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Determining the Optimal Paths in Networks with Rated Transition Time Costs

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Abstract. We formulate and study the problem of determining the optimal paths in networks with rated transition time costs on edges. Polynomial time algorithms for determining the optimal solution of this problem are proposed and grounded. The proposed algorithms generalize algorithms for determining the optimal paths in the weighted directed graphs.

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1 Introduction and Problem Formulation

In this paper we formulate and study an optimal path problem on networks that extends the minimum cost path problem in the weighted directed graphs.

Let $G = (X, E)$ be a finite directed graph with vertex set $X, |X| = n$ and edge set E where to each directed edge $e = (u, v) \in E$ a cost c_e is associated. Assume that for two given vertices x, y there exists a directed path $P(x, y) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \dots, x_k = y\}$ from x to y. For this directed path we define the total rated cost

$$C(x_0, x_k) = \sum_{t=0}^{k-1} \lambda^t c_{e_t},$$

where λ is a positive value. So, in this path the costs c_{e_t} of directed edges e_t are rated by $\lambda^t c_{te}$ when we pass from x to y. We consider the problem of determining a path from x to y with minimal total rated cost in the case with fixed number of transitions on the edges and in the case with free number of transitions on the edges. If $\lambda = 1$ then the formulated problem becomes the well known problem of determining the shortest path from x to y. The considered problem can be regarded as the problem of determining the optimal paths in a dynamic network determined by the graph $G = (X, E)$ with cost functions $c_e(t) = \lambda^t c_e$ on edges $e \in E$ that depend on time. Therefore if the number k of edges for the optimal path is fixed then we can apply the dynamic programming algorithm or time-expanded network method from [1, 3–5] which determines the solution of the problem using $O(|x|^3 k)$ elementary operations. In this paper we show that for the considered problem the linear programming approach can be applied which allows us to ground more efficient polynomial time algorithms for determining the optimal paths.

2 Algorithms for Solving the Problem with Free Number of Transitions on Edges

In this section we consider the optimal path problem without restrictions on the number of transitions on edges and show that it can be efficiently solved using the linear programming approach. The basic linear programming model we shall use for this problem is the following:
Minimize
$$\phi(\alpha) = \sum_{e \in E} c_e \alpha_e \tag{1}$$
subject to
$$\begin{cases} \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = 1, \quad u = x; \\ \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = \ 0, \ \ \forall u \in X \setminus \{x, y\}; \\ \qquad \alpha_e \geq 0, \ \ \forall e \in E, \end{cases} \tag{2}$$
where $E^-(u)$ is the set of directed edges that originate in the vertex $u \in X$ and $E^+(u)$ is the set of directed edges that enter u.

The following theorem holds.

Theorem 1. *If $\lambda \geq 1$ and in G there exists a directed path $P(x, y)$ from a given starting vertex x to a given final vertex y then for nonnegative costs c_e of edges $e \in E$ the linear programming problem (1), (2) has solutions. If α_e^* for $e \in E$ represents an optimal basic solution of this problem then the set of directed edges $E^* = \{e \in E | \alpha_e^* > 0\}$ determines an optimal directed path from x to y.*

Proof. Assume that $\lambda \geq 1$ and in G there exists at least a directed path $P(x, y) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \ldots, x_k = y\}$ from x to y. Denote by $E_P = \{e_0, e_1, e_2, \ldots, e_{k-1}\}$ the set of edges of directed path $P(x, y)$. Then it is easy to check that
$$\alpha_e = \begin{cases} \lambda^t, & if \ \ e = e_t \in E_P; \\ 0, & if \ \ e \in E \setminus E_P \end{cases} \tag{3}$$
represents a solution of system (2). Moreover we can see that if the directed path $P(x, y)$ does not contain directed cycles then the solution determined according to (3) corresponds to a basic solution of system (2). So, if in G there exists a directed path from x to y then the set of solutions of system (3) is not empty. Taking into account that the costs $c_e, \ e \in E$ are nonnegative we obtain that the optimal value of objective function (1) is bounded, i.e. the linear programming problem (1), (2) has solutions.

