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On the number of group topologies on countable groups

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Abstract. If a countable group G admits a non-discrete Hausdorff group topology, then the lattice of all group topologies of the group G admits:
– continuum c of non-discrete metrizable group topologies such that $\sup\{\tau_1, \tau_2\}$ is the discrete topology for any two of these topologies;
– two to the power of continuum of coatoms in the lattice of all group topologies.

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1 Introduction

This article is a continuation of article [1]. Main results of this article are Theorems 3.1 and 3.2.

Statements 3.1.2 and 3.1.3 of Theorem 3.1 are stronger than Theorem 14 from [1], and Statement 3.1.4 affirms that if a countable group G admits a non-discrete Hausdorff group topology, then the lattice of all group topologies of the group G admits two to the power of continuum of coatoms.

Moreover (see Theorem 3.2), for a countable group the requirement of the existence of a non-discrete Hausdorff group topology in Theorem 14 and Theorem 13 from [1] can be weakened to the requirement of the existence of a group topology in which the topological group does not have a finite basis of the filter of all neighborhoods of the unity element.

2 Notations and preliminaries

For proof of the main results we need the following notations and results:

Notations 2.1.

– $|A|$ is the cardinality of the set A;

– $\mathbb{N}$ is the set of all natural numbers;

– c is the continuum cardinality and $\omega(c)$ is the minimal transfinite number of the cardinality c;

– $\widetilde{\mathbb{N}}$ is a set of cardinality c of infinite subsets of the set $\mathbb{N}$ such that $A \bigcap B = \emptyset$ for any $A, B \in \widetilde{\mathbb{N}}$ and $A \neq B$ (the existence of such a set $\widetilde{\mathbb{N}}$ is proved in [3, Example 3.6.18]);

Notation 2.2. If $G(\cdot)$ is a group and x is some variable, then the free product of the group $G(\cdot)$ and of the free cyclic group generated by the element x is denoted by $G(x)$, i.e. $G(x)$ consists of elements of the form $g_1 \cdot x^{k_1} \cdot g_2 \cdot x^{k_2} \cdot \ldots \cdot g_s \cdot x^{k_s} \cdot g_{s+1}$, where $g_i \in G$ for $1 \leq i \leq s+1$ and k_j is an integer for $1 \leq j \leq s$.

Definitions 2.3.

– Elements of the group $G(x)$ are called *words in the variable x over the group $G(\cdot)$*;

– If $g \in G(\cdot)$ and $f(x) \in G(x)$, then the expression $f(x) = g$ will be called *an equation over the group $G(\cdot)$*;

– An element $a \in G(\cdot)$ is called *a root of the equation* $f(x) = g$ if $f(a) = g$.

Definitions 2.4.

– A partially ordered set $(X, \leq)$ is called *a lattice* if for any elements $a, b \in X$ there exist $\inf\{a, b\}$ and $\sup\{a, b\}$;

– A lattice $(X, \leq)$ is called *complete* if for any non-empty subset $S \subseteq X$ there exist $\inf S$ and $\sup S$;

– Lattices $(X, \leq)$ and $(Y, \leq)$ are called *anti-isomorphic* if there exists a bijective mapping $\Psi : (X, \leq) \to (Y, \leq)$ such that $\Psi(\inf\{a, b\}) = \sup\{\Psi(a), \Psi(b)\}$ and $\Psi(\sup\{a, b\}) = \inf\{\Psi(a), \Psi(b)\}$ for all elements $a, b \in X$.

The map $\Psi : (X, \leq) \to (Y, \leq)$ will be called *a lattice anti-isomorphism*;

– If a lattice $(X, \leq)$ has the greatest element 1, then an element $a \neq 1$ of the lattice $(X, \leq)$ is called *a coatom* if $b = 1$ for any element $b \in X$ such that $a < b$.

Notation 2.5. If $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are sequences of symmetrical subsets of a group $G(\cdot)$ (i.e. $(A_i)^{-1} = A_i$ and $(B_i)^{-1} = B_i$) such that $e \in \bigcap\limits_{i=1}^{\infty} B_i$, then for any natural number n by induction we define the set $F_n(B_1, \ldots, B_n; A_1, \ldots A_n)$: we take $F_1(B_1; A_1) = \{g \cdot h \cdot g^{-1} | g \in A_1, h \in B_1\} \bigcup B \cdot B$ and

$$F_{n+1}(B_1, \ldots, B_{n+1}; A_1, \ldots A_{n+1}) = F_1\big((B_1 \bigcup F_n(B_2, \ldots, B_{n+1}; A_2, \ldots A_{n+1}); A_1\big).$$

Theorem 2.6 (see for example [2, page 203 and page 205]). *A set Ω of subsets of a group $G(\cdot)$ is a basis of the filter of all neighborhoods of the unity element e for a Hausdorff group topology on the group $G(\cdot)$ if and only if the following conditions are satisfied:*

1) $\bigcap\limits_{V \in \Omega} V \supseteq \{e\}$;

2) *For every V_1 and $V_2 \in \Omega$ there exists $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \cap V_2$;*

3) *For every $V_1 \in \Omega$ there exists $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$;*

4) *For every $V_1 \in \Omega$ there exists $V_2 \in \Omega$ such that $V_2^{-1} \subseteq V_1$;*

