in this field is related to deriving formulas or estimates for the stability radius of multicriteria problems of Boolean and integer programming with linear criteria [12, 15–18].

In the present paper we continue the started in [19–25] research of varies types of the stability of multicriteria non-linear investment problems, formulation of which is based on Markowitz's classical portfolio theory. Here we obtained lower and upper attainable bounds for the stability radius of the multicriteria investment problem with Wald's maximin economic efficiency criteria and the Pareto-optimal principle in the case of the linear metric l_1 in the portfolio and the market state spaces, and the Chebyshev metric l_∞ in the efficiency criteria space. We notice that in [26] with such combination of metrics l_1 and l_∞ i the similar lower and upper bounds of the stability radius of the multicriteria investment problem with Wald's ordered minimax criteria were announced.

1 Problem statement and basic definitions

We consider the multicriteria discrete variant of Markowitz's investment managing problem [27]. To this end, we introduce the following notations. Let $N_n = \{1, 2, \dots, n\}$ be the set of alternative investment projects (assets); N_m be the set of possible market states (situation); $x = (x_1, x_2, \dots, x_n)^T \in X \subseteq \mathbf{E}^n$ be the investment portfolio with components $x_j = 1$ if investment project $j \in N_n$ is implemented, and $x_j = 0$ otherwise. Here $\mathbf{E} = \{0, 1\}$.

There are several approaches to evaluate the efficiency of investment projects (NPV, NFV, PI et al.), which take into account risk and uncertainty in different ways (see e. g. [28–31]). Let N_s be the set of project efficiency indicator. An investment portfolio x is evaluated by $\sum\limits_{j\in N_n} e_{ijk}x_j$, where e_{ijk} is the predicted economic efficiency of the indicator $k \in N_s$ of the investment project $j \in N_n$ in the case when the market is in the state $i \in N_m$. In this context the initial data of the problem is a 3-dimensional matrix of the project efficiency E of the size $m \times n \times s$ with elements e_{ijk} from $\mathbf{R}$.

Let the following vector objective function

$$f(x, E) = (f_1(x, E_1), f_2(x, E_2), \dots, f_s(x, E_s)),$$

be given on a set of investment portfolios X whose components are Wald's maximin criteria (extreme pessimism) [32]

$$f_k(x, E_k) = \min_{i\in N_m} E_{ik}x = \min_{i\in N_m} \sum_{j\in N_n} e_{ijk}x_j \to \max_{x\in X}, \qquad k \in N_s,$$

where $E_k \in \mathbf{R}^{m\times n}$ is the k-th cut of the 3-dimension matrix $E = [e_{ijk}] \in \mathbf{R}^{m\times n\times s}$, $E_{ik} = (e_{i1k}, e_{i2k}, ..., e_{ink})$ is the i-th row of that cut. Thus, the investor in the unstable economic state, following Wald's criteria, takes extreme caution and optimizes portfolio efficiency $E_{ik}x$ assuming that the market is in the worst state. Such caution is appropriate, because the investment is the exchange of a certain value today for an uncertain value in the future.

A multicriteria investment Boolean problem $Z^s(E)$, $s \geq 1$, means the problem of searching the Pareto set $P^s(E)$, i. e. the Pareto-optimal investment portfolios, where

$$P^s(E) = \{x \in X : \ P^s(x,E) = \emptyset\},$$
$$P^s(x,E) = \{x' \in X : \ x' \underset{E}{\succ} x\},$$
$$x' \underset{E}{\succ} x \quad \Leftrightarrow \quad g(x',x,E) \geq \mathbf{0}^{(s)} \ \ \& \ \ g(x',x,E) \neq \mathbf{0}^{(s)},$$
$$g(x',x,E) = (g_1(x',x,E_1), g_2(x',x,E_2), \dots, g_s(x',x,E_s)),$$
$$g_k(x',x,E_k) = f_k(x',E_k) - f_k(x,E_k) = \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x' - E_{ik}x), \ \ k \in N_s,$$
$$\mathbf{0}^{(s)} = (0,0,\dots,0)^T \in \mathbf{R}^s.$$

