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On spaces of densely continuous forms

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Abstract. We study the structure of the domain of the minimal upper semicontinuous extension of the set-valued mapping. It is proved that the set of all compact-valued upper semicontinuous mappings is closed in the space of all set-valued mappings. A similar assertion is true for the space of densely continuous forms.

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1 Introduction

Let $(Y,\mathcal{U})$ be a uniform space and X be a topological space. By $\exp(Y)$ or 2^Y we denote the space of all closed subsets of Y. The uniformity $\mathcal{U}$ is generated by a family of uniformly continuous pseudometrics $P(\mathcal{U})$. Consider that $\rho(y,z) \leq 1$ for all $\rho \in P(\mathcal{U})$ and $y, z \in Y$.

Let $\rho \in P(\mathcal{U})$. If $y \in Y$ and $L \subseteq Y$, then $N(y,\rho,r) = \{z \in Y : \rho(y,z) < r\}$ and $N(L,\rho,r) = \cup\{N(y,\rho,r) : y \in L\}$ for a real number $r > 0$. We put $\varphi(L,M) = \inf\{r : L \subseteq N(M,\rho,r)$, $M \subseteq N(L,\rho,r)$. If $\emptyset \in \{L,M\}$ and $L \neq M$, then $h\rho(L,M) = 1$. The families $hP(\mathcal{U}) = \{h\rho : \rho \in P(\mathcal{U})\}$ generate the uniformity $h(\mathcal{U})$ on $\exp(Y)$.

A set-valued mapping $g : X \to Y$ assigns to each point $x \in X$ a closed subset $g(x)$ of Y.

Let $g : X \to Y$ be a set-valued mapping. The mapping g is called:

– upper semicontinuous (us-continuous) at a point $x_0 \in X$ if for every open $V \subseteq Y$ with $g(x_0) \subseteq V$ there exists an open set U of X such that $x_0 \in U$ and $F(x) \subseteq V$ for any $x \in U$;

– lower semicontinuous (ls-continuous) at a point $x_0 \in X$ if for every open $V \subseteq Y$ with $g(x_0) \cap V \neq \emptyset$ there exists an open set U of X with $g(x) \cap V \neq \emptyset$ for each $x \in U$;

– continuous at a point $x \in X$ if g simultaneous by is us-continuous and ls-continuous at the point x;

– weakly continuous (w-continuous) if the graphic $Gr(g) = \cup\{\{x\} \times g(x) : x \in X\}$ is a closed subset of the space $X \times Y$;

– minimal if the graphic $Gr(g)$ is closed in $X \times Y$, the set $Dom(g) = \{x \in X : g(x) \neq \emptyset\}$ is dense in X and for each closed subset F of $Gr(g)$ such that $F \neq Gr(g)$ there exists a point $x \in Dom(g)$ such that $F \cap (\{x\} \times g(x)) = \emptyset$.

Remark 1. The set $Dom(g) = \{x \in X : g(x) \neq \emptyset\}$ is the domain of the mapping $g : X \to Y$. If the set $Dom(g)$ is dense in X and $x \in X \setminus Dom(g)$ then g is not us-continuous and not ls-continuous at the point $x \in X$.

A mapping $g : X \to Y$ is called:
– us-continuous if it is us-continuous at any point $x \in Dom(g)$;
– ls-continuous if it is ls-continuous at any point $x \in Dom(g)$;
– continuous if it is continuous at any point $x \in Dom(g)$.

Denote by $F(X, Y)$ the set of all single-valued mappings of the space X into the space Y, by $F(X, 2^Y)$ the set of all set-valued mappings of X into Y, by $C(X, Y) = \{g \in F(X, Y) : g \text{ is continuous}\}$ the set of all continuous mappings of X into Y.

Let $\mathcal{A}$ be a family of subsets of X which is closed under finite union and which covers X.

We define on $F(X, 2^Y)$ the topology of uniform convergence on sets in $\mathcal{A}$ as follows.

For any pseudometric $\rho \in P(\mathcal{U})$ and each $B \in \mathcal{A}$ on $F(X, 2^Y)$ define the pseudometric $\rho_B(f, g) = \sup\{h\rho(f(x), g(x)) : x \in B)\}$.