Now let us prove that an arbitrary basic solution of system (2) corresponds to a simple directed path $P(x, y)$ from x to y. Let $\alpha = (\alpha_{e_1}, \alpha_{e_2}, \ldots, \alpha_{e_m})$ be a feasible solution of problem (1), (2) and denote $E_\alpha = \{e \in E | \alpha_e > 0\}$. Then it is easy to observe that the set of directed edges $E_\alpha \subseteq E$ in G induces a subgraph

$G_\alpha = (X_\alpha, E_\alpha)$ in which vertex x is a source and y is a sink vertex. Indeed, if this is not so then we can determine a subset of vertices X'_α from X_α that can be reached in G_α from x and X'_α does not contain vertex y. In G_α we can select the subgraph $G'_\alpha = (X'_\alpha, E'_\alpha)$ induced by the subset of vertices X'_α and we can calculate

$$S = \sum_{u \in X'_\alpha} \sum_{e \in E^-(u)} \alpha_e,$$

where $E'^-(u) = \{e = (v, u) \in E' | v \in X'_\alpha\}$. It is easy to observe that the value S can be also expresses as follows

$$S = \sum_{u \in X'_\alpha} \sum_{e \in E^+(u)} \alpha_e,$$

where $E'^+(u) = \{e = (u, v) \in E' | v \in X'_\alpha\}$. If we sum the equalities from (3) that correspond to $u \in X'_\alpha$ then we obtain

$$\sum_{u \in X'_{alpha}} \sum_{e \in E^-(u)} \alpha_e - \lambda \sum_{u \in X'_{alpha}} \sum_{e \in E^+(u)} \alpha_e = 1$$

which involves $(1 - \lambda)S = 1$. However this couldn't take place because $\lambda \geq 1$ and $S \geq 0$, i.e. we obtain the contradiction. So, if $\alpha \geq 1$ then in G_α there exists at least a directed path from x to y. Taking into account that an arbitrary vertex u in G_α contains at least an entering edge $e = (v, u)$ and at least an outgoing directed edge $e = (u, w)$ we may conclude that G_α has a structure of directed graph, where x is a source and y is a sink.

Thus, to prove that a basic solution $\alpha = (\alpha_{e_1}, \alpha_{e_2}, \dots, \alpha_{e_m})$ corresponds to a directed graph G_α that has a structure of a simple directed path from x to y it is sufficient to show that G_α has a structure of an acyclic directed graph and G does not contain parallel directed paths $P'(u, w), P''(u, w)$ from a vertex $u \in X_\alpha$ to $w \in X_\alpha$. We can prove the first part of the mentioned property as follows. If α is a basic solution and G_α contains a directed cycles then there exists a directed path $P(x, y) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \dots, x_r, e_r, \dots x_k = y\}$ from x to y that contains a directed cycle $\{x_r, e_r, x_{r+1}, e_{r+1}, \dots, x_{r+s-1}, e_{r+s-1}, x_r\}$ with the set of edges $E^0 = \{e_r, e_{r+1}, \dots, e_{r+s-1}\}$. If we denote the set of edges of the directed path $P^1(x, x_r) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \dots, x_r\}$ from x to x_r by $E^1 = \{e_0, e_1, e_2, \dots, e_{r-1}\}$ and we denote the set of edges of the directed path $P^2(x_r, y) = \{x_r = x_{r+s}, e_{r+s}, x_{r+s+1}, e_{r+s+1}, \dots, x_k = y\}$ from $x_r = x_{r+s}$ to $x_k = y$ by $E^2 = \{e_{r+s}, e_{r+s+1}, \dots, e_{k-1}\}$ then for a small positive θ we can construct the following feasible solution

$$\alpha'_e = \begin{cases} \alpha_e, & \forall e \in E_\alpha \setminus (E^0 \cup E^2); \\ \alpha_{e_{r+i}} - \lambda^i \theta, & i = 0, 1, \dots, s-1; \\ \alpha_{e_{r+s+i}} - \lambda^{s+i}\theta + \lambda^i \theta, & i = 0, 1, \dots, k-r-s-1. \end{cases}$$

Here θ can be chosen in such a way that $\alpha'_e = 0$ at least for an edge $e \in E^0 \cup E^2$. So, the number of nonzero components of the solution $\alpha' = (\alpha'_{e_1}, \alpha'_{e_2}, \dots, \alpha'_{e_m})$ is less than the number of nonzero components of solution α.