5) *For every $V_1 \in \Omega$ and any element $g \in G$ there exists $V_2 \in \Omega$ such that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$.*

Moreover, this group topology is Hausdorff if and only if $\bigcap\limits_{V \in \Omega} V = \{e\}$

Proposition 2.7 (see [1]). *If $V_1, V_2, \dots$ and $S_1, S_2, \dots$ are some sequences of subsets of a group $(G, (\cdot))$, then (see I.5) for subsets $F_k(V_1, \dots, V_k; S_1, \dots, S_k)$ the following statements are true:*

2.7.1. *If $e \in V_1$, then $V_1 \subseteq V_1 \cdot V_1 \subseteq F_1(V_1; S_1)$ and $g \cdot V_1 \cdot g^{-1} \subseteq F_1(V_1; S_1)$ for any $g \in S_1$;*

2.7.2. *If $k \in \mathbb{N}$ and S_i and V_i are symmetric and finite sets for $1 \le i \le k$, then $F_k\big(V_1, \dots, V_k; S_1, \dots, S_k\big)$ is a symmetric and finite set;*

2.7.3. $F_k\big(\{e\}, \dots, \{e\}; S_1, \dots, S_k\big) = \{e\}$ *for any* $k \in \mathbb{N}$;

2.7.4. *If $U_i \subseteq V_i$ and $T_i \subseteq S_i$ for each $1 \le i \le k$, then*

$$F_k\big(U_1, \dots, U_k; T_1, \dots, T_k\big) \subseteq F_k\big(V_1, \dots, V_k; S_1, \dots, S_k\big);$$

2.7.5. *If $k, p \in \mathbb{N}$, and $e \in V_i$ for $i \le k$ and $V_{k+j} = \{e\}$ for $1 \le j \le p$, then*

$$F_k\left(V_1, \dots, V_k; S_1, \dots, S_k\right) = F_{k+p}\left(V_1, \dots, V_{k+p}; S_1, \dots, S_{k+p}\right);$$

2.7.6. *If an integer $k \ge 2$, then the equality*

$$F_k\big(V_1, \dots, V_k; S_1, \dots, S_k\big) =$$

$$F_k\Big(V_1 \cup F_{k-1}\big(V_2, \dots, V_k; S_2, \dots, S_k\big), \dots, V_{k-1} \cup F_1(V_k; S_k), V_k; S_1, \dots, S_k\Big)$$

is true;

2.7.7. *If $e \in V_i$ for each $1 \le i \le k$, then $V_t \subseteq F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ for each $1 \le t \le k$;*

2.7.8. *If $k, s \in \mathbb{N}$ and $e \in V_i$ for each $1 \le i \le k+s$, then*

$$F_{k+1}\Big(V_s, \dots, V_{k+s}; S_s, \dots, S_{k+s}\Big) \subseteq F_{k+s-t+1}\big(V_t, \dots, V_{k+s}; S_1, \dots, S_{k+s}\big)$$

for any $k, s, t \in \mathbb{N}$ and $t \le s$.

Notation 2.8. Let $G(\cdot) = \{e, g_1^{\pm 1}, \dots\}$ be a countable group, and for each positive integer n let be $S_n = \{g_1^{\pm 1}, g_2^{\pm 1}, \dots, g_n^{\pm 1}\}$.

For each pair of natural numbers (i, j) we define subsets $V_{(i,j)}$ and $S_{(i,j)}$ of the group $G(\cdot)$ and for each three natural numbers (i, j, k) such that $1 \le k \le j$ define a set $\Phi_{(i,j,k)}(x)$ of equations in the variable x over the group $G(\cdot)$ as follows:

$V_{(1,j)} = \{e\}$, $S_{(1,j)} = S_j$ and $\Phi_{(1,j,k)}(x) = \big\{x = c \big| c \in S_k\big\}$ for all $j, k \in \mathbb{N}$ and $k \le j$.

Suppose that for a natural number p the sets $V_{(i,j)}$, $S_{(i,j)}$ and $\Phi_{(i,j,k)}(x)$ are defined for $i \le p$ and all $j, k \in \mathbb{N}$ such that $k \le j$.

If $p+1$ is an even natural number, then we take:

$V_{(p+1,j)} = \{e\}$ for $j \ge p+1$;

$V_{(p+1,j)} = V_{(p,j)} \cup \{g, g^{-1}\}$, where $g \in G \setminus \bigcup\limits_{s=1}^{j} S_{(p,j)}$ (if $G \setminus \bigcup\limits_{s=1}^{j} S_{(p,j)} = \emptyset$, then we take $V_{(p+1,j)} = V_{(p,j)}$) for all $j < p+1$;

$\Phi_{(p+1,j,k)}(x) = \Phi_{(p,j,k)}(x)$ for all $k < j \in \mathbb{N}$;

$S_{(p+1,j)} = \big\{g \in G | g \in \bigcup\limits_{k=1}^{j} \Phi_{(p+1,j,k)}\big\}$ for all $j \in \mathbb{N}$.

