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On free groups in classes of groups with topologies

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Abstract. We study properties of free groups in distinct classes of groups with topologies. The conditions under which the quasi-metric on the space of generators X is extended to an invariant quasi-metric on a free group $F(X, \mathcal{V})$ in the fixed quasi-variety $\mathcal{V}$ of groups with topologies are given. This result is applied to the study:
– of free paratopological groups;
– of free quasitopological groups;
– of free semitopological groups;
– of free left topological groups.

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1 Introduction

By a space we understand a topological T_0-space. We use the terminology from [3, 9]. Let $\mathbb{N} = \{1, 2, ...\}$. By $cl_X H$ we denote the closure of a set H in a space X, $|A|$ is the cardinality of a set A.

A *paratopological* group is a group endowed with a topology such that the multiplication is jointly continuous. Recall that a *semitopological* group is a group with a topology such that the multiplication is separately continuous. Every paratopological group is a semitopological group. A semitopological group with a continuous inverse operation $x \to x^{-1}$ is called a *quasitopological* group. A *topological* group is a paratopological group with a continuous inverse operation $x \to x^{-1}$.

The space S of reals $\mathbb{R}$ with the topology generated by the open base consisting of the sets $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$, is called the Sorgenfrey line [9]. The Sorgenfrey line has the following properties [3]:

– S is an Abelian paratopological group with the Baire property;

– S is a hereditarily Lindelöf first-countable hereditarily separable non-metrizable space;

– S does not admit a structure of a topological group.

In this paper we study properties of free paratopological groups in a given quasi-variety of paratopological groups $\mathcal{W}$. The general theorem of existence of free paratopological (semitopological, quasitopological) groups in distinct classes of groups with topologies was proved in [7]. We follow [5, 7, 8, 11, 12] for the concept of a free object. The paratopological topology on a free group $F(X, \mathcal{W})$ is constructed by the Markov–Graev method [10, 13] developed in [15] for pseudo-quasi-metrics. We develop this method for free groups in the non-Burnside quasi-varieties of paratopological groups. In [15] the authors use the method of left (right) invariant pseudo-quasi-metrics. Since the topology generated by left (right) invariant pseudo-quasi-metrics may not be a paratopological topology [3, 4, 14, 15], this point of view may create dangerous moments. For this we use the method of invariant pseudo-quasi-metrics. The method of invariant pseudo-metrics on free objects was developed in [6, 10].

There exist distinct conditions under which a paratopological topology on a group is topological (see the references in [1–3, 15]). If G is a paratopological group and $x^n = e$ for some natural number n, then G is a topological group. By virtue of this fact, the method of invariant pseudo-quasi-metrics is useful in the non-Burnside quasi-varieties of paratopological groups. In the Burnside quasi-varieties of paratopological groups any invariant pseudo-quasi-metric is a pseudo-metric.

2 Quasi-metrics on groups

A function $\rho : X \times X \to R$ is called a *pseudo-quasi-metric* if $\rho(x, x) = 0$ and $0 \leq \rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$. If ρ is a pseudo-quasi-metric and $\rho(x, y) + \rho(y, x) > 0$ for all distinct $x, y \in X$, then ρ is called a *quasi-metric*.

Any pseudo-quasi-metric ρ generates a topology $\mathcal{T}(\rho)$ with the open base $\{B(x, \rho, r) = \{y \in X : \rho(x, y) < r\} : x \in X, r > 0\}$. The family $\mathcal{P}$ of pseudo-quasi-metric generates the topology $\mathcal{T}(P) = sup\{\mathcal{T}(\rho) : \rho \in \mathcal{P}\}$. If $\mathcal{P} = \emptyset$, then $\mathcal{T}(P) = \{\emptyset, X\}$. The topology $\mathcal{T}(\mathcal{P})$ is a T_0-topology if and only if for any two distinct points $x, y \in X$ we have $\rho(x, y) + \rho(y, y) > 0$ for some $\rho \in \mathcal{P}$.

If ρ is a pseudo-quasi-metric on a space X and the sets from $\mathcal{T}(\rho)$ are open in X, then we say that ρ is a *continuous pseudo-quasi-metric*.

Let U be an open subset of the space X. We put $\rho_U(x, y) = 1$ if $x \in U$ and $y \in X \setminus U$, and $\rho_U(x, y) = 0$ otherwise. Then $\mathcal{T}(\rho_U) = \{\emptyset, U, X\}$. Hence, any topology is generated by some family of pseudo-quasi-metrics.

Let G be a group and ρ be a pseudo-quasi-metric on G. The pseudo-quasi-metric ρ is called:

– *left* (respectively, *right*) *invariant* if $\rho(xa, xb) = \rho(a, b)$ (respectively, $\rho(ax, bx) = \rho(a, b)$) for all $x, a, b \in G$;

– *invariant* if it simultaneously is both left and right invariant.