In the portfolio space $\mathbf{R}^n$ and the market state space $\mathbf{R}^m$ we define the linear metric l_1, and in the efficiency criteria space $\mathbf{R}^s$ we define the Chebyshev metric l_∞, i. e. for any matrix $E \in \mathbf{R}^{m \times n \times s}$

$$\|E_{ik}\|_1 = \sum_{j \in N_n} |e_{ijk}|, \quad i \in N_m, \quad k \in N_s,$$
$$\|E_k\|_{11} = \sum_{i \in N_m} \|E_{ik}\|_1 = \sum_{i \in N_m} \sum_{j \in N_n} |e_{ijk}|, \quad k \in N_s,$$
$$\|E\|_{11\infty} = \max_{k \in N_s} \|E_k\|_{11} = \max_{k \in N_s} \sum_{i \in N_m} \|E_{ik}\|_1 = \max_{k \in N_s} \sum_{i \in N_m} \sum_{j \in N_n} |e_{ijk}|.$$

Thus, for any indexes $i \in N_m$ and $k \in N_s$,the following inequalities are true:

$$\|E_{ik}\|_1 \leq \|E_k\|_{11} \leq \|E\|_{11\infty}.$$

Apart from that, using the evident relation $E_{ik}x \geq -\|E_{ik}\|_1$, $x \in \mathbf{E}^n$, it is easy to see that for any portfolios x, x' the following inequalities hold:

$$E_{ik}x - E_{i'k}x' \geq -\|E_k\|_{11}, \quad i, \ i' \in N_m, \quad k \in N_s. \tag{1}$$

As usually [12,15,17], the stability radius of the problem $Z^s(E)$, $s \geq 1$, is defined as the number

$$\rho = \rho(m,n,s) = \begin{cases} \sup \ \Xi, & \text{if } \Xi \neq \emptyset, \\ 0, & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \ \forall E' \in \Omega(\varepsilon) \quad (P^s(E+E') \subseteq P^s(E))\},$$

$\Omega(\varepsilon) = \{E' \in \mathbf{R}^{m \times n \times s} : \ 0 < \|E'\|_{11\infty} < \varepsilon\}$ is the set of perturbing matrices, $P^s(E+E')$ is the Pareto set of the perturbed problem $Z^s(E+E')$. Thus, the stability radius defines an extreme level of perturbations of the elements of the matrix E such that new Pareto-optimal portfolios do not appear. In this context the stability of the problem $Z^s(E)$ is when the set Ξ is not empty, i. e. $\rho(m,n,s) > 0$.

Thus, the problem stability $Z^s(E)$ can be considered as the discrete analogue of the upper Hausdorff semicontinuity problem [5–7] at point E of the optimal mapping

$$P^s : \quad \mathbf{R}^{m \times n \times s} \to 2^{\mathbf{E}^n},$$

i.e. the point-set mapping which puts in correspondence the set of Pareto-optimal portfolios to each point of the space of problem parameters.

Obviously, if the equality $P^s(E) = X$ holds, the stability radius of the problem $Z^s(E)$ equals infinity. Therefore, in what follows, we will not consider this case and will call the problem $Z^s(E)$ for which the set $X \setminus P^s(E)$ is nonempty nontrivial one.

2 Stability radius bounds

For a nontrivial problem $Z^s(E)$ denote

$$\varphi = \varphi(m, n, s) = \min_{x \notin P^s(E)} \max_{x' \in P^s(x,E)} \min_{k \in N_s} \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x' - E_{ik}x).$$

Whereas for any portfolio $x \notin P^s(E)$ the set $P^s(x, E)$ is not empty, then we have the formula

$$\forall x \notin P^s(E) \quad \forall x' \in P^s(x, E) \qquad (x' \underset{E}{\succ} x).$$

Therefore, $\varphi \geq 0$.

Theorem 1. *Given $Z^s(E)$. The stability radius $\rho(m, n, s)$ of the multicriteria nontrivial investment problem $Z^s(E)$, $s \geq 1$, has the following lower and upper bounds:*

$$\varphi(m, n, s) \leq \rho(m, n, s) \leq mn\varphi(m, n, s).$$

Proof. To prove Theorem 1, we will first prove the inequality $\rho \geq \varphi$. This inequality is obvious if $\varphi = 0$. Let $\varphi > 0$. According to the definition of φ for any portfolio $x \notin P^s(E)$ there exists a Pareto-optimal portfolio $x^0 \in P^s(x, E)$ such that

$$\max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x) \geq \varphi, \quad k \in N_s.$$