Then $F(X, 2^Y)$ has the topology generated by the family of pseudometrics $\mathcal{A}(\mathcal{U}) = \{\rho_B : B \in \mathcal{A}, \rho \in P(\mathcal{U})\}$. The pseudometrics $\mathcal{A}(\mathcal{U})$ form on $F(X, 2^Y)$ a Hausdorff uniform structure and the space $F_{\mathcal{A}}(X, 2^Y)$ with this topology is completely regular and Hausdorff [3].

Whenever $\mathcal{A}$ consists of the all finite subsets of X, the topology generated by the uniform structure $\mathcal{A}(\mathcal{U})$ is the topology of pointwise convergence on $F(X, 2^Y)$ and this space is denoted by $F_p(X, 2^Y)$.

Since $\mathcal{A}$ consists of the all compact subsets of X, then the topology generated by the uniform structure $\mathcal{A}(\mathcal{U})$ is the topology of uniform convergence on compact sets and this space is denoted by $F_c(X, 2^Y)$.

Whenever $X \in \mathcal{A}$, then this topology is called the topology of uniform convergence and this space is denoted by $F_u(X, 2^Y)$.

On subspaces of the space $F_{\mathcal{A}}(X, 2^Y)$ we consider the topology generated by the uniform structure $\mathcal{A}(\mathcal{U})$ too.

2 Extensions of mappings

Fix a space X, a uniform space $(Y, \mathcal{U})$ with the uniformity $\mathcal{U}$ generated by the pseudometrics $P(\mathcal{U})$ and a compactification cY of Y.

Let $g : X \to Y$ be a set-valued mapping.

A set-valued mapping $\varphi : X \to Y$ is said to be an usc-extension of the mapping g if φ is a compact-valued us-continuous mapping and $g(x) \subseteq \varphi(x)$ for any $x \in X$.

A set-valued mapping $\varphi : X \to Y$ is said to be a minimal usc-extension of the mapping g if φ is an usc-extension of the mapping g and for any usc-extension $\psi : X \to Y$ of g we have $\varphi(x) \subseteq \psi(x)$ for any $x \in X$.

Remark 2. One can say that a set-valued mapping $\varphi : X \to Y$ is a maximal usc-extension of the mapping g if φ is an usc-extension of g and $Dom(\psi) \subseteq Dom(\varphi)$ for any usc-extension $\psi : X \to Y$ of g. If the space Y is compact, then the mapping $g_{max} : X \longrightarrow Y$, where $g_{max}(x) = Y$ for any $x \in X$, is the maximal usc-extension of any mapping $g : X \longrightarrow Y$. If the space Y is not compact and $\psi : X \longrightarrow Y$ is an usc-extension of the mapping $g : X \longrightarrow Y$, then $\psi(x) \neq Y$ for any $x \in X$. Fix a compact subset F of Y and put $\psi_F(x) = \psi(x) \cup F$ for each $\cup x \in X$. Then $\psi_F : X \longrightarrow Y$ is a usc-extension of the mapping g and $Gr(\psi) \subseteq Gr(\psi_F)$. Hence, for a non-compact space Y for each mapping $g : X \longrightarrow Y$ does not exist maximal usc-extension.

Proposition 1. *Let $g : X \to Y$ be a set-valued mapping and the domain $Dom(g)$ is dense in X. The following assertions are equivalent:*

1. *The mapping g has some usc-extension.*
2. *For g there exists a unique minimal usc-extension $m_c g : X \to Y$.*

Proof. The implication $2 \to 1$ is obvious. Assume that $\varphi : X \to Y$ is an usc-extension of g. Then φ is an us-continuous mapping of X into cY. Denote by $\pi_X : X \times cY \to X$ the projection $\pi_X(x, y) = x$ for all $(x, y) \in X \times cY$. Since cY is a compact space, the projection π_X is a perfect mapping. Denote by $\pi_{cY} : X \times cY \to cY$ the projection onto cY. The mapping π_{cY} is continuous.

Every subset $M \subseteq X \times cY$ is the graphic of some concrete set-valued mapping $\theta_M : X \to cY$, where $\theta_M(x) = \prod_{cY}(M \cap (\{x\} \times CY))$ for any $x \in X$. The mapping θ_M is us-continuous if and only if the set M is closed in the subspace $\pi_X(M) \times cY$.

In particular, if $\psi : X \to Y$ is an usc-extension of g, then $Gr(g) \subseteq Gr(\psi)$ and the set $Gr(\psi)$ is closed in the subspace $Dom(\psi) \times cY$. Hence $Gr(g) \subseteq Gr(\varphi)$ and the set $Gr(\varphi)$ is closed in $Dom(\varphi) \times cY$.