If ρ is a left (or right) invariant pseudo-quasi-metric on a paratopological group G, then ρ is continuous if and only if the set $B(e, \rho, r)$ is open in G for any $r > 0$.

If ρ is an invariant pseudo-quasi-metric on the group G, then $(G, \mathcal{T}(\rho))$ is a paratopological group and $\rho(x^{-1}, y^{-1}) = \rho(y, x)$ for any $x, y \in G$. Thus any family $\mathcal{P}$ of invariant pseudo-quasi-metrics generates a paratopological topology $\mathcal{T}(\mathcal{P})$ on the group.

A pseudo-quasi-metric ρ on a group G is called *a stable* pseudo-quasi-metric if $\rho(x_1x_2, y_1y_2) \leq \rho(x_1y_1) + \rho(x_2y_2)$ for all $x_1, x_2, y_1, y_2 \in G$ [6].

Proposition 1. *Let ρ be a pseudo-quasi-metric on a group G. The next assertions are equivalent:*

1. *ρ is invariant.*
2. *ρ is stable.*

Proof. Is obvious.

If ρ is a pseudo-quasi-metric on a group G and $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$, then ρ is *a pseudo-metric*. The pseudo-metric ρ is invariant if and only if $\rho(y^{-1},x^{-1}) = \rho(x,y) = \rho(zx,zy) = \rho(xz,yz)$ for all $x,y,z \in G$.

Definition 1. A subset H of a group G is called invariant if $xHx^{-1} = H$ for any $x \in G$.

Proposition 2. *Let U be an invariant open subset of a paratopological group G with a topology $\mathcal{T}$ and $e \in U$. We put $d_U(x,y) = 0$ if $x^{-1}y \in U$ and $d_U(x,y) = 1$ if $x^{-1}y \notin U$. Then d_U is an invariant pseudo-quasi-metric and $\mathcal{T}(d_U) \subseteq \mathcal{T}$.*

Proof. If $x \in U$, then $d_U(e,x) = 0$ and $d_U(e,y) = 1$ if $y \notin U$. Thus $B(e,d_U,r) = U$ for $0 < r \leq 1$ and $B(e,d_U,r) = G$ for $r > 1$. By construction, $d_U(x,y) = d_U(e,x^{-1}y) = d_U(e,(x^{-1}z^{-1})(zy)) = d_U(zx,zy)$ for all $x,y,z \in G$. Let $x,y \in G$. Then $x^{-1}y \in U$ if and only if $(z^{-1}x^{-1})(yz) \in U$ for any $z \in G$. Thus $d_U(xz,yz) = d_U(x,y)$. The proof is complete. □

Corollary 1. *For a paratopological group G the following assertions are equivalent:*

1. *The topology on G is generated by a family of invariant pseudo-quasi-metrics.*
2. *There exists an open base $\mathcal{B}$ of G at e such that any $U \in \mathcal{B}$ is invariant.*

Remark 1. Let U be an open subset of a paratopological group G with a topology $\mathcal{T}$ and $e \in U$. We put $d_{lU}(x,y) = 0$ if $x^{-1}y \in U$, and $d_{lU}(x,y) = 1$ if $x^{-1}y \notin U$, $d_{rU}(x,y) = 0$ if $xy^{-1} \in U$, and $d_U(x,y) = 1$ if $xy^{-1} \notin U$. Then d_{lU} is a continuous left invariant pseudo-quasi-metric on G and d_{rU} is a continuous right invariant pseudo-quasi-metric on G. Thus:

- the topology of a paratopological group G is generated by a family of left invariant pseudo-quasi-metrics;

- the topology of a paratopological group G is generated by a family of right invariant pseudo-quasi-metrics.

As was established by A. S. Mishchenko [14] (see also [4]), the topology, generated by a family of left (or right) invariant pseudo-metrics, may not be a paratopological topology.

3 Free paratopological groups

A class $\mathcal{V}$ of groups with topologies is called *a quasi-variety* of groups if:

(F1) the class $\mathcal{V}$ is multiplicative;

(F2) if $G \in \mathcal{V}$ and A is a subgroup of G, then $A \in \mathcal{V}$;

(F3) every space $G \in \mathcal{V}$ is a T_0-space.

Let $\mathcal{S}$ be a set of multiplicative and hereditary properties of groups with topologies. A class $\mathcal{V}$ of groups with topologies is called *an $\mathcal{S}$-complete* quasi-variety of groups with topologies if it is a quasi-variety with the next property:

(F4) if $G \in \mathcal{V}$, then G is a group with topology with the properties $\mathcal{S}$.

(F5) if $G \in \mathcal{V}$ and T is a T_0-topology on G with the properties $\mathcal{S}$, then $(G, T) \in \mathcal{V}$ too.