If $p+1$ is an odd natural number, then we take:

$V_{(p+1,j)} = \{e\}$ for $j \geq p+1$;

$V_{(p+1,j)} = F_{p+1-j}\big(V_{(p,j+1)}, \ldots, V_{(p,p+1)}; S_{j+1}, \ldots, S_{p+1}\big) \bigcup V_{(p,j)}$ for $j < p+1$;

$\Phi_{(p+1,j,j)}(x) = \big\{x = g \big| g \in S_j\big\}$ for all $j \in N$ and $\Phi_{(p+1,j,k)}(x) = \big\{f(x) = g \big|\ f(x) \in F_{j,k}\big(V_{(p+1,k+1)}, \ldots, V_{(p+1,j-1)}, V_{(p,j)} \cup \{x, x^{-1}\}; S_{k+1}, \ldots, S_j\big)$ and $g \in S_k\big\}$ for any $k, j \in \mathbb{N}$ and $k < j$;

$S_{(p+1,j)} = S_{(p,j)}$ for every $j \in \mathbb{N}$.

So, we identified subsets of $V_{(i,j)}$ and $S_{(i,j)}$ of the group $G(\cdot)$ for each pair of positive integers (i,j) and the set $\Phi_{(i,j,k)}(x)$ of equations on the group $G(\cdot)$ for each triples of positive integers (i,j,k) such that $1 \leq k \leq j$.

Theorem 2.9 (see [1, Theorem 11]). *If a countable group $G(\cdot)$ admits a non-discrete Hausdorff group topology τ and $M = \big\{f_1(x) = a_1, \ldots, f_m(x) = a_m\big\}$ is a finite set of equations over the group $G(\cdot)$ for which the unity element e is not a root of any of these equations, then in the topological group (G, τ) there exists a neighborhood W of the unity element e such that each its element is not a root of any of these equations.*

From Theorem 2.6 follows

Theorem 2.10. *If Ω is a set of group topologies on a group $G(\cdot)$ and for each topology $\tau \in \Omega$ in a topological group $(G(\cdot), \tau)$ a basis $\mathbf{B}_\tau$ of the filter of all neighborhoods of the unity element e is given, then the set*

$$\big\{ \bigcap_{\tau \in M} V_\tau | M \text{ is a finite subset in } \Omega \text{ and } V_\tau \in \mathbf{B}_\tau \big\}$$

is a basis of the filter of all neighborhoods of the unity element in the topological group $(G(\cdot), \sup \Omega)$.

From the definition of the prototype of any topology follows

Theorem 2.11. *Let $f : G(\cdot) \to \overline{G}(\cdot)$ be some group homomorphism from the group $G(\cdot)$ in the group $\overline{G}(\cdot)$. If $\overline{\tau}$ is a group topology on the group $\overline{G}(\cdot)$ and τ is the prototype of the topology $\overline{\tau}$ relative to the homomorphism f (i. e. $\tau = \{f^{-1}(\overline{U}) | \overline{U} \in \overline{\tau}\}$), then τ is a group topology on the group $G(\cdot)$ and for any basis $\overline{\mathbf{B}}$ of the filter of neighborhoods of the unity element in the topological group $(\overline{G}(\cdot), \overline{\tau})$, the set $\mathbf{B} = \{f^{-1}(\overline{V}) | \overline{V}) \in \overline{B}\}$ is a basis of the filter of neighborhoods of the unity element in the topological group $(G(\cdot), \tau)$.*

Similarly to the proof of step II of Theorem 13 in [1], is proved:

Theorem 2.12. *Let $G(\cdot) = \{e, g_i^{\pm 1} | \in \mathbb{N}\}$ be a countable group and let $\{h_k = g_{i_k} | k \in \mathbb{N}\}$ be a sequence of elements of the group $G(\cdot)$ such that*

$$h_i \notin F_n\big(\{e, g_1^{\pm 1}\}, \dots, \{e, g_{i-1}^{\pm 1}\}, \{e\}\{e, g_{i+1}^{\pm 1}, \dots, \{e, g_n^{\pm 1}\}; S_1, \dots, S_n\}\big)$$

for every $i, n \in \mathbb{N}$. *Then the following statements are true:*

2.12.1. *If C is an infinite subset of the set of all natural numbers $\mathbb{N}$ and*

$$U_{i,C} = \begin{cases} \{h_{k_i}, e, h_{k_i}^{-1}\} & \text{if} \quad i \in C, \\ \{e\} & \text{if} \quad i \notin C \end{cases}$$

for every $i \in \mathbb{N}$, then the set

$$\big\{\widehat{U}_i(C) \mid \widehat{U}_i(C) = \bigcup_{j=1}^{\infty} F_{j+1}\big(U_{i,C}, \dots U_{i+j,C}; S_i, \dots, S_{i+j}\big), i \in \mathbb{N}\big\}$$

is a basis of the filter of neighborhoods of the unity element for some group topology $\tau(C)$ in the group $G(\cdot)$;

2.12.2. *If A, B are subsets of the set $\mathbb{N}$ such that $A \setminus B$ and $B \setminus A$ are infinite subsets, then the topologies $\tau(A)$ and $\tau(B)$ are incomparable.*

Definition 2.13. An element $d \in X$ is called a maximal element in a partially ordered set $(X, \leq)$ if $d = z$ for any element z in X such that $d \leq z$.