Denote by Φ the closure of the set $Gr(g)$ in the space $X \times cY$.

Then the set $\Phi_1 = \Phi \cap (Dom(\varphi) \times cY)$ is the closure of $Gr(g)$ in $Dom(\varphi) \times cY$. The mapping $w : X \to cY$, where $Gr(w) = \Phi$, is us-continuous. Moreover, $g(x) \subseteq w(x) \subseteq \varphi(x) \subseteq Y$ for any $x \in Dom(g)$. Let $H = \{x \in X : w(x) \subseteq Y\}$. By construction, $Dom(g) \subseteq Dom(\varphi) \subseteq H$. Denote by $m_c g : X \to Y$ the mapping with the domain $Dom(m_c g) = H$ and $m_c g(x) = w(x)$ for any $x \in H$. Since $Dom(w) = X$, the mapping $m_c g$ is correctly defined. Obviously, $m_c g$ is an usc-extension of g.

Let $\psi : X \to Y$ be a usc-extension of g. Since $Gr(g) \subseteq Gr(\psi)$, the set $\Phi \cap Gr(\psi)$ is the closure of the set $Gr(g)$ in $Dom(\psi) \times cY$. Thus $m_c g(x) = w(x) \subseteq \psi(x)$ for any $x \in H \cap Dom(\psi) = Dom(m_c g) \cap Dom(\psi)$ and $Dom(\psi) \subseteq Dom(m_c g)$. Hence $m_c g$ is the minimal usc-extension of g. The existence of the minimal usc-extension is proved. The uniqueness of the minimal usc-extension is obvious. The proof is complete. □

Let $g : X \to Y$ be a set-valued mapping. The mapping $m_e g : X \to Y$ with the graphic $Gr(m_e g) = cl_{X \times Y} Gr(g)$ is called the minimal w-continuous extension of the mapping g.

Proposition 2. *Let $g : X \to Y$ be a set-valued mapping and the set $Dom(g)$ is dense in X. Then:*

1. If $\varphi : X \to Y$ is a w-continuous mapping and $g(x) \subseteq \varphi(x)$ for any $x \in Dom(g)$, then $m_e g(x) \subseteq \varphi(x)$ for any $x \in X$.

2. If $m_c g$ is the minimax usc-extension of g, then $m_c g(x) \subseteq m_e g(x)$ for any $x \in X$.

Proof. Follows from the coincidence of the closures of the sets $Gr(g)$, $Gr(m_c g)$, and $Gr(m_e g)$ in $X \times cY$. □

Corollary 1. *If the space Y is compact, then $m_c g = m_e g$ for any set-valued mapping $g : X \to Y$ with the dense domain $Dom(g)$ in X.*

Remark 3. Let $m_c g : X \to Y$ be the minimax usc-extension of a set-valued mapping $g : X \to Y$ with the dense domain $Dom(g)$ in X. If $x \notin Dom(g)$, then we say that X is an essential point of usc-discontinuity of the mapping g. If $x \in Dom(m_c g) \setminus Dom(g)$, then x is inessential point of usc-discontinuits of the mapping g.

3 m-metric and m-Baire spaces

Let m be an infinite cardinal number.

A uniform space $(Y,\mathcal{U})$ is an m-metric space if the uniform structure U is generated by a family $P(\mathcal{U})$ of pseudometrics of cardinality $\leq m$. In this case we assume that the cardinality $|P(\mathcal{U})| \leq m$ and for any $\rho_1, \rho_2 \in P(U)$ there exists $\rho \in P(U)$ such that $\sup\{\rho_1(x, y), \rho_2(x, y)\} \leq \rho(x, y)$ for all $x, y \in Y$.

A set L of a space X is called a G_m-set if L is the intersection of m open subsets of X. For $m = \aleph_0$ the G_m-set is called a G_δ-set. The complement of a G_m-set is an F_m-set and of G_δ-set is an F_σ-set.

A subset A of a space X is called m-meager if A is the union of m nowhere dense subsets of X. The space X is called an m-Baire space if every non-empty open subset of X is not m-meager.

For a space X the next three assertions are equivalent:

1 bm) X is an m-Baire space.

2 bm) The intersection of m open and dense subsets of X is dense in X.

3 bm) The intersection of m dense G_m-subsets is dense in X.