Now let us show that for a basic solution the graph G_α couldn't contain parallel directed paths from vertex x_r to vertex $w \in X_\alpha$. We prove this again by contradiction. We assume that in G_α we have two directed paths $P'(x_r, w) = \{x_r, e'_r, x'_{r+1}, \dots, e'_k, x'_k = w\}$ and $P''(x_r, y) = (x_r, y)\{x_r, e_{r+1}, x''_{r+1}, \dots, e''_l, x''_l = w\}$ from x to w with the corresponding edge sets $E' = \{e'_r, e'_{r+1}, \dots, e'_k\}$ and $E'' = \{e''_r, e''_{r+1}, \dots, e''_k\}$. Then for a small positive θ we can construct the following solution

$$\alpha'_e = \begin{cases} \alpha_e, & if \;\; e \in E_\alpha \setminus (E' \cup E''); \\ \alpha_{e'_{r+i}} - \lambda^i \theta, & if \;\; e = e'_{r+i} \in E', \;\; i = 0, 1, \dots, k-r; \\ \alpha_{e''_{r+i}} + \lambda^i \theta, & if \;\; e = e_{r+i}" \in E'', \;\; i = 0, 1, \dots, l-r. \end{cases}$$

Here we can chose θ in such a way that $\alpha'_{e_l} = 0$ at least for an edge $e_l \in E' \cup E''$, i. e. we obtain that the number of nonzero components of the solution α' is less then the number of nonzero components of α. Thus, if α is a basic solution then the corresponding graph G_α has a structure of a simple directed path from x to y. This means that if $\alpha^* = (\alpha^*_{e_1}, \alpha^*_{e_2}, \dots, \alpha^*_{e_m})$ is an optimal basic solution $\alpha^* = (\alpha^*_{e_1}, \alpha^*_{e_2}, \dots, \alpha^*_{e_m})$ of problem (1), (2) then the set of directed edges $E^* = \{e \in E | \alpha^*_e > 0\}$ determines an optimal directed path from x to y. □

Corollary 1. *If $\alpha \geq 1$ and vertex y is reachable in G from x then for an arbitrary basic solution α of system (2) the corresponding graph G_α has a structure of directed path from x to y.*

Corollary 2. *Assume that $0 < \lambda < 1$ and the graph G contains directed cycles. Then for a basic solution α of system (2) either the corresponding graph G_α has a structure of directed path from x to y or this graph does not contain directed paths from x to y; in the second case G_α contains a unique directed cycle that can be reached from x by using a unique directed path that connects vertex x with this cycle. Moreover, if G_α does not contain directed paths from x to y then it consists of the set of vertices and edges $\{x = x_0, e_0, x_1 e_1, x_2, e_2, \;\dots,\; x_r, e_r, x_{r+1}, e_{r+1}, \dots, x_{r+s-1}, e_{r+s-1}, x_r\}$ with a unique directed cycle $\{x_r, e_r, x_{r+1}, e_{r+1}, \dots, x_{r+s-1}, e_{r+s-1}, x_r\}$ where the nonzero components α_e of α can be expressed as follows*

$$\alpha_e = \begin{cases} \lambda^t, & if \;\; e = e_t, \;\; t = 0, 1, \dots, r-1; \\ \lambda^{r+i}/(1-\lambda^s), & if \;\; e = e_{r+i}, \;\; i = 0, 1, \dots, s-1. \end{cases} \tag{4}$$

Remark 1. If $0 < \lambda < 1$ then the linear programming problem (1), (2) may have an optimal basic solution α^* for which the graph G_{α^*} does not contain a directed path from x to y. This corresponds to the case when in G the optimal path from x to y does not exist.

Now we show that the linear programming model (1), (2) can be extended for the problem of determining the optimal paths from every $x \in X\{y\}$ to y. We can see that if $\lambda \geq 1$ then there exists the tree of optimal paths from every $x \in X\{y\}$ to y and this tree of optimal paths can be found on the basis of the following theorem.

Theorem 2. *Assume that $\lambda \geq 1$ and in G for an arbitrary $u \in X \setminus \{y\}$ there exists at least a directed path $P(u, y)$ from u to y. Additionally we assume that the costs c_e of edges $e \in E$ are nonnegative. Then the linear programming problem:*
Minimize

$$\phi(\alpha) = \sum_{e \in E} c_e \alpha_e \tag{5}$$

subject to

$$\begin{cases} \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = 1, \qquad \forall u \in X \setminus \{y\}, \\ \qquad \alpha_e \geq 0, \quad \forall e \in E \end{cases} \tag{6}$$

has solutions. Moreover, if $\alpha^* = (\alpha^*_{e1}, \alpha^*_{e2}, \ldots, \alpha^*_{em})$ is an optimal basic solution of problem (5), (6) then the set of directed edges $E^* = \{e \in E | \alpha^*_e > 0\}$ determines a tree of optimal directed paths G_{α^*} from every $u \in X \setminus \{y\}$ to y.

