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On a four-dimensional hyperbolic manifold with finite volume

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Abstract. In article [1] the authors construct and classify all the hyperbolic space-forms H^n/Γ where Γ is a torsion-free subgroup of minimal index in the congruence two subgroup Γ_2^n for $n = 3, 4$. In the present paper some hyperbolic 3- and 4-manifolds are constructed that are absent in [1].

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In the works [1] and [2] some four-dimensional hyperbolic manifolds with finite volume were constructed. They were obtained by identifying the faces of the regular 24-cells in H^4 with all vertices being on the absolute. The present article is devoted to the construction of a four-dimensional hyperbolic manifold with finite volume by identifying the faces of a four-dimensional hyperbolic polyhedron with all the vertices being on the absolute. This polyhedron is not regular and its construction is non-trivial.

1. The construction of a four-dimensional polyhedron

In the four-dimensional space H^4 consider the regular 24-cells R. As it is known this polyhedron has 24 three-dimensional faces, 96 two-dimensional faces, 96 edges, and 24 vertices. The three-dimensional faces of the polyhedron are regular octahedra, two-dimensional faces are regular triangles. Inscribe in the polyhedron R a three-dimensional sphere S^3. Denote its radius by r. If we begin to enlarge the radius r of the sphere S^3, the polyhedron R will increase, but its dihedral angles at the two-dimensional faces will decrease. Continuing the process, we ultimately come to the case when for some r_0 all the vertices of the polyhedron R become infinitely removed, i. e. they get out on the absolute. In this case the three-dimensional faces are regular octahedra with all the vertices being on the absolute. Then the dihedral angles at the two-dimensional faces will be equal to $\pi/2$. Indeed, consider a three-dimensional horosphere centered at a vertex of the polyhedron R. Choose the radius of the horosphere such that the horosphere intersects only one-dimensional edges of the polyhedron which go to the center of the horosphere. Then the intersection of the horosphere and the polyhedron R is a cube. But the dihedral angles at the two-dimensional faces of the polyhedron R are equal to the dihedral angles at the edges of the obtained cube. Since the metric on the horosphere is Euclidean, the dihedral angles of the cube are equal to $\pi/2$, i. e. the dihedral angles at the two-dimensional faces of the polyhedron R are equal to $\pi/2$. If we continue to enlarge the radis of the three-dimensional sphere, the vertices of the polyhedron R will get out on the absolute. We obtain a four-dimensional polyhedron R_2 with all the vertices being infinitely removed. The polyhedron R_2 has three-dimensional faces of two kinds: 24 cubes with all the vertices being on the absolute and 24 truncated octahedra with all the vertices being infinitely removed. The polyhedron R_2 has 94 infinitely removed vertices, 288 one-dimensional edges, two-dimensional faces of two kinds: 144 squares with all the vertices being on the absolute and 96 triangles with all the vertices being infinitely removed. Dihedral angles at two-dimensional faces of this polyhedron are of two kinds: dihedral angles at the squares are equal to $\pi/2$, dihedral angles at the triangles are equal to $\pi/3$, both facts can be easy proved. Label infinitely removed vertices of the polyhedron R_2 by the numbers from 1 to 94. Write all three-dimensional faces of the obtained polyhedron. First write cubes with all the vertices being on the absolute:

$K_{25}(1,7,8,6,30,16,15,22)$ $K_{26}(13,21,91,39,5,1,2,12)$
$K_{27}(7,2,10,3,28,17,19,44)$ $K_{28}(3,9,4,8,24,26,49,35)$
$K_{29}(4,6,5,11,54,33,31,40)$ $K_{30}(18,17,27,68,92,14,15,23)$
$K_{31}(9,10,12,11,52,48,42,41)$ $K_{32}(13,20,64,38,36,16,14,93)$
$K_{33}(18,19,21,20,67,71,45,46)$ $K_{34}(22,23,25,24,34,29,94,73)$
$K_{35}(50,26,25,96,69,51,28,27)$ $K_{36}(29,30,31,32,75,95,36,37)$
$K_{37}(74,56,35,34,32,76,55,33)$ $K_{38}(37,38,39,40,59,77,65,58)$
$K_{39}(66,46,91,58,57,63,43,41)$ $K_{40}(45,70,51,44,42,43,60,47)$
$K_{41}(47,48,49,50,72,61,53,56)$ $K_{42}(53,62,78,55,54,52,57,59)$
$K_{43}(61,60,86,89,90,62,63,87)$ $K_{44}(66,67,64,65,85,87,81,80)$
$K_{45}(70,71,81,86,83,69,68,79)$ $K_{46}(83,82,88,89,72,96,73,74)$
$K_{47}(78,90,85,77,75,76,88,84)$ $K_{48}(92,94,82,79,80,93,95,84)$

The polyhedron R_2 has also 24 truncated octahedra with all the vertices being on the absolute. Write these faces:

$O_1(1,2,3,4,5,6,7,8,9,10,11,12)$
$O_2(15,16,30,22,23,14,36,29,92,93,95,94)$
$O_3(1,7,15,16,13,2,17,14,20,21,19,18)$
$O_4(7,8,22,15,17,3,24,23,25,26,28,27)$
$O_5(8,6,30,22,24,4,31,29,34,35,33,32)$
$O_6(6,1,16,30,36,13,5,31,37,38,39,40)$
$O_7(12,10,42,41,91,2,44,43,45,46,21,19)$
$O_8(10,9,48,42,44,3,49,47,50,51,28,26)$
$O_9(9,11,52,48,49,4,54,53,55,56,35,33)$
$O_{10}(11,12,41,52,54,57,91,5,39,40,59,58)$
$O_{11}(62,61,60,63,57,53,47,43,52,48,42,41)$
$O_{12}(13,21,91,39,38,20,46,58,65,66,67,64)$

$$O_{13}(17, 19, 44, 28, 27, 18, 45, 51, 69, 68, 71, 70)$$
$$O_{14}(72, 74, 73, 96, 50, 56, 34, 25, 26, 24, 35, 49)$$
$$O_{15}(78, 76, 75, 77, 59, 55, 32, 37, 40, 54, 33, 31)$$
$$O_{16}(18, 20, 67, 71, 68, 14, 64, 81, 80, 79, 92, 93)$$
$$O_{17}(25, 27, 69, 96, 73, 83, 68, 23, 92, 94, 82, 79)$$
$$O_{18}(38, 37, 77, 65, 64, 36, 75, 85, 84, 80, 93, 95)$$
$$O_{19}(45, 46, 67, 71, 70, 43, 66, 81, 86, 60, 63, 87)$$
$$O_{20}(32, 34, 74, 76, 75, 29, 73, 88, 82, 84, 95, 94)$$
$$O_{21}(50, 51, 69, 96, 83, 70, 47, 72, 89, 86, 60, 61)$$
$$O_{22}(55, 56, 74, 76, 78, 53, 72, 88, 90, 62, 61, 89)$$
$$O_{23}(58, 59, 77, 65, 66, 57, 78, 85, 90, 62, 63, 87)$$
$$O_{24}(89, 88, 85, 81, 83, 86, 90, 87, 80, 84, 79, 82)$$