Hence, considering inequality (1), for any matrix $E' \in \mathbf{R}^{m \times n \times s}$ and any index $k \in N_s$ we have

$$g_k(x^0, x, E_k + E'_k) = \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x + E'_{i'k}x^0 - E'_{ik}x)$$
$$\geq \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x) - \|E'_k\|_{11} \geq \varphi - \|E'_k\|_{11}.$$

Therefore, assuming that $E' \in \Omega(\varphi)$, we obtain $g_k(x^0, x, E_k + E'_k) > 0$, $k \in N_s$. This means that $x^0 \underset{E+E'}{\succ} x$, i. e. x is not the Pareto-optimal portfolio of the perturbed problem $Z^s(E + E')$. Summarizing and taking into account $x \notin P^s(E)$, we conclude that

$$\forall E' \in \Omega(\varphi) \quad (P^s(E + E') \subseteq P^s(E)).$$

Hence, the inequality $\rho(m, n, s) \geq \varphi(m, n, s)$ is true.

Then let us prove the inequality $\rho \leq mn\varphi$. According to the definition of the number φ there exists a portfolio $x^* \notin P^s(E)$ such that for any portfolio $x \in P^s(x^*, E)$ there exists an index $l = l(x) \in N_s$ such that

$$\max_{i \in N_m} \min_{i' \in N_m} (E_{i'l}x - E_{il}x^*) \leq \varphi. \tag{2}$$

Then we assume $\varepsilon > mn\varphi$ and consider the perturbing matrix $E^0 = [e^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$, elements of which we define as follows:

$$e^0_{ijk} = \begin{cases} \delta, & \text{if } i \in N_m,\ x^*_j = 1,\ k \in N_s, \\ -\delta & \text{otherwith,} \end{cases}$$

where $\varphi < \delta < \varepsilon/mn$. We note that the elements of the matrix E^0 do not depend on a portfolio x, and therefore they do not depend on an index l. Taking into account the structure of the matrix E^0, we obtain

$$\|E^0_{ik}\|_1 = n\delta, \quad i \in N_m, \quad k \in N_s,$$

$$\|E^0\|_{11\infty} = \|E^0_k\|_{11} = mn\delta, \quad k \in N_s.$$

Therefore, $E^0 \in \Omega(\varepsilon)$. Moreover, all the rows E^0_{ik}, $i \in N_m$ of any cuts E^0_k, $k \in N_s$, are the same and consist of the components δ and $-\delta$. We denote the same row by A and obtain

$$A(x - x^*) = -\delta\|x - x^*\|_1 \le -\delta < -\varphi \le 0. \tag{3}$$

Hence, considering (2) and the structure of the perturbing matrix E^0, we conclude that for any portfolio $x \in P^s(x^*, E)$ the following relations are true:

$$g_l(x, x^*, E_l + E^0_l) = \min_{i\in N_m}(E_{il} + A)x - \min_{i\in N_m}(E_{il} + A)x^*$$
$$= \max_{i\in N_m}\min_{i'\in N_m}(E_{i'l}x - E_{il}x^*) + A(x - x^*) < 0.$$

Therefore, we obtain

$$\forall x \in P^s(x^*, E) \quad (x \notin P^s(x^*, E + E^0)). \tag{4}$$

Let now the portfolio $x \notin P^s(x^*, E)$. Then the following two cases are possible.

Case 1. $g(x, x^*, E) = \mathbf{0}^{(s)}$. Then according to relations (3) for any index $k \in N_s$ we have

$$g_k(x, x^*, E_k + E^0_k) = \min_{i\in N_m}\ (E_{ik} + A)x - \min_{i\in N_m}\ (E_{ik} + A)x^*$$
$$= g_k(x, x^*, E_k) + A(x - x^*) < 0.$$

Case 2. There exists an index $p \in N_s$ such that $g_p(x, x^*, E_p) < 0$. Then using again (3) we obtain $g_p(x, x^*, E_p + E^0_p) < 0$.

Thus, $x \notin P^s(x^*, E+E^0)$ if $x \notin P^s(x^*, E)$. Considering (4), as a result we obtain $P^s(x^*, E + E^0) = \emptyset$, i.e. x^* is a Pareto-optimal portfolio of the perturbed problem $Z^s(E + E^0)$. Since $x^* \notin P^s(E)$ we may conclude that

$$\forall\varepsilon > mn\varphi \quad \exists E^0 \in \Omega(\varepsilon) \qquad (P^s(E + E^0) \not\subseteq P^s(E)).$$

Hence, the inequality $\rho(m, n, s) \le mn\varphi(m, n, s)$ is true. □

Corollary 1. *The stability radius $\rho(m, n, s)$ equals zero if and only if $\varphi(m, n, s)$ equals zero.*

3 Attainability of the lower bound

Let us show that the lower bound for the problem stability radius, indicated in Theorem 1, is attainable.