A quasi-variety $\mathcal{V}$ of paratopological groups is called *an $\mathcal{S}$-complete variety* of paratopological groups if it is an $\mathcal{S}$-complete quasi-variety with the next property:

(F6) if $g : A \longrightarrow B$ is a continuous homomorphism of a paratopological group $A \in \mathcal{V}$ onto a T_0-paratopological group B with the property $\mathcal{S}$, then $B \in \mathcal{V}$.

Denote by I_p the property to be a paratopological group with an invariant bases at the identity e. If $\mathcal{S}_p$ is the property to be a paratopological group, then an $\mathcal{S}_p$-complete variety is called a complete variety and an $\mathcal{S}_p$-complete quasi-variety is called a complete quasi-variety of paratopological groups.

Let X be a non-empty topological space and $\mathcal{V}$ be a quasi-variety of groups with topologies. In any space X the basic point $p_X \in X$ is fixed, i.e. any space is pointed.

A free group of a space X in a class $\mathcal{V}$ is a pair $(F(X, \mathcal{V}), e_X)$ with the properties:

– $F(X, \mathcal{V}) \in \mathcal{V}$, $e_X : X \to F(X, \mathcal{V})$ is a continuous mapping and $e = e_X(p_X)$ is the neutral element of the group $F(X, \mathcal{V})$;

– the set $e_X(X)$ generates the group $F(X, \mathcal{V})$;

– for any continuous mapping $f : X \to G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\bar{f} : F(X, \mathcal{V}) \to G$ such that $f = \bar{f} \circ e_X$.

An abstract free group of a space X in the class $\mathcal{V}$ is a pair $(F^a(X, \mathcal{V}), a_X)$ with the properties:

– $F^a(X, \mathcal{V}) \in X, a_X : X \to F^a(X, \mathcal{V})$ is a mapping and $e = a_x(p_X)$;

– the set $a_X(X)$ generates the group $F^a(X, \mathcal{V})$;

– for any mapping $g : X \to G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\widehat{g} : F^a(X, \mathcal{V}) \to G$ such $g = \widehat{g} \circ a_X$.

In the proof of the following assertion we use the Kakutani's method [11].

Theorem 1 (see [7]). *Let $\mathcal{V}$ be a quasi-variety of groups with topologies. Then for each space X there exist:*

- a unique free group $(F(X, \mathcal{V}), e_X)$;

- a unique abstract free group $(F^a(X, \mathcal{V}), a_X)$;

- a unique continuous homomorphism $r_X : F^a(X, \mathcal{V}) \to F(X, \mathcal{V})$ of $F^a(X, \mathcal{V})$ onto $F(X, \mathcal{V})$ such that $e_X = r_X \circ a_X$.

Proof. Let τ be an infinite cardinal number and $|X| \leq \tau$. Then the class $\{f_\alpha : X \to G_\alpha : \alpha \in A\}$ of all mappings $f_\alpha : X \to G_\alpha$ with $G_\alpha \in \mathcal{V}$ and $|G_\alpha| \leq \tau$ is a set.

Let $B = \{\beta \in A : f_\beta : X \to G_\beta \textit{ is continuous}\}$. Consider the diagonal product $a_X = \Delta\{f_\alpha : \alpha \in A\} : X \to H_1 = \prod\{G_\alpha : \alpha \in A\}$ and the diagonal product $e_X = \Delta\{f_\alpha : \alpha \in A\} : X \to H_2 = \prod\{G_\alpha : \alpha \in B\}$. Let $F^a(X, \mathcal{V})$ be the subgroup of H_1 generated by the set $a_X(X)$ and $F(X, \mathcal{V})$ be the subgroup of H_2 generated by the set $e_X(X)$. Since $B \subseteq A$ there exists a unique continuous projection

$r_X : F^a(X, \mathcal{V}) \to F(X, \mathcal{V})$ such that $e_X = r_X \circ a_X$. The objects $(F(X, \mathcal{V}), e_X)$, $(F^a(X, V), a_X)$ and r_X are constructed. The proof is complete. $\square$

The group $F(X, \mathcal{V})$ is called *abstract free* if r_X is a continuous isomorphism. The next problems are important in the theory of universal algebras with topologies (see [5, 7, 12]) .

Problem 1. Under which conditions the free group $F(X, V)$ is abstract free?

Problem 2. Under which conditions the mapping $e_X : X \to F(X, V)$ is an embedding?

These problems for varieties of topological algebras were posed by A. I. Mal'cev [12].

Remark 2 (see [5,6,8]). A quasi-variety $\mathcal{V}$ is non-trivial if in $\mathcal{V}$ there exists an infinite group G. If the variety V is non-trivial, then:

– a_X is a one-to-one mapping of X onto $a_X(X)$.

Moreover, if $\mathcal{V}$ is a non-trivial I_p-complete quasi-variety or a non-trivial S_p-complete quasi-variety, then:

– for any completely regular space X the mapping e_X is an embedding of X into $F(X, \mathcal{V})$ and the free group $F(X, V)$ is abstract free (see [5, 7]).