2. The construction of a four-dimensional hyperbolic manifold

Indicate motions (isometries) that identify faces of the polyhedron:

$$\varphi_1 : \begin{matrix} (1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12) \\ (89, & 88, & 85, & 81, & 83, & 86, & 90, & 87, & 80, & 84, & 79, & 82); \end{matrix}$$

$$\varphi_2 : \begin{matrix} (15, & 16, & 30, & 22, & 23, & 14, & 36, & 29, & 92, & 93, & 95, & 94) \\ (62, & 61, & 60, & 63, & 57, & 53, & 47, & 43, & 52, & 48, & 42, & 41); \end{matrix}$$

$$\varphi_3 : \begin{matrix} (1, & 7, & 15, & 16, & 13, & 2, & 17, & 14, & 20, & 21, & 19, & 18) \\ (55, & 56, & 74, & 76, & 78, & 53, & 72, & 88, & 90, & 62, & 61, & 89); \end{matrix}$$

$$\varphi_4 : \begin{matrix} (7, & 8, & 22, & 15, & 17, & 3, & 24, & 23, & 25, & 26, & 28, & 27) \\ (58, & 59, & 77, & 65, & 66, & 57, & 78, & 85, & 90, & 62, & 63, & 87); \end{matrix}$$

$$\varphi_5 : \begin{matrix} (8, & 6, & 30, & 22, & 24, & 4, & 31, & 29, & 34, & 35, & 33, & 32) \\ (45, & 46, & 67, & 71, & 70, & 43, & 66, & 81, & 86, & 60, & 63, & 87); \end{matrix}$$

$$\varphi_6 : \begin{matrix} (6, & 1, & 16, & 30, & 36, & 13, & 5, & 31, & 37, & 38, & 39, & 40) \\ (50, & 51, & 69, & 96, & 83, & 70, & 47, & 72, & 89, & 86, & 60, & 61); \end{matrix}$$

$$\varphi_7 : \begin{matrix} (12, & 10, & 42, & 41, & 91, & 2, & 44, & 43, & 45, & 46, & 21, & 19) \\ (32, & 34, & 74, & 76, & 75, & 29, & 73, & 88, & 82, & 84, & 95, & 94); \end{matrix}$$

$$\varphi_8 : \begin{matrix} (10, & 9, & 48, & 42, & 44, & 3, & 49, & 47, & 50, & 51, & 28, & 26) \\ (38, & 37, & 77, & 65, & 64, & 36, & 75, & 85, & 84, & 80, & 93, & 95); \end{matrix}$$

$$\varphi_9 : \begin{matrix} (9, & 11, & 52, & 48, & 49, & 4, & 54, & 53, & 55, & 56, & 35, & 33) \\ (18, & 20, & 67, & 71, & 68, & 14, & 64, & 81, & 80, & 79, & 92, & 93); \end{matrix}$$

$$\varphi_{10} : \begin{matrix} (11, & 12, & 41, & 52, & 54, & 57, & 91, & 5, & 39, & 40, & 59, & 58) \\ (25, & 27, & 69, & 96, & 73, & 83, & 68, & 23, & 92, & 94, & 82, & 79); \end{matrix}$$

$$\varphi_{11}: \begin{array}{llllllllllll} (13, & 21, & 91, & 39, & 38, & 20, & 46, & 58, & 65, & 66, & 67, & 64) \\ (72, & 74, & 73, & 96, & 50, & 56, & 34, & 25, & 26, & 24, & 35, & 49); \end{array}$$

$$\varphi_{12}: \begin{array}{llllllllllll} (17, & 19, & 44, & 28, & 27, & 18, & 45, & 51, & 69, & 68, & 71, & 70) \\ (78, & 76, & 75, & 77, & 59, & 55, & 32, & 37, & 40, & 54, & 33, & 31); \end{array}$$

$$\varphi_{13}: \begin{array}{llllllll} (1, & 7, & 8, & ,6, & 30, & 16, & 15, & 22) \\ (66, & 46, & 91, & 58, & 57, & 63, & 43, & 41); \end{array}$$

$$\varphi_{14}: \begin{array}{llllllll} (13, & 21, & 91, & 39, & 5, & 1, & 2, & 12) \\ (66, & 67, & 64, & 65, & 85, & 87, & 81, & 80); \end{array}$$

$$\varphi_{15}: \begin{array}{llllllll} (7, & 2, & 10, & 3, & 28, & 17, & 19, & 44) \\ (50, & 26, & 25, & 96, & 69, & 51, & 28, & 27); \end{array}$$

$$\varphi_{16}: \begin{array}{llllllll} (3, & 9, & 4, & 8, & 24, & 26, & 49, & 35) \\ (83, & 82, & 88, & 89, & 72, & 96, & 73, & 74); \end{array}$$

$$\varphi_{17}: \begin{array}{llllllll} (4, & 6, & 5, & 11, & 54, & 33, & 31, & 40) \\ (74, & 56, & 35, & 34, & 32, & 76, & 55, & 33); \end{array}$$

$$\varphi_{18}: \begin{array}{llllllll} (18, & 17, & 27, & 68, & 92, & 14, & 15, & 23) \\ (45, & 70, & 51, & 44, & 42, & 43, & 60, & 47); \end{array}$$

$$\varphi_{19}: \begin{array}{llllllll} (9, & 10, & 12, & 11, & 52, & 48, & 42, & 41) \\ (13, & 20, & 64, & 38, & 36, & 16, & 14, & 93); \end{array}$$

$$\varphi_{20}: \begin{array}{llllllll} (18, & 19, & 21, & 20, & 67, & 71, & 45, & 46) \\ (70, & 71, & 81, & 86, & 83, & 69, & 68, & 79); \end{array}$$

$$\varphi_{21}: \begin{array}{llllllll} (22, & 23, & 25, & 24, & 34, & 29, & 94, & 73) \\ (61, & 60, & 86, & 89, & 90, & 62, & 63, & 87); \end{array}$$

$$\varphi_{22}: \begin{array}{llllllll} (29, & 30, & 31, & 32, & 75, & 95, & 36, & 37) \\ (53, & 62, & 78, & 55, & 54, & 52, & 57, & 59); \end{array}$$

$$\varphi_{23}: \begin{array}{llllllll} (37, & 38, & 39, & 40, & 59, & 77, & 65, & 58) \\ (78, & 90, & 85, & 77, & 75, & 76, & 88, & 84); \end{array}$$

$$\varphi_{24}: \begin{array}{llllllll} (47, & 48, & 49, & 50, & 72, & 61, & 53, & 56) \\ (92, & 94, & 82, & 79, & 80, & 93, & 95, & 84). \end{array}$$

Consider cycles of two-dimensional faces of the polyhedron R_2. As the dihedral angles at the quadrangular faces are equal to $\pi/2$, in order that the cycles of these faces be inessential each of them must contain four faces. Write cycles of these faces. We will present cycles of faces as follows: write a face, then write the motion that transfers this face into another face, then again a face, again a motion, and so on.