A space X is a Baire space if it is an $\aleph_0$-Baire space.

A space X is called m-complete if X is a G_m-subset of some compactification cX of X.

Proposition 3. *Let $(Y,\mathcal{U})$ be an m-complete m-metric space, $g : X \to Y$ be a set-valued mapping with a dense domain $Dom(g)$ in X and $m_c g : X \to Y$ be the usc-extension of g. Then $Dom(m_c g)$ is a dense G_m-set of X.*

Proof. Let Φ be the closure of the set $Gr(g)$ in $X \times cY$, where cY is a compactification of Y, and $\omega : X \to cY$ be the us-continuous mapping with the graphic $Gr(\omega) = \Phi$. Then Φ is the closure of the set $Gr(m_c g)$ in $X \times cY$ too. By construction, $Dom(m_c g) = \{x \in X : \omega(x) \subseteq Y\}$ and $m_c g(x) = \omega(x)$ for all $x \in Dom(m_c g)$. Fix a family $\{U_\alpha : \alpha \in A\}$ of open subsets of Y for which $|A| \leq m$ and $Y = \cap\{U_\alpha : \alpha \in A\}$. For any $\alpha \in A$ the set $V_\alpha = \{x \in X : \omega(x) \subseteq U_\alpha\}$ is open in X and $Dom(m_c g) \subseteq V_\alpha$. Let $L = \cap\{V_\alpha : \alpha \in A\}$. By construction, $Dom_c(g) \subseteq L$ and L is a G_m-set of X. Suppose that $x \notin Dom_c(g)$. Then there exist a point $y \in \omega(x) \setminus Y$ and $\alpha \in A$ for which $y \notin U_\alpha$. Then $x \notin V_\alpha$ and $x \notin L$. Therefore $L = Dom_c(g)$. The proof is complete. □

Proposition 4. *Let $g : X \to Y$ be a minimal us-continuous mapping of a space X into an m-metric space $(Y,\mathcal{U})$. Then $Dom_s(g) = \{x \in X : g(x)$ is a singleton set$\}$ is a G_m-subset of $Dom(g)$.*

Proof. We can assume that $X = Dom(g)$. Consider the pseudometrics $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$ which generate the uniformity $\mathcal{U}$ on Y. Assume that $|A| \leq m$.

For every $n \in N = \{1, 2, \ldots\}$, $\alpha \in A$ and $y \in Y$ we put $V(y, \alpha, n) = \{x \in X : g(x) \subseteq N(y, \rho_\alpha, 2^{-n})\}$ and $V(\alpha, n) = \cup\{V(y, \alpha, n) : y \in Y\}$. Since the set $N(y, \rho_\alpha, 2^{-n})$ is open in Y and the mapping g is us-continuous, the set $V(\alpha, n)$ is open in X. The set $L = \cap\{V(\alpha, n) : \alpha \in A, n \in N\}$ is a G_m-set of X.

If $x \in Dom_s(g)$ and $g(x) = y \in Y$, then $x \in V(y, \alpha, n)$ for all $\alpha \in A$ and $n \in N$. Hence $Dom_s(g) \subseteq L$. Let $x \notin Dom_s(g)$. Then there exist two distinct points $y_0, z_0 \in g(x), \alpha \in A$ and $n \in N$ for which $\rho_\alpha(y_0, z_0) > 2^{-n} > 0$. In this case $g(x) \setminus V(y, \alpha, n) \neq \emptyset$ for any $y \in Y$. Thus $x \notin V(\alpha, h)$. Therefore $L = Dom_s(g)$. The proof is complete. □

Corollary 2 *Let $g : X \to Y$ be a us-continuous mapping of an m-Baire space X into an m-complete m-metric uniform space $(Y,\mathcal{U})$. The following assertions are equivalent:*

1. *The mapping $m_e g : X \to Y$, Y is minimal, i.e. $g : Dom(g) \to Y$ is a minimal mapping.*
2. *$Dom_s(g)$ is a dense G_m-set of X.*

Proof. Implication $2 \to 1$ is obvious. Let $g : Dom(g) \to Y$ be minimal. Then the mapping $m_c g : X \to Y$ is minimal. Proposition 4 completes the proof. □

4 Spaces of dense forms

Fix an infinite cardinal number m, an m-Baire space X and an m-complete m-metric space $(Y,\mathcal{U})$ with uniformity $\mathcal{U}$ generated by the family $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$ of pseudometrics, where $|A| \leq m$.