Theorem 2.14 (see [3, page 28, the Kuratowski-Zorn's lemma]). *If $(X, \leq)$ is a partially ordered set such that for any linearly ordered subset $(A, \leq) \subseteq (X, \leq)$ there exists an element $a \in X$ such that $x \leq a$ for every $x \in A$, then for any $y \in X$ there exists a maximal element $d \in X$ in the partially ordered set $(X, \leq)$ such that $y \leq d$.*

Proposition 2.15(see [3, Corollary 3.6.12]). *If $(\beta\mathbb{N}, \tau)$ is Stone-Čech compacification, then the following statements are true:*

2.15.1. The set $\mathbb{N}$ is a dense subset of the topological space $(\beta\mathbb{N}, \tau)$;

2.15.2. The topological space $(\beta\mathbb{N}, \tau)$ is Hausdorff;

2.15.3. The cardinality of the set $\beta\mathbb{N}$ is equal to 2^c.

Proposition 2.16. *For any element $a \in \beta\mathbb{N} \setminus \mathbb{N}$ and any neighborhood U of the element a in the topological space $(\beta\mathbb{N}, \tau)$, the set $U \bigcap \mathbb{N}$ is infinite.*

Proof. Assume the contrary, i. e. that some element $a \in \beta\mathbb{N} \setminus \mathbb{N}$ has a neighborhood U such that $U \bigcap \mathbb{N}$ is a finite set.

Since every finite set is closed in any Hausdorff space and $a \notin \mathbb{N}$, then $V = U \setminus (\mathbb{N} \bigcap U)$ is a neighborhood of the element a in the topological space $(\beta\mathbb{N}, \tau)$, and $V \bigcap \mathbb{N} = \emptyset$.

This contradicts the Statement 2.15.1. □

3 Basic results

Theorem 3.1. *Let a countable group $G(\cdot)$ admit some Hausdorff non-discrete group topology τ_0 such that the topological group (G, τ_0) has a countable basis of the filter of all neighborhoods of the unity element. Then:*

3.1.1. *The group $G(\cdot)$ admits a continuum of non-discrete group topologies stronger than τ_0 and such that the following conditions are true:*

– *the space of the topological group is Hausdorff;*

– *the unity element has a countable basis of the filter of all neighborhoods;*

– *any two of these topologies are comparable.*

3.1.2. *The group $G(\cdot)$ admits a continuum of non-discrete group topologies stronger than τ_0 and such that for each of these topologies the following conditions are true:*

– *the space of the topological group is Hausdorff;*

– *the unity element has a countable basis of the filter of all neighborhoods;*

– *sup$\{\tau_1, \tau_2\}$ is the discrete topology for any two of these topologies $\tau_1 \neq \tau_2$;*

3.1.3. *There exist 2^c (two to the power of continuum) non-discrete group topologies stronger than τ_0 and such that sup$\{\tau_1, \tau_2\}$ is the discrete topology for any two of these topologies $\tau_1 \neq \tau_2$;*

3.1.4. *There exist 2^c coatoms in the lattice of all group topologies on the group $G(\cdot)$.*

Proof. Proof of Statement 3.1.1 see in the proof of Theorem 14 in [1].

Proof of Statement 3.1.2. Let $G = \{e, g_1^{\pm 1}, \dots\}$ be a numbering of elements of the group $G(\cdot)$ and let $S_n = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ for any $n \in \mathbb{N}$. Then there exists a countable basis $\{V_1, V_2, \dots\}$ of the filter of neighborhoods of the unity element in the topological group (G, τ_0) which consists of symmetric subsets such that $V_k \cap S_k = \emptyset$ and $g \cdot V_{k+1} \cdot g^{-1} \subseteq V_k$ for any $k \in \mathbb{N}$ and any $g \in S_k$.

It is easily proved by induction on k that $F_k(V_{i+1}, \dots, V_{i+k}; S_{i+1}, \dots, S_{i+k}) \subseteq V_i$ for any $i, k \in \mathbb{N}$.

The proof of the theorem will be realized in several steps.

Step I. Construction of an auxiliary sequence $h_1, h_2, \dots$ of elements of $G(\cdot)$ and of an increasing sequence $k_1, k_2, \dots$ of natural numbers.

By induction on n we construct a sequence $k_1, k_2, \dots$ of natural numbers such that $k_i \geq i$ for every $i \in \mathbb{N}$ and a sequence $h_1, h_2, \dots$ of elements of the set $G \backslash \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ for any positive integer i and for any subsets of $A \subseteq \{k_1, \dots, k_n\}$ and $B \subseteq \{k_1, \dots, k_n\}$ such that $A \bigcap B = \emptyset$ and the condition holds:

$$F_n(U_{1,A}, \dots, U_{n,A}; S_1, \dots, S_n) \bigcap F_n(U_{1,B}, \dots, U_{n,B}; S_1, \dots, S_n) = \{e\},$$

where

$$U_{i,C} = \begin{cases} \{h_i, e, h_i^{-1}\} & \text{if} \quad i \in C, \\ \{e\} & \text{if} \quad i \notin C \end{cases}$$

for any subset $C \subseteq \{1, \ldots, n\}$.

If $n = 1$, then we take $k_1 = 2$ and h_1 an arbitrary element of the set $V_2 \setminus \{e\}$.

If A and B are subsets of $\{1\}$ such that $A \bigcap B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$, and hence (see Statement 2.7.3), either $F_1(U_{1,A}; S_1) = F_1(\{e\}; S_1) = \{e\}$, or $F_1(U_{1,B}; S_1) = F_1(\{e\}; S_1) = \{e\}$. So $F_1(U_{1,A}; S_1) \bigcap F_1(U_{1,B}; S_1) = \{e\}$, and hence, for the natural number $k_1 = 2$ and the element h_1 all conditions specified above are true.