$(O_1 \cap K_{25})(1,7,8,6)$ φ_{13} $(O_{12} \cap K_{39})(66,46,91,58)$ φ_{11} $(O_{14} \cap K_{34})$
$(24,34,73,25)$ φ_{21} $(O_{24} \cap K_{43})(89,90,87,86)$ φ_1^{-1} $(O_1 \cap K_{25})(1,7,8.6)$;

$(O_3 \cap K_{25})(1,7,15,16)\ \varphi_{13}\ (O_{19} \cap K_{39})(66,46,43,63)\ \varphi_5^{-1}\ (O_5 \cap K_{29})$
$(31,6,4,33)\ \varphi_{17}\ (O_{22} \cap K_{37})(55,56,74,76)\ \varphi_3^{-1}\ (O_3 \cap K_{25})(1,7,15,16);$

$(O_6 \cap K_{25})(1,6,30,16)\ \varphi_{13}\ (O_{23} \cap K_{39})(66,58,57,63)\ \varphi_4^{-1}\ (O_4 \cap O_{27})$
$(17,7,3,28)\ \varphi_{15}\ (O_{21} \cap K_{35})(51,50,96,69)\ \varphi_6^{-1}\ (O_6 \cap K_{25})(1,6,30,16);$

$(O_5 \cap K_{25})(8,6,30,22)\ \varphi_{13}\ (O_10 \cap K_{39})(91,58,57,41)\ \varphi_{10}\ (O_{17} \cap K_{45})$
$(68,79,83,69)\ \varphi_{20}^{-1}\ (O_{19} \cap K_{33})(45,46,67,71)\ \varphi_5^{-1}\ (O_5 \cap K_{25})(8,6,30,22);$

$(O_4 \cap K_{25})(8,7,15,22)\ \varphi_{13}\ (O_7 \cap K_{39})(91,46,43,41)\ \varphi_7\ (O_{20} \cap K_{47})$
$(75,84,88,76)\ \varphi_{23}^{-1}\ (O_{23} \cap K_{38})(59,58,65,77)\ \varphi_4^{-1}\ (O_4 \cap K_{25})(8,7,15,22);$

$(O_2 \cap K_{25})(15,16,30,22)\ \varphi_{13}\ (O_{11} \cap K_{39})(43,63,57,41)\ \varphi_2^{-1}\ (O_2 \cap K_{34})$
$(29,22,23,94)\ \varphi_{21}\ (O_{11} \cap K_{43})(62,61,60,63)\ \varphi_2^{-1}\ (O_2 \cap K_{25})(15,16,30,22);$

$(O_1 \cap K_{26})(1,2,12,5)\ \varphi_{14}\ (O_{24} \cap K_{44})(87,81,80,85)\ \varphi_1^{-1}\ O_1 \cap K_{28})$
$(8,4,9,3)\ \varphi_{16}\ (O_{24} \cap K_{46})(89,88,82,83)\ \varphi_1^{-1}\ (O_1 \cap K_{26})(1,2,12,5);$

$(O_3 \cap K_{26})(1,2,21,13)\ \varphi_{14}\ (O_{19} \cap K_{44})(87,81,67,66)\ \varphi_5^{-1}\ (O_5 \cap K_{36})$
$(32,29,30,31)\ \varphi_{22}\ (O_{22} \cap K_{42})(55,53,62,78)\ \varphi_3^{-1}\ (O_3 \cap K_{26})(1,2,21,13);$

$(O_6 \cap K_{26})(1,5,39,13)\ \varphi_{14}\ (O_{23} \cap K_{44})(87,85,65,66)\ \varphi_4^{-1}\ (O_4 \cap K_{30})$
$(27,23,15,17)\ \varphi_{18}\ (O_{21} \cap K_{40})(51,47,60,70)\ \varphi_6^{-1}\ (O_6 \cap K_{26})(1,5,39,13);$

$(O_7 \cap K_{26})(91,21,2,12)\ \varphi_{14}\ (O_{16} \cap K_{44})(64,67,81,80)\ \varphi_9^{-1}\ (O_9 \cap K_{42})$
$(54,52,53,55)\ \varphi_{22}^{-1}\ (O_{20} \cap K_{36})(75,95,29,32)\ \varphi_7^{-1}\ (O_7 \cap K_{26})(91,21,2,12);$

$(O_{12} \cap K_{26})(91,21,13,39)\ \varphi_{14}\ (O_{12} \cap K_{44})(64,67,66,65)\ \varphi_{11}\ (O_{14} \cap K_{28})$
$(49,35,24,26)\ \varphi_{16}\ (O_{14} \cap K_{46})(73,74,72,96)\ \varphi_{11}^{-1}\ (O_{12} \cap K_{26})(91,21,13,39);$

$(O_{10} \cap K_{26})(91,12,5,39)\ \varphi_{14}\ (O_{18} \cap K_{44})(64,80,85,65)\ \varphi_8^{-1}\ (O_8 \cap K_{40})$
$(44,51,47,42)\ \varphi_{18}^{-1}\ (O_{17} \cap K_{30})(68,27,23,92)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{26})(91,12,5,3);$

$(O_1 \cap K_{27})(2,7,3,10)\ \varphi_{15}\ (O_{14} \cap K_{35})(26,50,96,25)\ \varphi_{11}^{-1}\ (O_{12} \cap K_{38})$
$(65,38,39,58)\ \varphi_{23}\ (O_{24} \cap K_{47})(88,90,85,84)\ \varphi_1^{-1}\ (O_1 \cap K_{27})(2,7,3,10);$

$(O_3 \cap K_{27})(2,7,17,19)\ \varphi_{15}\ (O_8 \cap K_{35})(26,50,51,28)\ \varphi_8\ (O_{18} \cap K_{48})$
$(95,84,80,93)\ \varphi_{24}^{-1}\ (O_{22} \cap K_{41})(53,56,72,61)\ \varphi_3^{-1}\ (O_3 \cap K_{27})(2,7,17,19);$