Proposition 3. *Let G be a paratopological group, $n \in \mathbb{N}$ and $x^n = e$ for any $x \in G$. Then G is a topological group.*

Proof. Since $x^{-1} = x^{n-1}$ and the mapping $x \to x^{n-1}$ is continuous, the mapping $x \to x^{-1}$ is continuous. The proof is complete. $\square$

Let $\mathcal{V}$ be a quasi-variety of paratopological groups, X be a space and $e \in X$. On the free group $F^a(X, \mathcal{V}, e)$ with the identity e consider the maximal paratopological topology $\mathcal{T}(X, \mathcal{V}, e)$ for which the identical mapping $a_X : X \to F^a(X, \mathcal{V}, e)$ is continuous.

Proposition 4. *Let $\mathcal{V}$ be a quasi-variety of semitopological groups, X be a space and $e, e_1 \in X$. Then:*

1. The semitolopological groups $F(X, \mathcal{V}, e)$ and $F(X, \mathcal{V}, e_1)$ are topologically isomorphic.

2. The semitolopological groups $(F^a(X, \mathcal{V}, e), a_X, \mathcal{T}(X, \mathcal{V}, e))$ and $(F^a(X, \mathcal{V}, e_1), b_X, \mathcal{T}(X, \mathcal{V}, e_1))$ are topologically isomorphic.

Proof. Consider the natural continuous mappings $e_X : X \to F(X, \mathcal{V}, e)$ and $l_X : X \to F(X, \mathcal{V}, e_1)$. We can assume that e_X and l_X are embeddings and $e_X(x) = l_X(x) = x$ for any $x \in X$. There exist two continuous homomorphisms $\varphi : F(X, \mathcal{V}, e) \to F(X, \mathcal{V}, e_1)$ and $\psi : F(X, \mathcal{V}, e_1) \to F(X, \mathcal{V}, e)$ such that $\varphi(x)) = xe^{-1}$ and $\psi(x) = e_X(xe_1^{-1})$ for any $x \in X$. Since ψ is a homomorphism, $\psi(\varphi(x)) = \psi(x \cdot e^{-1}) = \psi(x) \cdot \psi(e^{-1}) = \psi(x) \cdot \psi(e)^{-1} = (xe_1^{-1}) \cdot (e \cdot e_1^{-1})^{-1} = x$ for $x \in X \subseteq F(X, \mathcal{V}, e)$. Hence the composition $\varphi \circ \psi$ is a continuous homomorphism such that $(\psi \circ \varphi)(x) = x$ for any $x \in X$. Thus $\psi \circ \varphi$ is the identical isomorphism and $\psi = \varphi^{-1}$. The assertion 1 is proved. The proof of the assertion 2 is similar. The proof is complete. $\square$

4 Construction of the group $F^a(X, \mathcal{V})$

Fix a non-trivial quasi-variety $\mathcal{V}$ of paratopological groups.

Consider a space X. Then we can assume that $X \subseteq F^a(X, \mathcal{V})$ as a subset and $a_X(x) = x$ for each $x \in X$. In particular $e = p_X$ is the neutral element of the group $F^a(X, \mathcal{V})$. In this case $e \in X \subseteq F^a(X, \mathcal{V})$. The set X is called an alphabet.

Let $\widetilde{X} = X \cup X^{-1}$. Obviously, if $x = p_X$, then $x^{-1} = x = e$.

If $n \geq 1$ and $x_1, x_2, ..., x_n \in \widetilde{X}$, then the symbol $x_1x_2...x_n$ is called a word of the length n in the alphabet X.

Any word $x_1x_2...x_n$, where $x_1, x_2, ..., x_n \in \widetilde{X}$, represents a unique element $[x_1x_2...x_n] = x_1 \cdot x_2 \cdot ... \cdot x_n \in F^a(X, \mathcal{V})$.

A given element $b \in F^a(X, \mathcal{V})$ is represented by many words. There exists a word of the minimal length which represents the given element b. The length n of this word is called the length of the element b and we put $l(b) = n$.

If an element $b \in F^a(X, \mathcal{V})$ is represented by the words $x_1x_2...x_n$, $y_1y_2...y_m$ of the minimal length, then $n = m$ and $\{x_1, x_2, ..., x_n\} = \{y_1, y_2, ..., y_m\}$. In this case we say that the word $x_1x_2...x_n$ is irreducible and that $Sup(b) = X \cap \{x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}\}$ is the support of the element b. The set $Sup^*(b) = \{e, x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}\}$ is the generalized support of the element b. Obviously, $Sup(e) = \{e\}$ and $e \notin Sup(b)$ if $b \neq e$. If $e \in Y \subseteq X$, $b \in F^a(X, \mathcal{V})$ and $F^a(Y, \mathcal{V})$ is the subgroup of $F^a(X, \mathcal{V})$ generated by the set Y, then $b \in F^a(Y, \mathcal{V})$ if any only if $Sup(b) \subseteq Y$. If $\mathcal{V}$ is the variety of all paratopological groups, then any $b \in F^a(X, \mathcal{V})$ is represented by a unique word of the minimal length. Moreover, in this case any irreducible word is of the minimal length.