Suppose that we have already defined positive integers $k_1 < k_2, \ldots < k_n$ such that $k_i > i$ and elements $h_1, h_2, \ldots, h_n$ from the set $G \setminus \{e\}$ such that all conditions specified above are true.

For any sets $A \subseteq \{1, \ldots, n\}$ and $B \subseteq \{1, \ldots, n\}$ such that $A \bigcap B = \emptyset$ we consider the set $\Psi_{(A,B)}(x)$ of equations over the group $G(\cdot)$ of the form $f(x) = g$, where

$$f(x) \in F_{n+1}(U_{1,A}, \ldots, U_{n,A}, \{x, e, x^{-1}\}; S_1, \ldots, S_{n+1})$$

and $g \in F_n(U_{1,B}, \ldots, U_{n,B}\}; S_1, \ldots, S_n) \setminus \{e\}$.

We prove that the unity element e is not a root for any equation of the set $\Psi_{(A,B)}(x)$.

Assume the contrary, i.e. that $f(e) = g$ for some equation $f(x) = g$ from the set $\Psi_{(A,B)}(x)$. Then $g \in F_{k_n}(U_{1,B}, \ldots, U_{n,B}\}; S_1, \ldots, S_n) \setminus \{e\}$, and (see Statement 2.7.5)

$$g = f(e) \in F_{n+1}(U_{1,A}, \ldots, U_{n,A}, \{e\}; S_1, \ldots, S_{n+1}) = F_n(U_{1,A}, \ldots, U_{n,A}, ; S_1, \ldots, S_n),$$

and hence,

$$g \in F_n(U_{1,A}, \ldots, U_{n,A}, ; S_1, \ldots, S_n) \bigcap \big(F_n(U_{1,B}, \ldots, U_{n,B}\}; S_1, \ldots, S_n) \setminus \{e\}\big).$$

We have the contradiction with the inductive assumption that

$$F_n(U_{1,A}, \ldots, U_{n,A}; S_1, \ldots, S_n) \bigcap F_n(U_{1,B}, \ldots, U_{n,B}; S_1, \ldots, S_n) = \{e\},$$

and hence the unity element e is not a root of any equation of the set $\Psi_{(A,B)}(x)$.

Then, by Theorem 2.9 there exists a neighborhood $W_{(A,B)}$ of the unity element in the topological group $(G(\cdot), \tau_0)$ such that any element $h \in W_{\{A,B\}}$ is not a root of any equation of the set $\Psi_{(A,B)}(x)$.

If now $\widetilde{M} = \big\{(A,B) | A, B \subseteq \{1, \ldots, n+1\} \text{ and } A \bigcap B = \emptyset\big\}$, then from the finiteness of the set $\widetilde{M}$ it follows that $\bigcap\limits_{(A,B)\in\widetilde{M}} W_{(A,B)}$ is a neighborhood of the unity element in a topological group $(G(\cdot), \tau_0)$, and any element $h \in \bigcap\limits_{(A,B)\in\widetilde{M}} W_{(A,B)} \setminus \{e\}$ is not a root of any equation of the set $\bigcup\limits_{(A,B)\in\widetilde{M}} \Psi_{(A,B)}(x)$.

Since the set $\{V_1, V_2, \dots\}$ is a basis of the filter of neighborhoods of the unity element in the topological group $(G(\cdot), \tau_0)$, then there exists a natural number $k_{n+1} > n$ such that $V_{k_{n+1}} \subseteq \bigcap_{(A,B)\in\widetilde{M}} W_{(A,B)}$.

We take any element $h_{n+1} \in V_{k_{n+1}} \subseteq V_{n+1}$, and prove that the conditions specified above are true also for the number $n+1$, i.e. these conditions are satisfied for the sequence of natural numbers $k_1, k_2, \dots k_{n+1}$ and the sequence of elements $h_1, \dots, h_{n+1}$.

Since $k_{n+1} > n+1$ and $h_{n+1} \in V_{k_{n+1}}$, then it remains only to prove that

$$F_{n+1}(U_{1,A}, \dots, U_{n+1,A}; S_1, \dots, S_{n+1} \bigcap F_{n+1}(U_{1,B}, \dots, U_{n+1,B}; S_1, \dots, S_{n+1} = \{e\},$$

for any of subsets A, B of the set $\{1, \dots, n+1\}$ for which $A \bigcap B = \emptyset$.

Assume the contrary, i.e. that

$$F_{n+1}(U_{1,A}, \dots, U_{n+1,A}; S_1, \dots, S_{n+1}) \bigcap F_{n+1}(U_{1,B}, \dots, U_{n+1,B}; S_1, \dots, S_{n+1}) \neq \{e\}$$

for some subsets $A, B \subseteq \{1, \dots, n+1\}$ such that $A \bigcap B = \emptyset$.

Then either $A \not\subseteq \{1, 2, \dots, n\}$ or $B \not\subseteq \{1, \dots, n\}$.

Suppose, for definiteness, that $A \not\subseteq \{1, \dots, n\}$.

Since $A \bigcap B = \emptyset$ then $B \subseteq \{1, \dots, n\}$ and since $n+1 \in A$ then $U_{n+1,A} = \{h_{n+1}, e, h_{n+1}\}$.