$(O_7 \cap K_{27})(2,10,44,19)\ \varphi_{15}\ (O_4 \cap K_{35})(26,25,27,28)\ \varphi_4\ (O_{23} \cap K_{43})$
$(62,90,87,63)\ \varphi_{21}^{-1}\ (O_{20} \cap K_{34})(29,34,73,94)\ \varphi_7^{-1}\ (O_7 \cap K_{27})(2,10,44,19);$

$(O_8 \cap K_{27})(28,3,10,44)\ \varphi_{15}\ (O_{17} \cap K_{35})(69,96,25,27)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{31})$
$(41,52,11,12)\ \varphi_{19}\ (O_{18} \cap K_{32})(93,36,38,64)\ \varphi_8^{-1}\ (O_8 \cap K_{27})(28,3,10,44);$

$(O_{13} \cap K_{27})(28,17,19,44)\ \varphi_{15}\ (O_{13} \cap K_{35})(69,51,28,27)\ \varphi_{12}\ (O_{15} \cap K_{38})$
$(40,37,77,59)\ \varphi_{23}\ (O_{15} \cap K_{47})(77,78,76,75)\ \varphi_{12}^{-1}\ (O_{13} \cap K_{27})(28,17,19,44);$

$(O_8 \cap K_{28})(3,9,49,26)\ \varphi_{16}\ (O_{17} \cap K_{46})(83,82,73,96)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{42})$
$(57,59,54,52)\ \varphi_{22}^{-1}\ (O_{18} \cap K_{36})(36,37,75,95)\ \varphi_8^{-1}\ (O_8 \cap K_{28})(3,9,49,26);$

$(O_5 \cap K_{28})(4,8,24,35)\ \varphi_{16}\ (O_{22} \cap K_{46})(88,89,72,74)\ \varphi_3^{-1}\ (O_3 \cap K_{30})$
$(14,18,17,15)\ \varphi_{18}\ (O_{19} \cap K_{40})(43,45,70,60)\ \varphi_5^{-1};(O_5 \cap K_{28})(4,8,24,35);$

$(O_9 \cap K_{28})(4,9,49,35)\ \varphi_{16}\ (O_{20} \cap K_{46})(88,82,73,74)\ \varphi_7^{-1}\ (O_7 \cap K_{40})$
$(43,45,44,42)\ \varphi_{18}^{-1}\ (O_{16} \cap K_{30})(14,18,68,92)\ \varphi_9^{-1}\ (O_9 \cap K_{28})(4,9,49,35);$

$(O_4 \cap K_{28})(3,8,24,26)\ \varphi_{16}\ (O_{21} \cap K_{46})(83,89,72,96)\ \varphi_6^{-1}\ (O_6 \cap K_{36})$
$(36,37,51,30)\ \varphi_{22}\ (O_{23} \cap K_{42})(57,59,78,62)\ \varphi_4^{-1}\ (O_4 \cap K_{28})(3,8,24,26);$

$(O_1 \cap K_{29})(4,6,5,11)\ \varphi_{17}\ (O_{14} \cap K_{37})(74,56,35,34)\ \varphi_{11}^{-1}\ (O_{12} \cap K_{33})$
$(21,20,67,46)\ \varphi_{20}\ (O_{24} \cap K_{45})(81,86,83,79)\ \varphi_1^{-1}\ (O_1 \cap K_{29})(4,6,5,11);$

$(O_9 \cap K_{29})(4,11,54,33)\ \varphi_{17}\ (O_{20} \cap K_{37})(74,34,32,76)\ \varphi_7^{-1}\ (O_7 \cap K_{31})$
$(42,10,12,41)\ \varphi_{19}\ (O_{16} \cap K_{32})(14,20,64,93)\ \varphi_9^{-1}\ (O_9 \cap K_{29})(4,11,54,33);$

$(O_6 \cap K_{29})(5,6,31,40)\ \varphi_{17}\ (O_9 \cap K_{37})(35,56,55,33)\ \varphi_9\ (O_{16} \cap K_{48})$
$(92,79,80,93)\ \varphi_{24}^{-1}\ (O_{21} \cap K_{41})(47,50,72,61)\ \varphi_6^{-1}\ (O_6 \cap K_{29})(5,6,31,40);$

$(O_{10} \cap K_{29})(5,11,54,40)\ \varphi_{17}\ (O_5 \cap K_{37})(35,34,32,33)\ \varphi_5\ (O_{19} \cap K_{43})$
$(60,86,87,63)\ \varphi_{21}^{-1}\ (O_{17} \cap K34)(23,25,73,94)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{29})(5,11,54,40);$

$(O_{15} \cap K_{29})(31,40,54,33)\ \varphi_{17}\ (O_{15} \cap K_{37})(55,33,32,76)\ \varphi_{12}^{-1}\ (O_{13} \cap K_{33})$
$(18,71,45,19)\ \varphi_{20}\ (O_{13} \cap K_{45})(70,69,68,71)\ \varphi_{12}\ (O_{15} \cap K_{29})(31,40,54,33);$

$(O_2 \cap K_{30})(14,15,23,92)\ \varphi_{18}\ (O_{11} \cap K_{40})(43,60,47,42)\ \varphi_2^{-1}\ (O_2 \cap K_{36})$
$(29,30,36,95)\ \varphi_{22}\ (O_{11} \cap K_{42})(53,62,57,52)\ \varphi_2^{-1}\ (O_2 \cap K_{30})(14,15,23,92);$

$(O_{13} \cap K_{30})(17,18,68,27)\ \varphi_{18}\ (O_{13} \cap K_{40})(70,45,44,51)\ \varphi_{12}\ (O_{15} \cap K_{36})$
$(31,32,75,37)\ \varphi_{22}\ (O_{15} \cap K_{42})(78,55,54,59)\ \varphi_{12}^{-1}\ (O_{13} \cap K_{30})(17,18,68,27);$

$(O_1 \cap K_{31})(9,10,12,11)\ \varphi_{19}\ (O_{12} \cap K_{32})(13,20,64,38)\ \varphi_{11}\ (O_{14} \cap K_{41})$
$(72,56,49,50)\ \varphi_{24}\ (O_{24} \cap K_{48})(80,84,82,79)\ \varphi_1^{-1}\ (O_1 \cap K_{31})(9,10,12,11);$

$(O_8 \cap K_{31})(9,10,42,48)\ \varphi_{19}\ (O_3 \cap K_{32})(13,20,14,16)\ \varphi_3\ (O_{22} \cap K_{47})$
$(78,90,88,76)\ \varphi_{23}^{-1}\ (O_{18} \cap K_{38})(37,38,65,77)\ \varphi_8^{-1}\ (O_8 \cap K_{31})(9,10,42,48);$