Theorem 2. *There exists a class of multicriteria investment problems $Z^s(E)$, $s \geq 1$ such that for the stability radius of every problem of this class the following formula is true:*

$$\rho(m, n, s) = \varphi(m, n, s). \tag{5}$$

Proof. We will consider the class of problems $Z^s(E)$ such that the following terms are right:

$$X = \{x^0, x^*\}, \quad P^s(x^*, E) = \{x^0\},$$

i.e. $x^0 \underset{E}{\succ} x^*$, $x^* \notin P^s(E)$, $x^0 \in P^s(E)$. Then there exists an index $l \in N_s$ such that

$$g_l(x^0, x^*, E_l) = \varphi. \tag{6}$$

We also suppose that there exists an index $p \in N_n$ such that $x^0_p = 1$ and $x^*_p = 0$.

Further we introduce the notation

$$i(x^0) = \arg\min\{E_{il}x^0 : \ i \in N_m\},$$
$$i(x^*) = \arg\min\{E_{il}x^* : \ i \in N_m\}.$$

The numbers $i(x^0)$ and $i(x^*)$ can be either the same or different. The further proof does not depend on it.

For any number $\varepsilon > \varphi$ we define the elements of the perturbing matrix $E^0 = [e^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$ by the rule

$$e^0_{ijk} = \begin{cases} -\delta, & \text{if } \ i = i(x^0), \quad j = p, \quad k = l, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where

$$\varphi < \delta < \varepsilon. \tag{8}$$

Then the next equalities are obvious:

$$E^0_{i(x^0)l}x^0 = -\delta, \tag{9}$$

$$E^0_{il}x^0 = 0, \quad i \in N_m \setminus \{i(x^0)\}, \tag{10}$$

$$E^0_{il}x^* = 0, \quad i \in N_m, \tag{11}$$

$$\|E^0\|_{11\infty} = \|E^0_l\|_{11} = \|E^0_{il}\|_1 = \delta, \quad i \in N_m.$$

Therefore, $E^0 \in \Omega(\varepsilon)$.

Using (9) and (10), we obtain

$$f_l(x^0, E_l + E^0_l) = \min\Big\{(E_{i(x^0)l} + E^0_{i(x^0)l})x^0, \ \min_{i\neq i(x^0)}(E_{il} + E^0_{il})x^0\Big\} =$$

$$= \min\Big\{ f_l(x^0, E_l) - \delta,\ \min_{i \neq i(x^0)} E_{il} x^0 \Big\} = f_l(x^0, E_l) - \delta. \tag{12}$$

And from (11) the following relations are true:

$$f_l(x^*, E_l + E_l^0) = \min\Big\{ (E_{i(x^*)l} + E^0_{i(x^*)l})x^*,\ \min_{i \neq i(x^*)} (E_{il} + E^0_{il})x^* \Big\} =$$

$$= \min\Big\{ f_l(x^*, E_l),\ \min_{i \neq i(x^*)} E_{il} x^* \Big\} = f_l(x^*, E_l).$$

Hence, consistently applying (12), (6) and (8), we have

$$g_l(x^0, x^*, E_l + E_l^0) = g_l(x^0, x^*, E_l) - \delta = \varphi - \delta < 0.$$

Therefore, $x^0 \notin P^s(x^*, E + E^0)$, i.e. $P^s(x^*, E + E^0) = \emptyset$. It proves that x^* is a Pareto-optimal investment portfolio of the perturbed problem $Z^s(E + E^0)$. Thence, because of $x^* \notin P^s(E)$ we derive

$$\forall \varepsilon > \varphi \quad \exists E^0 \in \Omega(\varepsilon) \qquad (P^s(E + E^0) \not\subseteq P^s(E)).$$

Thus, $\rho(m, n, s) \leq \varphi(m, n, s)$. Hence, by Theorem 1 the formula (5) is true. □

Remark 1. If $m = 1$ then $i(x^0) = i(x^*)$. Therefore, as we noted earlier, the proof of Theorem 2 given above is true in this case. Hence, there exists a class of multicriteria linear Boolean programming problems $Z^s_B(E)$ whose stability radius equals $\varphi(1, n, s)$.