Proof. Let $\alpha = (\alpha_{e_1}, \alpha_{e_2}, \ldots, \alpha_{e_m})$ be a feasible solution of problem (5), (6) and consider the set of directed edges $E_\alpha = \{e \in E | \alpha_e > 0\}$ that corresponds to this solution. Then in the graph $G_\alpha = (X, E_\alpha)$ induced by the set of edges E_α the vertex y is attainable from every $x \in X$. An arbitrary basic solution α of system (6) corresponds to a graph G_α which has a structure of directed tree with sink vertex y. Moreover the optimal value of the objective function of the problem is bounded. Therefore if we find an optimal basic solution α^* of the problem (5), (6) then we determine the corresponding tree of optimal paths G_{α^*}. □

If the graph $G = (X, E)$ does not contain directed cycles then Theorem 1 and Theorem 2 can be extended for an arbitrary positive λ, i.e. in this case the following theorem holds.

Theorem 3. *If $G = (X, E)$ has a structure of an acyclic directed graph with sink vertex y then for an arbitrary $\lambda \geq 0$ and arbitrary costs $c_e, e \in E$ there exists the solution of the linear programming problem (1), (2). Moreover, if α^* is an optimal basic solution of this problem then the set of directed edges $E^* = \{e \in E | \alpha^*_e > 0\}$ determines an optimal directed path from x to y.*

Proof. The proof of this theorem is similar to the proof of Theorems 2. In this case the set of edges E_α for a basic solution of problem (5), (6) induces the graph $G_\alpha = (X_\alpha, E_\alpha)$ that has a structure of directed tree with sink vertex y. Therefore the set of edges E_{α^*} for an optimal basic solution of problem (5), (6) corresponds to a directed tree $G_{\alpha^*} = (X_{\alpha^*}, E_{\alpha^*})$ of optimal paths from every $u \in X$ to sink vertex y. □

As we have shown (see Corollary 2 and Remark 1) if $0 < \lambda < 1$ and the graph $G = (X, E))$ contains directed cycles then the linear programming problem (1), (2) may not find the optimal path from x to y even for the case with positive costs $c_e, \forall e \in E$ because such an optimal path in G may not exist. Below we illustrate an example of the problem with $\lambda = 1/2$ and the network represented in Figure 1. In the considered network the vertices are represented by circles and edges by arcs. Inside the circles the numbers of the vertices are written and near the arcs the values α^*_e that corresponds to the optimal solution of the problem with $x = 4,\ y = 1$ and $c_{(4,2)} = 1,\ c_{(2,1)} = 10,\ c_{(2,3)} = 1,\ c_{(3,2)} = 1$ are written. The optimal basic solution of the linear programming problem (1), (2) for the considered example is $\alpha^*_{(4,2)} = 1,\ \alpha^*_{(2,1)} = 0,\ \alpha^*_{(2,3)} = 2/3,\ \alpha^*_{(3,2)} = 1/3$ and the graph G_{α^*} is induced by the set of edges $\{(4,2),(2,3),(3,2)\}$. Here we can see that the values $\alpha^*_{(4,2)} = 1,\ \alpha^*_{(2,3)} = 2/3,\ \alpha^*_{(3,2)} = 1/3$ satisfy condition (4). The corresponding graph G_{α^*} does not contain the directed path from vertex 4 to 1, i.e the optimal path from vertex 4 to 1 does not exist.

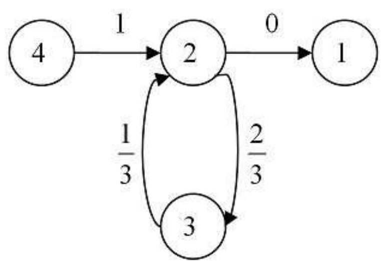

Figure 1.

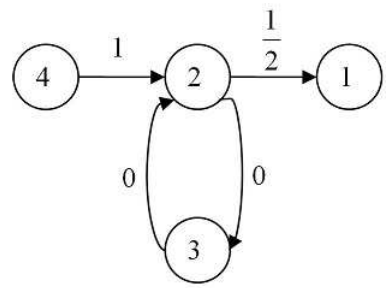

Figure 2.