$(O_9 \cap K_{31})(9,11,52,48)\ \varphi_{19}\ (O_6 \cap K_{32})(13,38,36,16)\ \varphi_6\ (O_{21} \cap K_{45})$
$(70,86,83,69)\ \varphi_{20}^{-1}\ (O_{16} \cap K_{33})(18,20,67,71)\ \varphi_9^{-1}\ (O_9 \cap K_{31})(9,11,52,48);$

$(O_{11} \cap K_{31})(41,42,48,52)\ \varphi_{19}\ (O_2 \cap K_{32})(93,14,16,36)\ \varphi_2\ (O_{11} \cap K_{41})$
$(48,53,61,47)\ \varphi_{24}\ (O_2 \cap K_{48})(94,95,93,92)\ \varphi_2\ (O_{11} \cap K_{31})(41,42,48,52);$

$(O_3 \cap K_{33})(18,19,21,20)\ \varphi_{20}\ (O_{19} \cap K_{45})(70,71,81,86)\ \varphi_5^{-1}\ (O_5 \cap K_{34})$
$(24,22,29,34)\ \varphi_{21}\ (O_{22} \cap K_{43})(89,61,62,90)\ \varphi_3^{-1}\ (O_3 \cap K_{33})(18,19,21,20);$

$(O_7 \cap K_{33})(19,21,46,45)\ \varphi_{20}\ (O_{16} \cap K_{45})(71,81,79,68)\ \varphi_9^{-1}\ (O_9 \cap K_{41})$
$(48,53,56,49)\ \varphi_{24}\ (O_{20} \cap K_{48})(94,95,84,82)\ \varphi_7^{-1}\ (O_7 \cap K_{33})(19,21,46,45);$

$(O_4 \cap K_{34})(22,23,25,24)\ \varphi_{21}\ (O_{21} \cap K_{43})(61,60,86,89)\ \varphi_6^{-1}\ (O_6 \cap K_{38})$
$(40,39,38,37)\ \varphi_{23}\ (O_{23} \cap K_{47})(77,85,90,78)\ \varphi_4^{-1}\ (O_4 \cap K_{34})(22,23,25,24);$

$(O_{10} \cap K_{38})(40,39,58,59)\ \varphi_{23}\ (O_{18} \cap K_{47})(77,85,84,75)\ \varphi_8^{-1}\ (O_8 \cap K_{41})$
$(48,47,50,49)\ \varphi_{24}\ (O_{17} \cap K_{48})(94,92,79,82)\ \varphi_{10}\ (O_{10} \cap K_{38})(40,39,58,59).$

The dihedral angles at the triangular faces of the polyhedron are equal to $\pi/3$. Therefore a cycle of these faces will be inessential if it contains six faces. Write cycles of these faces:

$(O_1 \cap O_3)(1,2,7)\ \varphi_1\ (O_{22} \cap O_{24})(89,88,90)\ \varphi_3^{-1}\ (O_3 \cap O_{16})$
$(18,14,20)\ \varphi_9^{-1}\ (O_1 \cap O_9)(9,4,11)\ \varphi_1\ (O_{16} \cap O_{24})(80,81,79)$
$\varphi_9^{-1}\ (O_9 \cap O_{22})(55,53,56)\ \varphi_3^{-1}\ (O_1 \cap O_3)(1,2,7);$

$(O_1 \cap O_6)(1,5,6)\ \varphi_1\ (O_{21} \cap O_{24})(89,83,86)\ \varphi_6^{-1}\ (O_6 \cap O_{18})$
$(37,36,38)\ \varphi_8^{-1}\ (O_1 \cap O_8)(9,3,10)\ \varphi_1\ (O_{18} \cap O_{24})(80,85,84)$
$\varphi_8^{-1}\ (O_8 \cap O_{21})(51,47,50)\ \varphi_6^{-1}\ (O_1 \cap O_3)(1,5,6);$

$(O_1 \cap O_5)(8,6,4)\ \varphi_1\ (O_{19} \cap O_{24})(87,86,81)\ \varphi_5^{-1}\ (O_5 \cap O_{20})$
$(32,34,29)\ \varphi_7^{-1}\ (O_1 \cap O_7)(12,10,2)\ \varphi_1\ (O_{20} \cap O_{24})(82,84,88)$
$\varphi_7^{-1}\ (O_7 \cap O_{19})(45,46,43)\ \varphi_5^{-1}\ (O_1 \cap O_5)(8,6,4);$

$(O_1 \cap O_4)(8,7,3)\ \varphi_1\ (O_{23} \cap O_{24})(87,90,85)\ \varphi_4^{-1}\ (O_4 \cap O_{17})$
$(27,25,23)\ \varphi_{10}^{-1}\ (O_1 \cap O_{10})(12,11,5)\ \varphi_1\ (O_{17} \cap O_{24})(82,79,83)$
$\varphi_{10}^{-1}\ (O_{10} \cap O_{23})(59,58,57)\ \varphi_4^{-1}\ (O_1 \cap O_4)(8,7,3);$

$(O_2 \cap O_3)(15,16,14)\ \varphi_2\ (O_{11} \cap O_{23})(62,61,53)\ \varphi_3^{-1}\ (O_3 \cap O_7)$
$(21,19,2)\ \varphi_7\ (O_2 \cap O_{20}(95,94,29)\ \varphi_2\ (O_7 \cap O_{11})(41,42,43)$
$\varphi_7\ (O_{20} \cap O_{22}(76,74,88)\ \varphi_3^{-1}\ (O_2 \cap O_3)(15,16,14);$

$(O_2 \cap O_4)(15,22,23)\ \varphi_2\ (O_{11} \cap O_{23})(62,63,57)\ \varphi_4^{-1}\ (O_4 \cap O_8)$
$(26,28,3)\ \varphi_8\ (O_2 \cap O_{18})(95,93,36)\ \varphi_2\ (O_8 \cap O_{11})(42,48,47)$
$\varphi_8\ (O_{18} \cap O_{23})(65,77,85)\ \varphi_4^{-1}\ (O_2 \cap O_4)(15,22,23);$

$(O_2 \cap O_4)(22,29,30)\ \varphi_2\ (O_{11} \cap O_{19})(63,43,60)\ \varphi_5^{-1}\ (O_5 \cap O_9)$
$(33,4,35)\ \varphi_9\ (O_2 \cap O_{16})(93,14,92)\ \varphi_2\ (O_9 \cap O_{11})(48,53,52)$
$\varphi_9\ (O_{16} \cap O_{19})(71,81,67)\ \varphi_5^{-1}\ (O_2 \cap O_4)(22,29,30);$