A set-valued mapping $g : X \to Y$ is called a dence set-valued continuous form from X to Y if $Dom(g)$ is a dense subset of X and $g = m_c g$.

A set-valued mapping $g : X \to Y$ is called a dense continuous form from X to Y if $Dom_s(g)$ is a dense subset of X and $g = m_c g$.

Remark 4. From Corollary 2 it follows that for a set-valued mapping $g : X \to Y$ the following assertions are equivalent:

1. g is a dense continuous form from X to Y.

2. There exists a dense subspace Z of X and a continuous single-valued mapping $f : Z \to Y$ such that $g = m_c f$.

Hence our definition of a dense continuous form coincides with the definition of a dense continuous form from [5].

Denote by $DUC(X, Y)$ the family of all us-continuous compact-valued mappings $g : X \to Y$ for which the domain $Dom(g)$ is dense in X, by $DU(X, Y)$ the family of all dense set-valued continuous forms from X to Y, by $DC(X, Y)$ the family of all single-valued mappings $g \in DUC(X, Y)$ and by $D(X, Y)$ the family of all dense continuous forms from X to Y. It is obvious that $D(X, Y) \subseteq DU(X, Y) \subseteq DUC(X, Y)$.

There exists a single-valued mapping $e : DUC(X, Y) \to DU(X, Y)$, where $e(g) = m_c g$ for any $g \in DUC(X, Y)$.

5 Completeness of the spaces of set-valued dense continuous forms

Fix a space X and a complete uniform space $(Y,\mathcal{U})$ with the uniformity $\mathcal{U}$ generated by the family of pseudometrics $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$.

Let $FC(X, Y)$ be the set of all compact-valued us-continuous mappings of X into Y. On X fix a family Γ of subsets which is closed under finite union and which covers X.

On $F(X, exp(Y))$ consider the topology and the uniformity generated by the pseudometrics $\Gamma(\mathcal{U}) = \{\rho_{\alpha B} : \alpha \in A, B \in \Gamma\}$.

Theorem 1. *Let $X \in \Gamma$. Then the set $FC(X, Y)$ is closed in the space $F_\Gamma(X, exp(Y))$.*

Proof. Let $g \in F(X, exp(Y) \setminus FC(X, Y)$.

Case 1. $g(x_0)$ is not a compact set for some point $x_0 \in X$.

Since Y is a complete uniform space, in this case there exist $\alpha \in A$, $\varepsilon > 0$ and an infinite sequence $\{y_n \in g(x_0) : n \in N\}$ such that $\rho_\alpha(y_n, y_m) \geq \varepsilon$ for all $n, m \in N$ and $n \neq m$. Fix $B \in \Gamma$ for which $x_0 \in B$ and $\delta < e^{-1}$ such that $0 < 3\delta < \varepsilon$.

Let $f \in F(X : \exp(Y))$ and $\rho_{\alpha B}(f, g) < \delta$. Then for any $n \in N$ there exists a point $z_n \in f(x_0)$ such that $\rho_\alpha(y_n, z_n) < \delta$. In this case $\rho_\alpha(z_{n,} z_m) \geq \delta$ for all $n, m \in N$ and $n \neq m$. Thus the set $f(x_0)$ is not precompact in Y. Since Y is complete, the set $f(x_0)$ is not compact. Therefore the set $V = \{f \in F(X, \exp(Y)) : \rho_{\alpha B}(g, f) < \delta\}$ is open in $F_\Gamma(X, \exp(Y))$, $g \in V$ and $V \cap FC(X, Y) = \emptyset$.

Case 2. $g(x)$ is a compact set of Y for each $x \in X$.

In this case there exists a point $x_0 \in Dom(g)$ such that g is not us-continuous at x_0. Thus there exists an open subset U of Y such that $g(x_0) \subseteq U$ and for any neighborhood W of x_0 in X there exists a point $x \in W \cap Dom(g)$ for which $g(x) \setminus U \neq \emptyset$.

Since the set $g(x_0)$ is compact there exists $\varepsilon > 0$ and $\alpha \in A$ such that $N(g(x_0), \rho_\alpha, 4\varepsilon) \subseteq U$. Suppose that $\rho_{\alpha x}(g, f) < \varepsilon$ and $f \in FC(X, Y)$. In this case $f(x_0) \subseteq N(g(x_0), p_\alpha, \varepsilon)$, the set $W = \{x \in X : f(x) \subseteq N(g(x_0), p_\alpha, \varepsilon)\}$ is open in X and $x_0 \in W$. There exists $x \in W$ such that $g(x) \setminus U \neq \emptyset$. Fix $y \in g(x) \setminus U$.