Then since the element $h_{n+1} \neq e$ and it is not a root of any equation of the set $\Psi_{(A,B)}(x)$ then from the definition of the set $\Psi_{(A,B)}(x)$ and Statement I.7.5 it follows that $\{e\} =$

$$F_{n+1}(U_{1,A}, \dots, U_{n,A}, \{h_{n+1}, e, h_{n1}^{-1}\}; S_1, \dots, S_{n+1}) \bigcap F_n(U_{1,B}, \dots, U_{n,B}; S_1, \dots, S_n) =$$

$$F_{n+1}(U_{1,A}, \dots, U_{n,A}, \{h_{n+1}, e, h_{n+1}^{-1}\}; S_1, \dots, S_{n+1}) \bigcap$$

$$F_{n+1}(U_{1,B}, \dots, U_{n,B}, \{e\}; S_1, \dots, S_{n+1}) =$$

$$F_{n+1}(U_{1,A}, \dots, U_{n+1,A}; S_1, \dots, S_{n+1}) \bigcap F_{n+1}(U_{1,B}, \dots, U_{n+1,B}; S_1, \dots, S_{n+1}) \neq \{e\}.$$

We have a contradiction, and hence the conditions specified above are true for the sequence of natural numbers $k_1, k_2, \dots k_{n+1}$ and the sequence of elements $h_1, h_2, \dots, h_{n+1}$.

So, we have constructed the sequence $k_1, k_2, \dots$ of natural numbers $k_i \geq i$ and the sequence $h_1, h_2, \dots$ of elements of the set $G \setminus \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ and which satisfy the following condition:

$$F_m(U_{1,A}, \dots, U_{m,A}; S_1, \dots, S_m) \bigcap F_m(U_{1,B}, \dots, U_{m,B}; S_1, \dots, S_m) = \{e\}$$

for any positive integer m and any subsets $A, B \subseteq \{1, \dots, m\}$ such that $A \bigcap B = \emptyset$.

Step II. Construction of a set T of group topologies of cardinality of continuum and such that $sup\{\tau_1, \tau_2\}$ is the discrete topology for all different topologies $\tau_1, \tau_2 \in$ T.

If $j \in \mathbb{N}$, $A = \{k_j\}$ and $B = \mathbb{N} \setminus \{k_j\}$, then $A \bigcap B) = \emptyset$, and hence,

$$F_n(U_{1,A}, \dots, U_{n,A}; S_1, \dots, S_n) \bigcap F_n(U_{1,B}, \dots, U_{n,B}; S_{k_1}, \dots, S_{k_n}) = \{e\}$$

for any positive integer n. Then

$$h_j \notin F_n\Big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{j-1}, h_{j-1}^{-1}\}, \{e\}, \{e, h_{j+1}, h_{j+1}^{-1}\}, \dots, \{e, h_n, h_n^{-1}\};$$

$S_1, S_2, \dots, S_n\Big)$ and hence, the sequence $k_1, k_2, \dots$ of natural numbers and the sequence of elements $h_1, h_2, \dots$ satisfy the conditions specified in the proof of Theorem 13 from [1].

Now we consider:

– the set $U_{i,A} = \{e\}$ if $k_i \notin A$ and $U_{i,A} = \{h_i, e, h_i^{-1}\}$ if $k_i \in A$ for any positive integer i and any set $A \in \widetilde{\mathbb{N}}$ (for the definition of the set $\widetilde{\mathbb{N}}$, see I.1);

– the set $U_{(i+1,j),A} = F_j(U_{i+1,A}, \dots, U_{i+j,A}; S_{i+1}, \dots, S_{i+j})$ for every pair (i, j) of natural numbers.

Then (see [1, Step II of the proof of Theorem 13]) the set $\{\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(ij),A} | i \in N\}$ is a basis of the filter of neighborhoods of the unity element for some group topology $\tau(A)$ on the group $G(\cdot)$ such that the following conditions are true:

– the space of the topological group $(G(\cdot), \tau(A))$ is Hausdorff;

– the unity element has a countable basis of the filter of neighborhoods;

– the topology $\tau(A)$ is stronger than the topology τ_0 for any set $A \in \widetilde{\mathbb{N}}$.

We show that if $A, B \in \widetilde{\mathbb{N}}$ and $A \neq B$, then $sup\{\tau(A), \tau(B)\}$ is the discrete topology.

In fact, since $A \bigcap B = \emptyset$, then $\{e\} \subseteq U_{(1,m),A} \bigcap U_{(1,s),B} \subseteq$

$$F_{n+m}(U_{1,A}, \dots, U_{n+m,A}; S_1, \dots, S_{n+m}) \bigcap F_{n+s}(U_{1,B}, \dots, U_{n+s,B}; S_1, \dots, S_{n+s}) \subseteq$$

$$F_{n+m+s}(U_{1,A}, \dots, U_{n+m+s,A}; S_1, \dots, S_{n+m+s}) \bigcap$$
$$F_{n+m+s}(U_{1,B}, \dots, U_{n+m+s,B}; S_1, \dots, S_{n+m+s}) = \{e\}$$

for any positive integers m and s, and hence,

$$\hat{U}_1(A) \bigcap \hat{U}_1(B) = \big(\bigcup_{j=1}^{\infty} U_{(1,j),A}\big) \bigcap \big(\bigcup_{j=1}^{\infty} U_{(1,j),B}\big) = \{e\}.$$

Since $\hat{U}_1(A)$ and $\hat{U}_1(B)$ are neighborhoods of the unity element in the topological group $(G(\cdot) sup\{\tau(A), \tau(B)\})$, then $\{e\} = \hat{U}_1(A) \bigcap \hat{U}_1(B)$ is a neighborhood of the identity in the topological group $(G(\cdot), \sup\{\tau(A), \tau(B)\})$, and hence, $\sup\{\tau(A), \tau(B)\}$ is the discrete topology for any different sets $A, B \in \widetilde{\mathbb{N}}$.