$(O_2 \cap O_6)(16,36,30)\ \varphi_2\ (O_{11} \cap O_{21})(61,47,60)\ \varphi_6^{-1}\ (O_6 \cap O_{10})$
$(40,5,39)\ \varphi_{10}\ (O_2 \cap O_{17})(94,23,92)\ \varphi_2\ (O_{10} \cap O_{11}(41,57,52)$
$\varphi_{10}\ (O_{17} \cap O_{21})(69,83,96)\ \varphi_6\ (O_2 \cap O_6)(16,36,30);$

$(O_3 \cap O_6)(1,16,13)\ \varphi_3\ (O_{15} \cap O_{22})(55,76,78)\ \varphi_{12}^{-1}\ (O_3 \cap O_{13})$
$(18,19,17)\ \varphi_3\ (O_{21} \cap O_{22})(89,61,72)\ \varphi_6^{-1}\ (O_6 \cap O_{15})(37,40,31)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{21})(51,69,70)\ \varphi_6^{-1}\ (O_3 \cap O_6)(1,16,13);$

$(O_3 \cap O_4)(15,17,7)\ \varphi_3\ (O_{14} \cap O_{22})(74,72,56)\ \varphi_{11}^{-1}\ (O_3 \cap O_{12})$
$(21,13,20)\ \varphi_3\ (O_{22} \cap O_{23})(62,78,90)\ \varphi_4^{-1}\ (O_4 \cap O_{14})(26,24,25)$
$\varphi_{11}^{-1}\ (O_{12} \cap O_{23})(65,66,58)\ \varphi_4^{-1}\ (O_3 \cap O_4)(15,17,7);$

$(O_4 \cap O_5)(8,22,24)\ \varphi_4\ (O_{15} \cap O_{23})(59,77,78)\ \varphi_{12}^{-1}\ (O_4 \cap O_{13})$
$(27,28,17)\ \varphi_4\ (O_{19} \cap O_{23})(87,63,66)\ \varphi_5^{-1}\ (O_5 \cap O_{15})(32,33,31)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{19})(45,71,70)\ \varphi_5^{-1}\ (O_4 \cap O_5)(8,22,24);$

$(O_5 \cap O_{14})(34,35,24)\ \varphi_5\ (O_{19} \cap O_{21})(86,60,70)\ \varphi_6^{-1}\ (O_6 \cap O_{12})$
$(38,39,13)\ \varphi_{11}\ (O_{14} \cap O_{21})(50,96,72)\ \varphi_6^{-1}\ (O_5 \cap O_6)(6,30,31)$
$\varphi_5\ (O_{12} \cap O_{19})(46,67,66)\ \varphi_{11}\ (O_5 \cap O_{14})(34,35,24);$

$(O_7 \cap O_8)(10,42,44)\ \varphi_7\ (O_{14} \cap O_{20})(34,74,73)\ \varphi_{11}^{-1}\ (O_7 \cap O_{12})$
$(46,21,91)\ \varphi_7\ (O_{18} \cap O_{20})(84,95,75)\ \varphi_8^{-1}\ (O_8 \cap O_{14})(50,26,49)$
$\varphi_{11}^{-1}\ (O_{12} \cap O_{18})(38,65,64)\ \varphi_8^{-1}\ (O_7 \cap O_8)(10,42,44);$

$(O_7 \cap O_{10})(12, 41, 91)\ \varphi_7\ (O_{15} \cap O_{20})(32, 76, 75)\ \varphi_{12}^{-1}\ (O_7 \cap O_{13})$
$(45, 19, 44)\ \varphi_7\ (O_{17} \cap O_{20})(82, 94, 73)\ \varphi_{10}^{-1}\ (O_{10} \cap O_{15})(59, 40, 54)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{17})(27, 69, 68)\ \varphi_{10}^{-1}\ (O_7 \cap O_{10})(12, 41, 91);$

$(O_8 \cap O_9)(9, 48, 49)\ \varphi_8\ (O_{15} \cap O_{18})(37, 77, 75)\ \varphi_{12}^{-1}\ (O_8 \cap O_{13})$
$(51, 28, 44)\ \varphi_8\ (O_{16} \cap O_{18})(80, 93, 64)\ \varphi_9^{-1}\ (O_9 \cap O_{15})(55, 33, 54)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{16})(18, 71, 68)\ \varphi_9^{-1}\ (O_8 \cap O_9)(9, 48, 49);$

$(O_9 \cap O_{10})(54, 52, 11)\ \varphi_9\ (O_{12} \cap O_{16})(64, 67, 20)\ \varphi_{11}\ (O_9 \cap O_{14})$
$(49, 35, 56)\ \varphi_9\ (O_{16} \cap O_{17})(68, 92, 79)\ \varphi_{10}^{-1}\ (O_{10} \cap O_{12})(91, 39, 58)$
$\varphi_{11}\ (O_{14} \cap O_{17})(73, 96, 25)\ \varphi_{10}^{-1}\ (O_9 \cap O_{10})(54, 52, 11).$

Finally write cycles of one-dimensional edges of the polyhedron R_2:

$(1, 2)\ \varphi_{14}\ (87, 81)\ \varphi_1^{-1}\ (8, 4)\ \varphi_{16}\ (89, 88)\ \varphi_3^{-1}\ (18, 14)$
$\varphi_{18}\ (45, 43)\ \varphi_7\ (82, 88)\ \varphi_{16}^{-1}\ (9, 4)\ \varphi_1\ (80, 81)$
$\varphi_{14}^{-1}\ (12, 2)\ \varphi_7\ (32, 29)\ \varphi_{22}\ (55, 53)\ \varphi_3^{-1}\ (1, 2);$

$(1, 5)\ \varphi_{14}\ (87, 85)\ \varphi_1^{-1}\ (8, 3)\ \varphi_{16}\ (89, 83)\ \varphi_6^{-1}\ (37, 36)$
$\varphi_{22}\ (59, 57)\ \varphi_{10}\ (82, 83)\ \varphi_{16}^{-1}\ (9, 3)\ \varphi_1\ (80, 85)$
$\varphi_{14}^{-1}\ (12, 5)\ \varphi_{10}\ (27, 23)\ \varphi_{18}\ (51, 47)\ \varphi_6^{-1}\ (1, 5);$

$(1, 6)\ \varphi_{13}\ (66, 58)\ \varphi_{11}\ (24, 25)\ \varphi_{21}\ (89, 86)\ \varphi_6^{-1}\ (37, 38)$
$\varphi_{23}\ (78, 90)\ \varphi_3^{-1}\ (13, 20)\ \varphi_{19}^{-1}\ (9, 10)\ \varphi_1\ (80, 84)$
$\varphi_{24}^{-1}\ (72, 56)\ \varphi_3^{-1}\ (17, 7)\ \varphi_{15}\ (51, 50)\ \varphi_6^{-1}\ (1, 6);$