In Figure 2 the optimal solution of problem (1), (2) with $x = 4,\ y = 1$ and $c_{(4,2)} = 1,\ c_{(2,1)} = 1,\ c_{(2,3)} = 2,\ c_{(3,2)} = 2$ is represented. In this case the optimal basic solution of problem (1), (2) is $\alpha^*_{(4,2)} = 1,\ \alpha^*_{(2,1)} = 1/2,\ \alpha^*_{(2,3)} = 0,\ \alpha^*_{(3,2)} = 0$. The corresponding nonzero components of this solution generate in G the subgraph $G_{\alpha^*} = (X_\alpha, E_{\alpha^*})$, where $E_{\alpha^*} = \{(4,2),(2,1)\}$. The set of edges E_{α^*} generates a unique directed path from vertex 4 to 1, i.e. in the considered case there exists the optimal path from vertex 4 to 1.

If for problem (5), (6) we consider the dual problem then on the basis of duality theorems of linear programming we can prove the following result.

Theorem 4. *Assume that $\lambda \geq 1$ and the costs $c_e, e \in E$ are strict positive. Let $\beta^*_u,\ \forall u \in X$ be a solution of the following linear programming problem:*
Maximize

$$\psi(\beta) = \sum_{x \in X \setminus \{x\}} \beta_x \qquad (7)$$

subject to

$$\beta_u - \lambda\beta_v \le c_{u,v}, \forall (u,v) \in E^0, \tag{8}$$

where

$$E^0 = \{e = (u,v) \in E | u \in X \setminus \{y\}, v \in X.$$

If β_u^* $u \in X$ is an optimal basic solution of problem (7), (8) then an arbitrary tree $T = (X, E'_{\beta^*})$ with sink vertex y of the graph $G_{\beta^*} = (X, E_{\beta^*})$ induced by the set of directed edges

$$E_{\beta^*} = \{e = (x,y) \in E | \beta_u^* - \lambda\beta_v^* = c_{u,v}\}$$

represents the tree of optimal paths from $x \in X \setminus \{y\}$ to y. An optimal basic solution of problem (7), (8), can be found starting with $\beta_v^* = 0$ for $v = y$ and $\beta_u^* = \infty$ for $u \in X \setminus \{y\}$ and then repeat $|X| - 1$ tames the following calculation procedure: replace β_u^* for $u \in X \setminus \{y\}$ by $\beta_u^* = \min\limits_{v \in X(u)} \left\{\lambda\beta_v^* + c_{u,v}\right\}$, where $X(v) = \{u \in X | (u,v) \in E\}$.

Proof. Assume that α_e^*, $e \in E$ and β_u^*, $u \in X$ represent the optimal solutions of the primal linear programming problem (5), (6) and the dual linear programming problem (7), (8), respectively. Then according to dual theorems of linear programming these solutions satisfy the following condition:

$$\alpha_{u,v}^*(\beta_u^* - \lambda\beta_v^* - c_{u,v}) = 0 \quad \forall (u,v) \in E^0. \tag{9}$$

So, if α_e^*, $e \in E$ is an optimal basic solution then $\beta_u^* - \lambda\beta_v^* - c_{u,v} = 0$ for an arbitrary $e = (u,v) \in E_{\alpha^*}$. Taking into account that the corresponding graph G_{α^*} for an optimal basic solution α^* has a structure of the directed tree with sink vertex y then we obtain this tree coincides with the tree of optimal paths T_{β^*} that determines the solution β_u^*, $u \in X$ of the problem (7), (8).

Now let us prove that the procedure for calculating the values β_x^* determines correctly the optimal solution of the dual problem. Indeed, if in G the vertex y is attainable from each $v \in X$ then the rank of system (8) is equal to $|X| - 1$. This means that for an arbitrary optimal basic solution not more than $|X| - 1$ its components may be different from zero. Therefore we can take $\beta_y^* = 0$. After that taking into account the condition (9) we can find β_u^* for $u \in X \setminus \{y\}$ using the calculation procedure from the theorem starting with $\beta_v^* = 0$ for $v = y$ and $\beta_u^* = \infty$ for $u \in X \setminus \{y\}$. □

Thus, based on Theorem 4 we can find the tree of optimal paths in G for the problem with free number of transitions as follow.