Since $y \in N(g(x_0), p_\alpha, 4\varepsilon)$, then $N(y, p_\alpha, 4\varepsilon) \cap g(x_0) = \emptyset$. Since $\rho_{\alpha x}(g, f) < \varepsilon$, there exists $z \in f(x)$ such that $\rho_{\alpha X}(g, f) < \varepsilon$. Hence $z \notin N(g(x_0), \rho_\alpha, \varepsilon)$ and $x \notin W$, a contradiction. Therefore $N(g(x_0), \rho_\alpha, \varepsilon) \cap FC(X, Y) = \emptyset$ and the set $FC(X, Y)$ is closed. □

In the case 1 we have proved the following assertion.

Proposition 5. *The set $F^c(X, \exp Y)$ of all compact-valued mappings is closed in the space $F_\Gamma(X, \exp Y)$ for any family Γ.*

Proposition 6. *The set $DU(X, Y)$ is dense in the space $F^c_p(X, \exp Y)$ of all compact–valued mappings in the topology of pointwise convergence.*

Proof. Fix a mapping $g \in F^c(X, \exp Y), \alpha \in A, \varepsilon > 0$ and a finite subset $F = \{x_1, x_2 \dots x_m\}$ of X. Fix a point $b \in Y$ and the open subsets $\{v_1, v_2, \dots v_n\}$ of X such that $x_i \in V_i$ and $V_i \cap V_j = \emptyset$ for all $i, j \leq n$ and $i \neq j$.

We put $f(x) = g(x_i)$ for all $i \leq n$ and $x \in V_i$, and $f(x) = \cup\{g(x_i) : i \leq n\}$ for any $x \in (X \setminus U\{V_i : i \leq n\})$. Then f is us-continuous, $Dom(f) = X$ and $\rho_{\alpha F}(g, f) = 0 < \varepsilon$. The proof is complete. □

Proposition 7. *The set $F^d(X, \exp Y)$ of all set-valued mappings $g : X \to Y$ with a dense domain $Dom(g)$ in X is closed in $F_u(X, \exp(Y))$ in the topology of uniform convergence.*

Proof. Let $g : X \to Y$ be a set-valued mapping and the set $Dom(g)$ be not dense in X. Then the set $V = X \setminus cl_X\, Dom(g)$ is open and non–empty.

Fix $\alpha \in A$. If $L = Y$ and $L \neq \emptyset$, $h\rho_\alpha(\emptyset, L) = 1$. The set $U = \{f \in F(X, \exp(Y)) : h\rho_\alpha(g, f) < 1\}$ is open in $F_u(X, \exp(Y))$ and $g \in U$. Let $f \in F^d(X, \exp(Y))$. Since the set $Dom(f)$ is dense in X, there exists a point $x \in V \cap Dom(f)$. In this case $f(x) \neq \emptyset$ and $g(x) = \emptyset$. Hence $h\rho_\alpha(f(x), g(x)) = 1$ and $f \notin U$. Therefore $U \cap F^d(X, \exp Y) = \emptyset$. The proof is complete. □

Corollary 3. *The set $F^{cd}(X, \exp(Y))$ of all compact-valued mappings with the dense domain is dense in the space $F_u(X, \exp(Y))$.*

Proof. By virtue of Propositions 5 and 7, the set $F^{ed}(X, \exp(Y)) = F^c(X : \exp Y) \cap F^d(X, \exp(Y))$ is closed in $F_u(X, \exp(Y))$. □

Theorem 2. *The set $FC(X, Y)$ is closed in the space $F_u(X, \exp(Y))$.*

Proof. Let $\exp_c(Y)$ be the spaces of all compact subsets of Y in the topology generated by the pseudometrics $hP(U)$. The uniform space $\exp_c(Y)$ is complete [9]. Fix a Cauchy sequence $\{g_\mu : \mu \in M\}$, where M is a directed set. Since the space $\exp_c(Y)$ is complete, for any $x \in X$ in $\exp_c(Y)$ there exists the limit $g(x) = lim\{g_\mu(x) : \mu \in M$. In this case $g = \lim\{g_\mu : \mu \in M\}$ in the space $F_n(X, \exp(Y))$. Fix $\alpha \in A$. There exists $\lambda \in M$ such that $h\rho_\alpha(g(x), g_\mu(x)) < 1$ for all $\mu \geq \lambda$ and all $x \in X$. Thus

$Dom(g) = Dom(g_\mu)$ for all $\mu \geq \lambda$. We can assume that $Dom(g) = Dom(g_\mu) = X$ for all $\mu \in M$.