Let $\mathcal{V}_a$ be the variety of all T_0-paratopological Abelian groups and $\mathcal{V}_g$ be the variety of all T_0-paratopological groups.

For any $n \in \mathbb{N}$ denote by $\mathcal{B}_n$ the Burnside variety of all T_0-paratopological groups of the exponent (index) n: $G \in \mathcal{B}_n$ if and only if $x^n = e$ for each $x \in G$. The variety $\mathcal{B}_1$ is the unique trivial variety of paratopological groups. If $\mathcal{V}$ is an I_p-variety of Abelian paratopological groups, then either $\mathcal{V} = \mathcal{V}_a$, or $\mathcal{V} = \mathcal{V}_a \cap \mathcal{B}_n$ for some $n \in \mathbb{N}$.

If $\mathcal{V}$ is a quasi-variety of paratopological groups and $\mathbb{Z} \in \mathcal{V}$, where $\mathbb{Z}$ is the group of integers, then $\mathcal{V}$ is a quasi-variety of the exponent 0. Obviously, if $\mathcal{V}$ is an I_p-complete variety of the exponent 0, then $\mathcal{V}_a \subseteq \mathcal{V}$.

A class of paratopological Abelian groups is I_p-complete if and only if it is complete.

5 Extension of pseudo-quasi-metrics on free groups

Fix a non-trivial I_p-complete quasi-variety $\mathcal{V}$ of paratopological groups. Consider a non-empty set X with a fixed point $e \in X$. We assume that $e \in X \subseteq F^a(X, \mathcal{V})$ and e is the identity of the group $F^a(X, \mathcal{V})$.

Let ρ be a pseudo-quasi-metric on the set X. Denote by $Q(\rho)$ the set of all invariant pseudo-quasi-metrics d on $F^a(X, \mathcal{V})$ for which $d(x, y) \leq \rho(x, y)$ for all

$x, y \in X$. The set $Q(\rho)$ is non-empty, since it contains the trivial pseudo-quasi-metric $d(x, y) = 0$ for all x, y. For all $a, b \in F^a(X, \mathcal{V})$ we put $\widehat{\rho}(a, b) = sup\{d(a, b) : d \in Q(\rho)\}$. We say that $\widehat{\rho}$ is the maximal extension of ρ on $F^a(X, \mathcal{V})$.

Property 1. $\widehat{\rho}(a, a) = 0$ and $\widehat{\rho}(a, b) \leq \widehat{\rho}(a, c) + \widehat{\rho}(c, b)$ for all $a, b, c \in F^a(X, \mathcal{V})$.

Proof. We assume that $\infty + \infty = t + \infty = \infty + t = \infty \leq \infty$ and $t < \infty$ for any real number t. In these conditions the assertion of Property 1 follows from the construction of $\widehat{\rho}$. □

Property 2. $\widehat{\rho}(x, y) \leq \rho(x, y)$ for all $x, y \in X$.

Proof. Follows from the constructions of $\widehat{\rho}$. □

Property 3. $\widehat{\rho}(xa, xb) = \widehat{\rho}(ax, bx) = \widehat{\rho}(a, b)$ for all $x, a, b \in F^a(X, \mathcal{V})$.

Proof. Follows from the invariance of the pseudo-quasi-metrics $Q(\rho)$. □

Property 4. $\widehat{\rho}(a, b) = \widehat{\rho}(ab^{-1}, e) = \widehat{\rho}(e, a^{-1}b)$.

Proof. Follows from Property 3. □

Property 5. $\widehat{\rho}(a_1a_2, b_1b_2) \leq \widehat{\rho}(a_1, b_1) + \widehat{\rho}(a_2, b_2)$.

Proof. Follows from Proposition 1 and Property 3. □

Property 6. $\widehat{\rho}(a^{-1}, b^{-1}) = \widehat{\rho}(b, a)$ for all $a, b \in F^a(X, \mathcal{V})$.

Proof. We have $\widehat{\rho}(a^{-1}, b^{-1}) = \widehat{\rho}(aa^{-1}b, ab^{-1}b) = \widehat{\rho}(b, a)$. □

Property 7. $\widehat{\rho}(a, b) < \infty$ for all $a, b \in F^a(X, \mathcal{V})$.