$(1, 7)\ \varphi_{13}\ (66, 46)\ \varphi_{11}\ (24, 34)\ \varphi_{21}\ (89, 90)\ \varphi_3^{-1}\ (18, 20)$
$\varphi_{20}\ (70, 86)\ \varphi_6^{-1}\ (13, 38)\ \varphi_{19}^{-1}\ (9, 11)\ \varphi_1\ (80, 79)$
$\varphi_{24}^{-1}\ (72, 50)\ \varphi_6^{-1}\ (31, 6)\ \varphi_{17}\ (55, 56)\ \varphi_3^{-1}\ (1, 7);$

$(1, 16)\ \varphi_{13}\ (66, 63)\ \varphi_5^{-1}\ (31, 33)\ \varphi_{17}\ (55, 76)\ \varphi_{12}^{-1}\ (18, 19)$
$\varphi_{20}\ (70, 71)\ \varphi_5^{-1}\ (24, 22)\ \varphi_{21}\ (89, 61)\ \varphi_6^{-1}\ (37, 40)$
$\varphi_{23}\ (78, 77)\ \varphi_{12}^{-1}\ (17, 28)\ \varphi_{15}\ (51, 69)\ \varphi_6^{-1}\ (1, 16);$

$(1, 13)\ \varphi_{14}\ (87, 66)\ \varphi_4^{-1}\ (27, 17)\ \varphi_{18}\ (51, 70)\ \varphi_{12}\ (37, 31)$
$\varphi_{22}\ (59, 78)\ \varphi_4^{-1}\ (8, 24)\ \varphi_{16}\ (89, 72)\ \varphi_3^{-1}\ (18, 17)$
$\varphi_{18}\ (45, 70)\ \varphi_1 2\ (32, 31)\ \varphi_{22}\ (55, 78)\ \varphi_3^{-1}\ (1, 13);$

$(2, 10)\ \varphi_{15}\ (26, 25)\ \varphi_4\ (62, 90)\ \varphi_{21}^{-1}\ (29, 34)\ \varphi_5 (81, 86)\ \varphi_{20}^{-1}$
$(21, 20)\ \varphi_{11}\ (74, 56)\ \varphi_{17}^{-1}\ (4, 6)\ \varphi_5\ (43, 46)\ \varphi_{13}^{-1}$
$(15, 7)\ \varphi_4\ (65, 58)\ \varphi_{23}\ (88, 84)\ \varphi_1^{-1}\ (2, 10);$

$(2, 7)\ \varphi_{15}\ (26, 50)\ \varphi_8\ (95, 84)\ \varphi_{24}^{-1}\ (53, 56)\ \varphi_9\ (81, 79)$
$\varphi_{20}^{-1}\ (21, 46)\ \varphi_{11}\ (74, 34)\ \varphi_{17}^{-1}\ (4, 11)\ \varphi_9\ (14, 20)$
$\varphi_{19}^{-1}\ (42, 10)\ \varphi_8\ (65, 38)\ \varphi_{23}\ (88, 90)\ \varphi_1^{-1}\ (2, 7);$

$(2, 19)\ \varphi_{15}\ (26, 28)\ \varphi_4\ (62, 63)\ \varphi_{21}^{-1}\ (29, 94)\ \varphi_2\ (43, 41)$
$\varphi_{13}^{-1}\ (15, 22)\ \varphi_4\ (65, 77)\ \varphi_{23}\ (88, 76)\ \varphi_3^{-1}\ (14, 16)$
$\varphi_{19}^{-1}\ (42, 48)\ \varphi_2^{-1}\ (95, 93)\ \varphi_{24}^{-1}\ (53, 61)\ \varphi_3^{-1}\ (2, 19);$

$(2,21)\ \varphi_{14}\ (81,67)\ \varphi_5^{-1}\ (29,30)\ \varphi_{22}\ (53,62)\ \varphi_2^{-1}\ (14,15)$
$\varphi_{18}\ (43,60)\ \varphi_5^{-1}\ (4,35)\ \varphi_{16}\ (88,74)\ \varphi_7^{-1}\ (43,42)$
$\varphi_{18}^{-1}\ (14,92)\ \varphi_2\ (53,52)\ \varphi_{22}^{-1}\ (29,95)\ \varphi_7^{-1}\ (2,21);$

$(3,7)\ \varphi_{15}\ (96,50)\ \varphi_{11}^{-1}\ (39,38)\ \varphi_{23}\ (85,90)\ \varphi_4^{-1}\ (23,25)$
$\varphi_{21}\ (60,86)\ \varphi_5^{-1}\ (35,34)\ \varphi_{17}^{-1}\ (5,11)\ \varphi_1\ (83,79)$
$\varphi_{20}^{-1}\ (67,46)\ \varphi_5^{-1}\ (30,6)\ \varphi_{13}\ (57,58)\ \varphi_4^{-1}\ (3,7);$

$(3,10)\ \varphi_{15}\ (96,25)\ \varphi_{11}^{-1}\ (39,58)\ \varphi_{23}\ (85,84)\ \varphi_8^{-1}\ (47,50)$
$\varphi_{24}\ (92,79)\ \varphi_9^{-1}\ (35,56)\ \varphi_{17}^{-1}\ (5,6)\ \varphi_1\ (83,86)$
$\varphi_{20}^{-1}\ (67,20)\ \varphi_9^{-1}\ (51,11)\ \varphi_{19}\ (36,38)\ \varphi_8^{-1}\ (3,10);$

$(3,26)\ \varphi_{16}\ (83,96)\ \varphi_6^{-1}\ (36,30)\ \varphi_{22}\ (57,62)\ \varphi_2^{-1}\ (23,15)$
$\varphi_{18}\ (47,60)\ \varphi_6^{-1}\ (5,39)\ \varphi_{14}\ (85,65)\ \varphi_8^{-1}\ (47,42)$
$\varphi_{18}^{-1}\ (23,92)\ \varphi_2\ (57,52)\ \varphi_{22}^{-1}\ (36,95)\ \varphi_8^{-1}\ (3,26);$

$(3,28)\ \varphi_{15}\ (96,69)\ \varphi_{10}^{-1}\ (52,41)\ \varphi_{19}\ (36,93)\ \varphi_2\ (47,48)$
$\varphi_{24}\ (92,94)\ \varphi_{10}^{-1}\ (39,40)\ \varphi_{23}\ (85,77)\ \varphi_4^{-1}\ (23,22)$
$\varphi_{21}\ (60,61)\ \varphi_2^{-1}\ (30,16)\ \varphi_{13}\ (57,63)\ \varphi_4^{-1}\ (3,28);$