We determine the values β_u^* for $u \in X$ using the following steps:

Preliminary step (step 0): Fix $\beta_y^* = 0$, and $\beta_u^* = \infty$ for $u \in X \setminus \{y\}$;

General step (step k ($k \ge 1$)): For every $u \in X \setminus \{y\}$ replace the value β_u^* by $\beta_u^* = \min\limits_{v \in X(u)} \left\{\lambda\beta_v^* + c_{u,v}\right\}$, where $X(v) = \{u \in X | (u,v) \in E\}$. If $k < |X| - 1$ then go to next step; otherwise stop.

If β_u^* for $u \in X$ are known then we determine the set of directed edges E_{β^*} and the corresponding directed graph $G_{\beta^*} = (X, E_{\beta^*})$. After that we find a directed tree $T_{\beta^*} = (X, E'_{\beta^*})$ in G_{β^*}. Then T_{β^*} represents the tree of optimal paths from $x \in X$ to y.

It is ease to observe that the proposed algorithm allows us to solve the considered problems in general case with the same complexity as the problem with $\lambda = 1$, i.e this algorithm in the case $\lambda \geq 1$ extends the algorithm for shortest path problems (see [2, 3]).

3 Algorithms for Solving the Problem with Fixed Number of Transitions on Edges

The optimal path problem with fixed number of transitions from starting vertex to final one can be formulated and studied using the following linear programming model:
Minimize

$$\phi_{x,y}(\alpha) = \sum_{e \in E} c_e \alpha_e \tag{10}$$

subject to

$$\begin{cases} \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = \ 1, & u = x; \\ \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = \ 0, & \forall u \in X \setminus \{x, y\}; \\ \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = -\lambda^{k-1}, & u = y; \\ \alpha_e \geq 0, \ \forall e \in E. \end{cases} \tag{11}$$

This model is valid for an arbitrary $\lambda > 0$ ($\lambda \neq 1$). If we solve the linear programming problem (10), (11) then find an optimal solution α^* that determines the optimal value of objective function and the corresponding graph G_{α^*}. However such an approach for solving this problem does not allow to determine the order of the edges from G_{α^*} that form the optimal path $P(x, y)$ with fixed number of transitions from x to y. The algorithms based on linear programming in this case determine in polynomial time only the optimal cost of the optimal path and the corresponding graph G_{α^*}.

In order to determine the optimal path $P(x, y)$ with a given number of transitions K from x to y it is necessary to solve the sequence of $K|X - 1|$ linear programming problem (10), (11) with fixed starting vertex for $k = 1, 2, \ldots, K$ and for an arbitrary final vertex $y \in X \setminus \{x\}$.For each such a problem we determine the optimal value w $\phi_{x,y}(\alpha^{k^*})$ and the corresponding graph G_{α^k}. After that starting from final vertex y we find the optimal path $P(x, y)$ as follows: we fix a directed edge $e^{K-1} = (u^{K-1}, u^K = y)$ for which $\phi_{x,y}(\alpha^{K^*}) = \phi_{x,u^{K-1}}(\alpha^{K-1^*} + \lambda^{K-1} c_{e^{K-1}})$, then find a directed edge $e^{K-2} = (u^{K-2}, u^K = y)$ for which $\phi_{x,u^{K-1}}(\alpha^{K-1^*}) =$

$\phi_{x,u^{K-2}}(\alpha^{K-1^*} + \lambda^{K-2}c_{e^{K-2}})$ and so on. In such a way we find the vertices $x = u^o, u^1, \ldots, u^k = y$ of the path $P(x, y)$.

More useful algorithms for solving the problem with fixed number of transitions on edges of the network are the dynamic programming algorithms and the time-expanded network method from [4–6]. To apply these algorithms it is sufficient to consider the network with cost functions $c_e(t) = \lambda^t c_e$ on edges $e \in E$.

4 Conclusion

The optimal paths problem on networks with rated transition time costs on edges generalizes the shortest path problem in weighted directed graphs. The proposed linear programming approach for studying this problem allows to ground polynomial time algorithms for determining the optimal paths in networks with rated costs on edges. The elaborated algorithms generalizes algorithms for determining the optimal paths in weighted directed graphs and may be useful for determining the solution for the dynamic version of minimum cost flow problem on networks with the costs on edges that depend on flow and on time (the case with separable cost functions).

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