Proof. For some $n \in N$ we have $a = x_1^{\varepsilon_1}x_2^{\varepsilon_2}...x_n^{\varepsilon_n}$ and $b = y_1^{\delta_1}y_2^{\delta_2}...y_n^{\delta_n}$. Fix $i \leq n$. If $\varepsilon_i = \delta_i = 1$, then $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) = \widehat{\rho}(x_i, y_i) \leq \rho(x, y)$. If $\varepsilon_i = \delta_i = -1$, then $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) = \widehat{\rho}(x_i^{-1}, y_i^{-1}) = \widehat{\rho}(y_i, x_i) \leq \rho(y_i, x_i)$. If $\varepsilon_i = -\delta_i$, then $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) \leq \widehat{\rho}(x_i, y_i^{-1}) + \widehat{\rho}(x_i^{-1}, y_i) \leq \rho(e, x_i) + \rho(x_i, e) + \rho(e, y_i) + \rho(y_i, e)$. Hence $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) \leq \rho(x_i, y_i) + \rho(y_i, x_i) + \rho(e, x_i) + \rho(x_i, e) + \rho(e, y_i) + \rho(y_i, e) < \infty$. Then $\widehat{\rho}(a, b) \leq \sum\{\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) : i \leq n\} < \infty$. □

Example 1. Consider the variety $\mathcal{B}_2$ of paratopological groups. Any group $G \in \mathcal{B}_2$ is commutative. Fix a space X with the fixed point $e \in X$.

Let $\rho(z, z) = \rho(z, e) = 0$ for any $z \in X$ and $\rho(e, x) = \rho(x, y) = 1$ for all $x, y \in X, x \neq y, x \neq e \neq y$. Then ρ is a quasi-metric on X. In this case $x^{-1} = x$ for $x \in X$. If $x_1x_2...x_n$ is an irreducible word, then $|\{x_1, x_2, ..., x_n\}| = n$, i.e. $x_i \neq x_j$ for distinct $i, j \leq n$. Consider the maximal extension $\widehat{\rho}$ of the quasi-metric ρ on $G = F^a(X, \mathcal{B}_2)$. Then $\widehat{\rho}(x, y) = \widehat{\rho}(y^{-1}, x^{-1}) = \widehat{\rho}(y, x)$, i.e. $\widehat{\rho}$ is a pseudo-metric. Hence $\widehat{\rho}(x, e) = \widehat{\rho}(e, x) = \rho(x, e) = 0$ for any $x \in X$. Thus $0 \leq \widehat{\rho}(x, y) \leq \widehat{\rho}(x, e) + \rho(e, y) = 0$ for all $x, y \in X$. Therefore $\widehat{\rho}(a, b) = 0$ for all $a, b \in G$. We proved that the pseudo-quasi-metric $\widehat{\rho}$ is trivial.

Example 2. Let $\mathcal{A}_3$ be the variety of all paratopological Abelian groups with the identity $x^3 = e$. Fix a space X with the basic point $e \in X$. Let $b \in X \setminus \{e\}$. Then the words bb and b^{-1} are irreducible, $bb = b^{-1}$, the word bb is not of the minimal length and the word b^{-1} is of the minimal length.

Proposition 5. *Let ρ be a quasi-metric on $X, \rho(x^{-1}, y^{-1}) = \rho(y, x)$, $\rho(x, y^{-1}) = max\{\rho(x, e), \rho(e, y^{-1})\}$ and $\rho(y^{-1}, x) = max\{\rho(y^{-1}, e), \rho(e, x)\}$ for all $x, y \in X$. Then $\widehat{\rho}(a, b) = inf\{\sum\{\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) : i \leq n\} : n \in N, a \equiv x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot ... \cdot x_n^{\varepsilon_n}, b \equiv y_1^{\delta_1} \cdot y_2^{\delta_2} \cdot ... \cdot y_n^{\delta_n}\}$ for all $a, b \in F^a(X, \mathcal{V})$.*

Proof. Obviously $\widehat{\rho_1}(a, b) = inf\{\sum\{\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) : i \leq n, n \in N, a \equiv x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot ... \cdot x_n^{\varepsilon_n}, b \equiv y_1^{\delta_1} \cdot y_2^{\delta_2} \cdot ... \cdot y_n^{\delta_n}\}$ is an invariant pseudo-quasi-metric on $F^a(X, \mathcal{V})$ and $\widehat{\rho_1}(x, y) \leq \widehat{\rho}(x, y)$ for all $x, y \in F^a(X, \mathcal{V})$. If $a = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot ... \cdot x_n^{\varepsilon_n}$ and $b = y_1^{\delta_1} \cdot y_2^{\delta_2} \cdot ... \cdot y_n^{\delta_n}$, then $\widehat{\rho}(a, b) \leq \sum\{\widehat{\rho}(x_i^{\varepsilon_i}, t_i^{\delta_i}) : i \leq n\}$. Thus $\widehat{\rho}(a, b) \leq \widehat{\rho_1}(a, b)$. The proof is complete. □