We give a numerical example that illustrates the statement of Theorem 2.

Example. Let $m = 2$, $n = 3$, $s = 2$; $X = \{x^0, x^*\}$, $x^0 = (0, 1, 1)^T$, $x^* = (1, 1, 0)^T$; $E \in \mathbf{R}^{2\times3\times2}$ is the matrix with cuts

$$E_1 = \begin{pmatrix} 5 & 1 & 2 \\ 2 & 0 & 4 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 6 & 2 & 3 \\ 2 & 1 & 5 \end{pmatrix}.$$

Then $p = 3$, $f(x^0, E) = (E_1 x^0, E_2 x^0) = (3, 5)$, $f(x^*, E) = (E_1 x^*, E_2 x^*) = (2, 3)$, $g(x^0, x^*, E) = (1, 2)$. Hence, $x^* \notin P^2(E)$, $\{x^0\} = P^2(x^*, E)$, $l = 1$, $i(x^0) = 1$, $i(x^*) = 2$. Therefore, $\varphi = \varphi(2, 3, 2) = \min\{1, 2\} = 1$. Further we will show that $\rho(2, 3, 2) \leq \varphi = 1$.

Since $e^0_{i(x^0)pl} = e^0_{131}$ then defining the cuts E_1^0 and E_2^0 of the perturbing matrix E^0 according to the rule (7), we obtain

$$E_1^0 = \begin{pmatrix} 0 & 0 & -\delta \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\delta > \varphi = 1$. Then it is easy to see in view of $l = 1$ that

$$g_1(x^0, x^*, E_1 + E_1^0) = g_1(x^0, x^*, E_1) - \delta = 1 - \delta < 0.$$

Hence, $x^* \in P^2(E + E^0)$. This inclusion and $\|E^0\|_{11\infty} = \delta > 1$, $x^* \notin P^2(E)$ gives $\rho(2, 3, 2) \leq 1$. Therefore, considering Theorem 1, we conclude that $\rho(2, 3, 2) = \varphi(2, 3, 2) = 1$.

4 Attainability of the upper bound

Let us show that the upper bound of the stability radius of the problem $Z^s(E)$ is attainable for $m = s = 1$. It is easy to see that in the particular case for $m = 1$ our problem $Z^s(E)$ transforms into a multicriteria linear Boolean programming problem, which we will write in the convenient form

$$Z^s_B(E): \quad f_k(x, E_k) = E_k x \to \max_{x \in X}, \quad k \in N_s,$$

where $X \subseteq \mathbf{E}^n$, E_k is the k-th row of the matrix $E = [e_{kj}] \in \mathbf{R}^{s\times n}$. Such case can be interpreted as the situation when the investor has not another alternative market state. As earlier, the metric l_∞ is in the criteria space $\mathbf{R}^s$, and the metric l_1 is in the solution space $\mathbf{R}^n$.

Theorem 3. *For $m = s = 1$ there exists a class of scalar linear Boolean programin problems $Z^1_B(E)$, $E \in \mathbf{R}^{1\times n}$ such that for the stability radius of every problem of this class the following formula is true:*

$$\rho(1, n, 1) = n\varphi(1, n, 1). \tag{13}$$

Proof. Let us show that there exists a class with $X = \{x^*, x^1, x^2, \ldots, x^n\} \subset \mathbf{E}^n$, $n \geq 2$, where $x^* = \mathbf{0}^{(n)}$, $x^j = e^j$, $j \in N_n$. Here e^j is the j-th column of an identity matrix of size $n \times n$. Let $E = (a, a, \ldots, a) \in \mathbf{R}^n$ in view of $m = s = 1$, where $a > 0$. Therefore, we have $f(x^*, E) = Ex^* = 0$, $f(x^j, E) = Ex^j = a$, $j \in N_n$, i. e. $x^* \notin P^1(E)$, $x^j \in P^1(E) = P^1(x^*, E)$, $j \in N_n$. Hence according to the definition of $\varphi(1, n, 1)$ the inequality $\varphi = \varphi(1, n, 1) = a$ is valid.