From the fact that the topology $\tau(A)$ is non-discrete for any set $A \in \widetilde{\mathbb{N}}$ it follows that $\tau(A) \neq \tau(B)$ for any different sets $A, B \in \widetilde{\mathbb{N}}$.

Statement 3.1.2 is proved.

Proof of Statement 3.1.3. Since the set $A = U \bigcap \mathbb{N}$ is infinite (see Proposition 2.16) for each element $a \in \beta\mathbb{N} \setminus \mathbb{N}$ and any neighborhood U of the element a of the topological space $(\beta\mathbb{N}, \tau)$, then we consider the topology $\tau_{a,U} = \tau(U \bigcap \mathbb{N})$, which was defined in the proof of Statement 3.1.2.

If $a \in \beta\mathbb{N} \setminus \mathbb{N}$ and Ω_a is the set of all neighborhoods U of the element a in the topological space $(\beta\mathbb{N}, \tau)$, we consider the topology $\tau_a = sup\{\tau_{a,U} | U \in \Omega_a\}$, and show that $sup\{\tau_a, \tau_b\}$ is the discrete topology for any distinct elements $a, b \in \beta\mathbb{N} \setminus \beta\mathbb{N}$.

From the fact that the space $(\beta\mathbb{N}, \tau)$ is Hausdorff it follows that there exist neighborhoods U and V of points a and b, respectively, such that $U \bigcap V = \emptyset$, and hence, $(U \bigcap \mathbb{N}) \bigcap (V \bigcap \mathbb{N}) = \emptyset$. Then (see the end of the proof of Statement 3.1.2), $sup\{\tau_{a,U}, \tau_{a,V}\}$ is the discrete topology, and as $sup\{\tau_a, \tau_b\} \geq sup\{\tau_{a,U}, \tau_{a,V}\}$, then $sup\{\tau_a, \tau_b\}$ is the discrete topology.

Since for each $c \in \beta\mathbb{N} \setminus \mathbb{N}$ the topology τ_c is a non-discrete topology, then the set $\{\tau_c | c \in \beta\mathbb{N} \setminus \mathbb{N}\}$ has cardinality 2^c, and since $sup\{\tau_a, \tau_b\}$ is the discrete topology for any distinct elements $a, b \in \beta\mathbb{N} \setminus \mathbb{N}$ then Statement 3.1.3 is proved.

Proof of Statement 3.1.4. If $\mathcal{T}\prime$ is the set of all group topologies on the group $G(\cdot)$ and τ_d^* is the discrete topology, then $(\mathcal{T}\prime, \subseteq)$ is a complete lattice. From Theorem 2.10 it follows that for any linearly ordered subset $(\mathcal{T}, \subseteq)$ of non-discrete topologies the set $\{e\}$ is not a neighborhood of the unity element in the topological group $(G(\cdot), \sup \mathcal{T})$, and hence, $\sup \mathcal{T} \in \mathcal{T}\prime \setminus \{\tau_d^*\}$. Then, by Theorem 2.14, for any topology τ_a where $a \in \beta\mathbb{N} \setminus \mathbb{N}$, which is defined in the proof of Statement 3.1.3, there exists a maximum element τ'_a in partially ordered set $\in \mathcal{T}\prime \setminus \{\tau_d^*\}$ such that $\tau_a \leq \tau'_a$. Then for each $a \in \beta\mathbb{N} \setminus \mathbb{N}$ the topology τ'_a is a coatom in the lattice $(\mathcal{T}\prime, \subseteq)$.

Since $\tau_d^* = \sup\{\tau_a, \tau_b\} \leq \sup\{\tau'_a, \tau'_b\} \leq \tau_d^*$ for different $a, b \in \beta\mathbb{N} \setminus \mathbb{N}$, then $\tau'_a \neq \tau'_b$ for different $a, b \in \beta\mathbb{N} \setminus \mathbb{N}$, and hence, the set $\{\tau'_a \mid a \in \beta\mathbb{N} \setminus \mathbb{N}\}$ has the cardinality 2^c (two to the power of continuum).