We affirm that the mapping $g : X \to Y$ is us-continuous. Fix $x_0 \in X$ and an open subset U of Y for which $g(x_0) \subseteq U$. There exist $\alpha \in A$ and $0 < \varepsilon < 1$ such that $N(g(x_0), \rho_\lambda, 4\varepsilon) \subseteq U$. Fix now $\mu \in M$ for which $h\rho_\alpha(g(x), g_\mu(x)) < \varepsilon$ for all $x \in X$. The set $V = \{x \in X : g_\mu(x) \subseteq N(g_\mu(x_0), \rho_\alpha, \varepsilon)\}$ is open in X and $x_0 \in V$. If $x \in V$, then $h\rho_\alpha(g_\mu(x_0), g_\mu(x)) < \varepsilon$, $h\rho_\alpha(g(x_0), \rho_\mu(x_0)) < \varepsilon$ and $h\rho_\alpha(g(x_0), g_\mu(x)) < 2\varepsilon$. Sice $h\rho_\alpha(g(x), g_\mu(x)) < \varepsilon$, then $h\rho_\alpha(g(x_0), g(x)) < 3\varepsilon$ and $g(x) \subseteq N(g(x_0), \rho_\alpha, 4\varepsilon) \subseteq U$. Hence g is us-continuous at the point x_0. The proof is complete. □

Corollary 4. *The set $DU(X,Y)$ of all set-valued α continuous forms M which are closed in the space $F_u(X, \exp(Y))$ and in the uniformity of uniform convergence is a complete uniform space.*

6 Completeness of the space of dense continuous forms

Fix an infinite cardinal number m, an m-Baire space X and an m-complete m-metric space $(Y,\mathcal{U})$ with a complete uniformity $\mathcal{U}$ generated by the pseudometrics $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$, where $|A| \leq m$.

Theorem 3. *The set $D(X,Y)$ is closed in the space $F_u(X, \exp(Y))$.*

Proof. Since $D(X,Y) \subseteq DU(X,Y)$ and the set is closed in $F_u(X, \exp Y)$, then it is sufficient to prove that the set $D(X,Y)$ is closed in the space $DU_u(X,Y)$. Let $\{g_\mu \in D(X,Y) : \mu \in M\}$ be a Cauchy sequence where M is a directed set. Since Y is an m-metric space we can assume that $|M| \leq m$. Let $g = \lim\{g_\mu : \mu \in M\}$. From Theorem 2 it follows that g is a compact-valued us-continuous mapping. If $\alpha \in A$, then there exists $\lambda \in M$ such that $h\rho_\alpha(g(x), g_\mu(x)) < 1$ for all $x \in X$ and $\mu \geq \alpha$. Thus $Dom(g) = Dom(g_\mu)$ for all $\mu \geq \lambda$.

Therefore $g \in DU(X,Y)$ and we can assume that $Dom(g) = Dom(g_\mu) = X$ for all $\mu \in M$. From Corollary 2 it follows that $Dom_s(g_\mu) = \{x \in X : g_\mu(x)$ is a singleton set$\}$ is a dense G_m-set of X for any $\mu \in M$. Since $|M| \leq m$ and X is an m-Baire space, the subspace $Z = \cap\{Dom_s(g_\mu) : \mu \in M\}$ is a dense G_m-set of X. Thus $f_\mu = g_\mu|Z : Z \to Y$ is a single-valued continuous mapping of Z into Y for any $\mu \in M$.

Let $f = g|Z : Z \to Y$. Then $f = \lim\{f_\mu : \mu \in M\}$ and the uniform limit of single–valued mappings is a single–valued mappings. Thus $Z \subseteq Dom_s(g)$ and $Dom_s(g)$ is a dense subset of X. From Remark 4 it follows that $g \in D(X;Y)$. The proof is complete. □

Corollary 5. *The space $D(X;Y)$ in the uniformity of uniform convergence is complete.*

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