Let now $E' = (e'_1, e'_2, \ldots, e'_n)$ be a perturbing row vector from the row set $\Omega(na)$, i.e. $\|E'\|_1 = \sum\limits_{j\in N_n} |e'_j| < na$. It is easy to prove by contrary that there exists an index p such that $|e'_p| < a$. Therefore, we derive

$$g(x^p, x^*, E + E') = (E + E')(x^p - x^*) = a + e'_p > 0.$$

Hence we see that for any perturbing row $E' \in \Omega(n\varphi)$ the portfolio x^* is not a Pareto-optimal portfolio of the perturbed problem $Z^1(E + E')$. Thus, in view of $x^* \notin P^1(E)$ we get $\rho(1, n, 1) \geq n\varphi(1, n, 1)$. Therefore, according to Theorem 1 the equality (13) is true. □

From Theorems 1–3 following Remark 1 the well-known result followws.

Corollary 2 [33]. *The stability radius $\rho(1, n, s)$, $s \geq 1$, of the multicriteria nontrivial linear Boolean programing problem $Z^s_B(E)$ has the following lower and upper bounds:*

$$\varphi(1, n, s) \leq \rho(1, n, s) \leq n\varphi(1, n, s).$$

Remark 2. We note that in [18] lower and upper bounds of the stability radius of the multicriteria linear Boolean programing problem $Z^s_B(E)$, which is searching the Pareto set, were obtained when $X = \{x \in \mathbf{E}^n : \; Ax \leq b\}$, every problem parameter is under perturbation, i.e. both the elements of the matrix $E \in \mathbf{R}^{s\times n}$ and the elements of the matrix $A \in \mathbf{R}^{q\times n}$ and the vector $b \in \mathbf{R}^q$ are perturbed, while the same Chebyshev metric l_∞ is in every space of problem parameters $\mathbf{R}^n$, $\mathbf{R}^s$ and $\mathbf{R}^q$.

5 Stability conditions

Let us introduce the Slater set [34] of the problem $Z^s(E)$:

$$Sl^s(E) = \{x \in X : \quad Sl^s(x,E) = \emptyset\},$$

where $Sl^s(x,E) = \{x' \in X : \quad \forall k \in N_s \quad (g_k(x',x,E_k) > 0)\}$. It is obvious that $P^s(E) \subseteq Sl^s(E)$ and $P^s(x,E) \supseteq Sl^s(x,E)$ for any $E \in \mathbf{R}^{m\times n\times s}$ and $x \in X$.

Theorem 4. *For a multicriteria nontrivial investment problem $Z^s(E)$, $s \geq 1$, the statements below are equivalent:*
(i) problem $Z^s(E)$ is stable,
(ii) $P^s(E) = Sl^s(E)$,
(iii) $\varphi(m,n,s) > 0$.

Proof. $(i) \Rightarrow (ii)$. Assume that problem $Z^s(E)$ is stable but $P^s(E) \neq Sl^s(E)$. Then there exists an investment portfolio $x^* \in Sl^s(E) \setminus P^s(E)$. Therefore, $Sl^s(x^*,E) = \emptyset$ and $P^s(x^*,E) \neq \emptyset$. This means that

$$\forall x \in P^s(x^*,E) \quad \exists l \in N_s \qquad (g_l(x,x^*,E_l) = 0).$$

Hence, $\varphi(m,n,s) = 0$ and according to Corollary 1 $\rho(m,n,s) = 0$, which contradicts the stability of the problem $Z^s(E)$.

$(ii) \Rightarrow (iii)$. If $P^s(E) = Sl^s(E)$, then for any portfolio $x \notin P^s(E)$ the set $Sl^s(x,E)$ is empty. Therefore, there exists a portfolio $x^0 \in X$ such that the inequalities $g_k(x^0,x,E_k) > 0$, $k \in N_s$, are true, i.e. $x^0 \in P^s(x,E)$. Thus,

$$\forall x \notin P^s(E) \quad \exists x^0 \in P^s(x,E) \quad \forall k \in N_s \qquad (g_k(x^0,x,E_k) > 0).$$

Hence, $\varphi(m,n,s) > 0$.

$(iii) \Rightarrow (i)$. According to Theorem 1, this implication is obvious. □

Since $P^1(E) = Sl^1(E)$, from Theorem 4 follows

Corollary 3. *A scalar investment problem $Z^1(E)$ is stable for any matrix $E \in \mathbf{R}^{m\times n}$.*

Remark 3. Since any two norms are equivalent in finite-dimensional linear spaces [35], the result of Theorem 4 is true for any norms in the space $\mathbf{R}^{m\times n\times s}$ of problem parameters.

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