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Invariant transformations of loop transversals. 1a. The case of automorphism

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Abstract. One special class of invariant transformations of loop transversals in groups is investigated. Transformations from this class correspond to arbitrary automorphisms of transversal operations of loop transversals mentioned above.

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1 Introduction

The notion of a transversal in a group to its own subgroup is well known and has been studied during the last 70 years (since R. Baer's work [1]). Loop transversals (transversals whose transversal operations are loops) in some fixed groups to their own subgroups present special interest.

This investigation is a continuation and important part of [6]. In the present work we will investigate such transformations of loop transversals which correspond to the most symmetric transformation of transversal operations – to an automorphism. We will use the statements from [6] and obtain the basic results of this work as corollaries.

Let us remember some necessary definitions and preliminary statements.

2 Necessary definitions and statements

Definition 1. A system $< E, \cdot >$ is called a **left (right) quasigroup** if the equation $(a \cdot x = b)$ (the equation $(y \cdot a = b)$) has exactly one solution in the set E for any fixed $a, b \in E$. If for some element $e \in E$ we have

$$e \cdot x = x \cdot e = x \quad \forall x \in E,$$

then a left (right) quasigroup $< E, \cdot, e >$ is called a **left (right) loop** (the element $e \in E$ is called a **unit**). A left quasigroup $< E, \cdot >$ which is simultaneously a right quasigroup is called simply a **quasigroup**. Similarly, a left loop which is simultaneously a right loop is called a **loop**.

Definition 2. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left (right) cosets in G to H, and we assume $H_1 = H$. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H. If a left transversal T is simultaneously a right one, it is called a **two-sided transversal**.

On any left transversal T in a group G to its subgroup H it is possible to define the following operation (*transversal operation*) :

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h,\ h \in H.$$

Definition 3. If a system $< E, \overset{(T)}{\cdot}, 1 >$ is a loop, then such left transversal $T = \{t_x\}_{x\in E}$ is called a **loop transversal**.

At last remind the definitions of a left multiplicative group and of a left inner permutation group of a loop.

Definition 4. Let $< E, \cdot, e >$ be a loop. Then a group

$$LM(< E, \cdot, e >) \overset{def}{=} < L_a \mid a \in E >,$$

generated by all left translations L_a of loop $< E, \cdot, e >$, is called a **left multiplicative group** of the loop $< E, \cdot, e >$. Its subgroup

$$LI(< E, \cdot, e >) \overset{def}{=} < l_{a,b} \mid l_{a,b} = L^{-1}_{a\cdot b} L_a L_b, : a, b \in E >$$

generated by all permutations $l_{a,b}$, is called a **left inner permutation group** of the loop $< E, \cdot, e >$.

Definition 5 (see [2])**.** A mapping $\Phi = (\alpha, \beta, \gamma)$ (α, β, γ are permutations on a set E) of the operation $< E, \cdot >$ on the operation $< E, \circ >$ is called an **isotopy** if

$$\gamma(x \cdot y) = \alpha(x) \circ \beta(y) \quad \forall x, y \in E.$$

If $\Phi = (\gamma, \gamma, \gamma)$, then such an isotopy is called an **isomorphism**. If $\Phi = (\gamma, \gamma, \gamma)$, and $< E, \cdot > = < E, \circ >$ then such an isomorphism is called an **automorphism**.

3 The transformations which correspond to automorphisms of the transversal operations of loop transversals

Let $T = \{t_x\}_{x\in E}$ be a loop transversal in a group G to its subgroup H, and $< E, \overset{(T)}{\cdot}, 1 >$ is its transversal operation. Consider the following group:

$$M_G(T) \overset{def}{=} < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{G} \alpha^{-1} >,$$

it is generated by all permutations $\alpha \in St_1(S_E)$ which satisfy the condition

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{G} \alpha^{-1}.$$

Lemma 1. *The following propositions are true:*

1. $N_{St_1(S_E)}(\widehat{G}) \subseteq M_G(T) \subseteq St_1(S_E)$,
2. $M_G(T)$ *is maximal among subgroups* $M \subseteq St_1(S_E)$ *which satisfy the following property:*
$$LM(< E, \overset{(T)}{\cdot}, 1 >) = \bigcap_{\alpha \in M} (\alpha \widehat{G} \alpha^{-1}).$$

Proof. See Lemma 6 from [6]. □

Lemma 2. *Let* $\varphi : E \to E$ *be an automorphism of the loop* $< E, \overset{(T)}{\cdot}, 1 >$ *(note that* $\varphi(1) = 1$*). Then*

1. $\widehat{T} = h_0^{-1} \widehat{T} h_0$ *for some* $h_0 \in H^* = M_G(T)$*;*
2. $\varphi \equiv h_0$ *and* $LI(< E, \overset{(T)}{\cdot}, 1 >) \subseteq h_0 \widehat{H} h_0^{-1}$.

Proof. It is an evident corollary of the Lemma 7 from [6]. □

Lemma 3. *Let* $T = \{t_x\}_{x \in E}$ *be a fixed loop transversal in* G *to* H*. Let* $h_0 \in N_{St_1(S_E)}(H)$ *be an element such that:*

$$t_{x'} \overset{def}{=} h_0^{-1} t_x h_0 \quad \forall x \in E.$$

Then $\varphi \equiv h_0 \in Aut(< E, \overset{(T)}{\cdot}, 1 >)$.

Proof. It is an evident corollary of the Lemma 8 from [6]. □

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