6 Elementary spaces and free groups

Fix a non-trivial quasi-variety $\mathcal{V}$ of paratopological groups. Consider the space $E_\infty = \{0, 1, -1, 2, -2, \dots n, -n, \dots\}$ with the topology generated by the quasi-metric $\rho_\infty(x, y) = 1$ if $x < y$, and $\rho_\infty(x, y) = 0$ if $x \leq y$. Let $E_n = \{0, 1, -1, 2, -2, \dots n, -n\}$ and $\rho_n(x, y) = \rho_\infty(x, y)$ for all $x, y \in E_n$. Then (E_n, ρ_n) is a subspace of the quasi-metric space (E_∞, ρ_∞). Assume that $p_{E_\infty} = p_{E_n} = 0 \in E_n$ for each n. For any $n \in N$ consider the continuous retraction $r_n : E_\infty \to E_n$, where $r_n(x) = x$ for any $x \in E_n$, $r_n(x) = -n$ for any $x \leq -n$ and $r_n(x) = n$ for any $x \geq n$.

Proposition 6. *The following assertions are equivalent:*

1. *$e_{E_1} : E_1 \to F(E_1, \mathcal{V})$ is an embedding;*
2. *$e_{E_\infty} : E_\infty \to F(E_\infty, \mathcal{V})$ is an embedding;*
3. *$e_X : X \to F(X, \mathcal{V})$ is an embedding for any space X.*

Proof. Implications $3 \to 2 \to 1$ are obvious. Assume that e_{E_1} is an embedding. Fix a T_0-space X. There exist a cardinal number τ and an embedding $f : X \to E_1^\tau$, where $f(p_X) = 0$. We assume that $E_1 \subseteq F(E_1, \mathcal{V})$. Then $E_1^\tau \subseteq F(E_1, \mathcal{V})^\tau$. Consider the continuous homomorphism $\widehat{f} : F(X, \mathcal{V}) \to F(E_1, \mathcal{V})^\tau$ generated by the mapping f. Since $f = \widehat{f} \circ e_X$ is an embedding, e_X is an embedding too. The proof is complete. □

Lemma 1. *Let F be a finite set of a space X and $|F| = n \geq 1$. Then there exists a continuous mapping $s_F : X \to E_n \subseteq E_\infty$ such that $s_F(p_X) = 0$ and $s_F(x) \neq s_F(y)$ for all distinct $x, y \in F$.*

Proof. We can assume that $p_X \in F$. In any non-empty finite space Y there exists a point y such that the set $\{y\}$ is closed in Y. Thus in F there exists a well-ordering $F = \{x_1, x_2, ..., x_n\}$ such that the set $\{x_1, x_2, ..., x_i\}$ is closed in F for any $i \leq n$. Assume that $p_X = x_k$, where $1 \leq k \leq n$. We put $F_1 = cl_X\{x_1\}, F_2 = cl_X\{x_1, x_2\} \setminus cl_X\{x_1\}$,..., $F_{n-1} = cl_X\{x_1, x_2, ..., x_{n-1}\} \setminus cl_X\{x_1, x_2, ..., x_{n-2}\}$, $F_n = X \setminus cl_X\{x_1, x_2, ..., x_{n-1}\}$. Obviously, there exists a continuous mapping $s_F : X \to E_n$ such that $s_F(p_X = 0$, $s_F(x_i) = i - k < s_F(x_j) = j - k$ for $1 \leq j < i \leq n$ and $s_F(F_i) = s_F(\{x_i\})$ for any $i \leq n$. The proof is complete. □

Proposition 7. *For a quasi-variety $\mathcal{V}$ the following assertions are equivalent:*

1. *The free group $F(E_\infty, \mathcal{V})$ is abstract free.*

2. For each $n \in N$ the free group $F(E_n, \mathcal{V})$ is abstract free.
3. For each T_0-space X the free group $F(X, \mathcal{V})$ is abstract free.

Proof. Implications $3 \to 2 \to 1 \to 2$ are obvious. Implication $1 \to 3$ follows from Lemma 1. □

The next assertion is obvious.

Proposition 8. *For an I_p-complete quasi-variety $\mathcal{V}$ the following assertions are equivalent:*

1. The maximal extension d_∞ of the quasi-metric ρ_∞ on $F^a(E_\infty, \mathcal{V})$ is a quasi-metric and $\rho(x, y) = d_\infty(x, y)$ for all $x, y \in E_\infty$;

2. For any $n \in N$ the maximal extension d_n of the quasi-metric ρ_n on $F^a(E_n, \mathcal{V})$ is a quasi-metric and $\rho_n(x, y) = d_n(x, y)$ for all $x, y \in E_n$.

Proposition 9. *Let $\mathcal{V}$ be an I_p-complete quasi-variety, $n \in \mathbb{N}$ and the maximal extension d_n of the quasi-metric ρ_n on $F^a(E_n, \mathcal{V})$ is a quasi-metric. Then:*
1. $F(E_n, \mathcal{V})$ is an abstract free group;
2. $e_{E_n} : E_n \to F(E_n, \mathcal{V})$ is an embedding;
3. $d_n(x, y) = \rho_n(x, y)$ for all $x, y \in E_n$.