Statement 3.1.4 is proved and, hence, the theorem is completely proved. □

Theorem 3.2. *Let $G(\cdot)$ be a countable group and let $\mathcal{T}_0$ be the set of all group topologies on the group $G(\cdot)$ and $\mathcal{T}_1$ be the set of all group topologies on the group $G(\cdot)$ such that for any of these topologies the topological group $(G(\cdot), \tau)$ has a finite basis of the filter of all neighborhoods of unity element. Then the following statements are true:*

3.2.1. *The partially ordered set $(\mathcal{T}_1, \subseteq)$ is a lattice which is anti-isomorphic to the lattice $(\mathcal{N}, \subseteq)$ of all normal subgroups of the group $G(\cdot)$;*

3.2.2. *If $\mathcal{T}_0 \neq \mathcal{T}_1$, then in the group $G(\cdot)$ there exist continuum of group topologies in each of which the topological group has a countable basis of the filter of all neighborhoods of the unity element such that any two topologies are comparable;*

3.2.3. *If $\mathcal{T}_0 \neq \mathcal{T}_1$, then in the group $G(\cdot)$ there are 2^c (two to the power of continuum) of group topologies any two of which are incomparable.*

Proof. **Proof of Statement 3.2.1.** As for any normal subgroup N of the group $G(\cdot)$ the set $\{N\}$ satisfies all conditions of Theorem 2.6, then it is a basis if the filter of neighborhoods of the group $G(\cdot)$ for some group topology $\tau(N)$, and in this topology the topological group has a finite basis of the filter of all neighborhoods of the unity element, i. e. $\tau(N) \in \mathcal{T}_1$.

Now, if $\tau_0 \in \mathcal{T}_1$ and $\mathcal{B}$ is some finite basis of the filter of all neighborhoods of unity element in the topological group $(G(\cdot), \tau_0)$, then $N(\tau_0) = \bigcap_{V \in \mathcal{B}} V$ is an open normal subgroup of $G(\cdot)$, and hence, $N(\tau_0)$, is a neighborhood of the unity element in the topological group $(G(\cdot), \tau_0)$. Then $\tau(N(\tau_0)) = \tau_0$.

So, we have proved that $\mathcal{T}_1 = \{\tau(N) | N \in \mathcal{N}\}$. As $\tau(N_1) \leq \tau(N_2)$ if and only if $N_1 \supseteq N_2$, then $(\mathcal{T}_1, \leq)$ is a lattice, which is anti-isomorphic (see I.4) to the lattice $(\mathcal{N}, \subseteq)$.

Statement 3.2.1 is proved.

Proof of Statement 3.2.2. Let $G(\cdot)$ be a group such that $\mathcal{T}_0 \neq \mathcal{T}_1$ and $\tau_0 \in \mathcal{T}_0 \setminus \mathcal{T}_1$. If $\mathbf{B}$ is some basis of the filter of all neighborhoods of the unity element in the topological group $(G(\cdot), \tau_0)$, and $\mathbf{N} = \bigcap_{V \in \mathbf{B}} V$, then N is a closed normal subgroup of the topological group $(G(\cdot), \tau_0)$.

Since $\tau_0 \notin \mathcal{T}_1$, then $\mathbf{N}$ is not a neighborhood of the unity element in the topological group $(G(\cdot), \tau_0)$ (otherwise the set $\{\mathbf{N}\}$ would be a basis of the filter of neighborhoods of unity element in the topological group $(G(\cdot), \tau_0)$).

Then the factor-group $(\overline{G}(\cdot), \overline{\tau}_0) = (G(\cdot), \tau_0)/\mathbf{N}$ is a non-discrete, Hausdorff topological group, and by Statement 3.1.1, a set $\overline{\mathbf{T}}$ of cardinality of continuum of group topologies exists on the group $G(\cdot)/\mathbf{N}$, in each of which a topological group has a countable basis of the filter of neighborhoods of the unity element and any two of them are comparable.

Since the canonical homomorphism $f : G(\cdot) \to G(\cdot)/\mathbf{N}$ is a surjective map, then $f\big(f^{-1}(\overline{V})\big) = \overline{V}$ for any subset $\overline{V} \subseteq G(\cdot)/\mathbf{N}$.

Let now $\overline{\tau}_1, \overline{\tau}_2 \in \overline{\mathbf{T}}$, and let τ_1 and τ_2 be the prototype topologies $\overline{\tau_1}$ and $\overline{\tau_2}$ with respect to the homomorphism f, respectively.

If $\overline{\tau}_1 \leq \overline{\tau}_2$, then $\tau_1 = \{f^{-1}(\overline{U}) | \overline{U} \in \overline{\tau}_1\} \subseteq \{f^{-1}(\overline{V}) | \overline{V} \in \overline{\tau}_2\} = \tau_2$, and if $\tau_1 \leq \tau_2$, then $\overline{\tau}_1 = \{f\big(f^{-1}(\overline{U})\big) | \overline{U} \in \overline{\tau}_1\} = \{f(U) | U \in \tau_1\} \subseteq \{f^{-1}(V) | V \in \tau_2\} = \{f\big(f^{-1}(\overline{V})\big) | \overline{U} \in \overline{\tau}_2\} = \overline{\tau}_2$, and hence, $\tau_1 \leq \tau_2$ if and only if $\overline{\tau}_1 \leq \overline{\tau}_2$.

Then a set of cardinality of continuum of group topologies exists on the group $G(\cdot)$ in each of which the respective topological group has a countable basis of the filter of neighborhoods of the unity element and any two of them are comparable.

Statement 3.2.2 is proved.

Statement 3.2.3 can be proved by analogy with the proof of Statement 3.2.2 if you use Statement 3.1.3 and the fact that if $sup\{\overline{\tau}_1, \overline{\tau}_2\} = \tau_d^*$ for non-discrete topologies $\overline{\tau}_1$ and $\overline{\tau}_2$, then the topologies τ_1 and τ_2 are incomparable.

This theorem is proved. □

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