$(4,33)\ \varphi_{17}\ (74,76)\ \varphi_3^{-1}\ (15,16)\ \varphi_{13}\ (43,63)\ \varphi_2^{-1}\ (29,22)$
$\varphi_{21}\ (62,61)\ \varphi_3^{-1}\ (21,19)\ \varphi_{20}\ (81,71)\ \varphi_9^{-1}\ (53,48)$
$\varphi_{24}\ (95,94)\ \varphi_2\ (42,41)\ \varphi_{19}\ (14,93)\ \varphi_9^{-1}\ (4,33);$

$(5,40)\ \varphi_{17}\ (35,33)\ \varphi_5\ (60,63)\ \varphi_{21}^{-1}\ (23,94)\ \varphi_2\ (57,41)$
$\varphi_{13}^{-1}\ (30,22)\ \varphi_5\ (67,71)\ \varphi_{20}\ (83,69)\ \varphi_6^{-1}\ (36,16)$
$\varphi_{19}^{-1}\ (52,48)\ \varphi_2^{-1}\ (92,93)\ \varphi_{24}^{-1}\ (47,61)\ \varphi_6^{-1}\ (5,40);$

$(6,8)\ \varphi_{13}\ (58,91)\ \varphi_{13}\ (25,73)\ \varphi_{21}\ (86,87)\ \varphi_5^{-1}$
$(34,32)\ \varphi_{17}^{-1}\ (11,54)\ \varphi_9\ (20,64)\ \varphi_{19}^{-1}\ (10,12)\ \varphi_1(84,82)$
$\varphi_{24}^{-1}\ (56,49)\ \varphi_9(79,68)\ \varphi_{20}^{-1}\ (46,45)\ \varphi_5^{-1}\ (6,8);$

$(7,8)\ \varphi_{13}\ (46,91)\ \varphi_7\ (84,75)\ \varphi_{23}^{-1}\ (58,59)\ \varphi_{10}\ (79,82)$
$\varphi_{24}^{-1}\ (50,49)\ \varphi_{11}^{-1}\ (38,64)\ \varphi_{19}^{-1}\ (11,12)\ \varphi_{10}\ (25,27)$
$\varphi_{15}^{-1}\ (10,44)\ \varphi_7\ (34,73)\ \varphi_{21}\ (90,87)\ \varphi_1^{-1}\ (7,8);$

$(8,22)\ \varphi_{13}\ (91,41)\ \varphi_7\ (75,76)\ \varphi_{23}^{-1}\ (59,77)\ \varphi_{12}^{-1}\ (27,28)$
$\varphi_{15}^{-1}\ (44,19)\ \varphi_7\ (73,94)\ \varphi_{21}\ (87,63)\ \varphi_5^{-1}\ (32,33)$
$\varphi_{17}^{-1}\ (54,40)\ \varphi_{12}^{-1}\ (68,69)\ \varphi_{20}^{-1}\ (45,71)\ \varphi_5^{-1}\ (8,22);$

$(9,49)\ \varphi_{16}\ (82,73)\ \varphi_7^{-1}\ (45,44)\ \varphi_{18}^{-1}\ (18,68)\ \varphi_{12}\ (55,54)$
$\varphi_{22}^{-1}\ (32,75)\ \varphi_7^{-1}\ (12,91)\ \varphi_{14}\ (80,64)\ \varphi_8^{-1}\ (51,44)$
$\varphi_{18}^{-1}\ (27,68)\ \varphi_{12}\ (59,54)\ \varphi_{22}^{-1}\ (37,75)\ \varphi_8^{-1}\ (9,49);$

$(9,48)\ \varphi_{19}\ (13,16)\ \varphi_3\ (78,76)\ \varphi_{23}^{-1}\ (37,77)\ \varphi_{12}^{-1}\ (51,28)$
$\varphi_{15}^{-1}\ (17,19)\ \varphi_3\ (72,61)\ \varphi_{24}\ (80,93)\ \varphi_9^{-1}\ (55,33)$
$\varphi_{17}^{-1}\ (31,40)\ \varphi_{12}^{-1}\ (70,69)\ \varphi_{20}{-1}\ (18,71)\ \varphi_9^{-1}\ (9,48);$

$(12,41)\ \varphi_{19}\ (64,93)\ \varphi_9^{-1}\ (54,33)\ \varphi_{17}\ (32,76)\ \varphi_{12}^{-1}\ (45,19)$
$\varphi_{20}\ (68,71)\ \varphi_9^{-1}\ (49,48)\ \varphi_{24}\ (82,94)\ \varphi_{10}^{-1}\ (59,40)$
$\varphi_{23}\ (75,77)\ \varphi_{12}^{-1}\ (44,28)\ \varphi_{15}\ (27,69)\ \varphi_{10}^{-1}\ (12,41);$

$(13,21)\ \varphi_{14}\ (66,67)\ \varphi_{11}\ (24,35)\ \varphi_{16}\ (72,74)\ \varphi_3^{-1}\ (17,15)$
$\varphi_{18}\ (70,60)\ \varphi_6^{-1}\ (13,39)\ \varphi_{14}\ (66,65)\ \varphi_{11}\ (24,26)$
$\varphi_{16}\ (72,96)\ \varphi_6^{-1}\ (31,30)\ \varphi_{22}\ (78,62)\ \varphi_3^{-1}\ (13,21);$

$(95,75)\ \varphi_{22}\ (52,54)\ \varphi_9\ (67,64)\ \varphi_{14}^{-1}\ (21,91)\ \varphi 11\ (74,73)$
$\varphi_{16}^{-1}\ (35,49)\ \varphi_9\ (92,68)\ \varphi_{18}\ (42,44)\ \varphi_8\ (65,64)$
$\varphi_{14}^{-1}\ (39,91)\ \varphi_{11}\ (96,73)\ \varphi_{16}^{-1}\ (26,49)\ \varphi_8\ (95,75).$

As each cycle contains 12 edges, we have shown that cycles of one-dimensional edges are inessential, too. Thus we have shown that identifying the faces of the polyhedron R_2 by the motions $\varphi_1, \varphi_2, \ldots, \varphi_{24}$, the cycles both of two-dimensional faces and one-dimensional edges are inessential. Therefore the group Γ generated by the motions $\varphi_1, \varphi_2, \ldots, \varphi_{24}$ does not contain elements of finite order, i. e. Γ is torsion-free. Then the quotient space of the space H^4 by the group Γ is a four-dimensional hyperbolic manifold M with finite volume which is not closed. The manifold M has four cusps, i. e. four ends of the form $T^3 \times [0, \infty)$, where T^3 is a three-dimensional torus.

References

[1] Ratcliffe I. G., Tschantz S. T. *The Volume Spectrum of Hyperbolic 4-Manifolds.* Experimental Math., 2000, **9**, 101–125.

[2] Gutsul I. S. *Some hyperbolic manifolds.* Bul. Acad. Ştiinţe Repub. Moldova, Matematica, 2004, No. 3(46), 63–70.

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