Proof. There exists $r > 0$ such that $r \leq 1$ and $1 = \rho_n(x, y) \geq d_n(x, y) \geq r$ for any $x, y \in E_n$ for which $x < y$. Then $d'(x, y) = min\{1, r^{-1} d_n(x, y)\}$ is an invariant quasi-metric on $F^a(E_n, V)$ and $d'(x, y) = \rho_n(x, y)$ for all $x, y \in E_n$. Since $d'(x, y) \leq d_n(x, y)$ for all $x, y \in F^a(E_n, \mathcal{V})$, we have $d_n(x, y) = \rho_n(x, y)$ for all $x, y \in E_n$. The proof is complete. □

Corollary 2. *If $\mathcal{V}$ is an I_p-complete quasi-variety and d_∞ is a quasi-metric on $F^a(E_\infty, \mathcal{V})$, then $d_\infty(x, y) = \rho_\infty(x, y)$ for all $x, y \in E_\infty$.*

Corollary 3. *Let $\mathcal{V}$ be an I_p-complete quasi-variety. Assume that d_∞ is a quasi-metric on $F^a(E_\infty, \mathcal{V})$. Then for any T_0-space X the free group $F(X, \mathcal{V})$ is abstract free, $e_X : X \to F(X, \mathcal{V})$ is an embedding and on $F(X, \mathcal{V})$ there exists a T_0-topology T which is generated by some family of invariant pseudo-quasi-metrics and e_X is an embedding of X into $(F(X, \mathcal{V}), T)$.*

7 Free Abelian groups of spaces

Proposition 10. *The maximal extension d_∞ of the quasi-metric ρ_∞ on $F^a(E_\infty, \mathcal{V}_a)$ is a quasi-metric.*

Proof. On the group $\mathbb{Z}$ of integers consider the topology generated by the quasi-metric $\rho(x, y) = 1$ for $x < y$ and $\rho(x, y) = 0$ for $y \leq x$. Obviously, ρ is an invariant quasi-metric and $\mathbb{Z} \in \mathcal{V}_a$.

The group $G = \{(x_n : n \in \mathbb{Z}) \in \mathbb{Z}^{\mathbb{Z}} : \textit{the set } \{n \in \mathbb{Z} : x_n \neq 0\} \textit{ is finite}\}$, $G \in \mathcal{V}_a$ is Abelian and on G consider the topology generated by the invariant quasi-metric $d((x_n : n \in \mathbb{Z}), (y_n : n \in \mathbb{Z})) = sup\{d(x_n, y_n) : n \in \mathbb{Z}\}$. We put $a_0 = (x_n : n \in \mathbb{Z})$ and $x_n = 0$ for any $n \in \mathbb{Z}$. If $n \in \mathbb{Z}$ and $n \geq 1$, then $a_n : (x_n : n \in \mathbb{Z})$, where $x_i = 1$ for $i \in \{0, 1, 2, ..., n-1\}$ and $x_j = 0$ for each $j \in \mathbb{Z} \setminus \{0, 1, 2, ..., n-1\}$. If $n \in \mathbb{Z}$ and

$n \leq -1$, then $a_n = (x_n : n \in \mathbb{Z})$, where $x_i = -1$ for each $i \in \{-1, -2, ..., -n\}$ and $x_j = 0$ for any $j \in \mathbb{Z} \setminus \{-1, -2, ..., -n\}$. Consider the mapping $h : E_\infty \to \mathbb{Z}^{\mathbb{Z}}$, where $d(n) = a_n$ for any $n \in E_\infty$. By construction, $\rho_\infty(x, y) = d(h(x), h(y))$ for all $x, y \in E_\infty$. Thus h is an isometrical embedding of E_∞ in $\mathbb{Z}^{\mathbb{Z}}$. The set $d_\infty(E_\infty)$ generated the group G and the pair (G, h) is the abstract free group $(F^a(E_\infty, \mathcal{V}_a), a_{E_\infty})$ of the space E_∞. In this case $d(x, y) \leq d_\infty(x, y)$ for any $x, y \in G = F^a(E_\infty, V_\infty)$. Thus d_∞ is a quasi-metric. The proof is complete. $\square$

Corollary 4. *For any T_0-space X the free group $F(X, \mathcal{V}_a)$ is abstract free, $e_X : X \to F(X, \mathcal{V}_a)$ is an embedding and the topology of the space $F(X, \mathcal{V}_a)$ is generated by some family of invariant pseudo-quasi-metric.*

8 On the non-Burnside quasi-varieties

Let $\mathcal{V}$ be an I_p-complete quasi-variety of paratopological groups. Assume that $\mathbb{Z} \in \mathcal{V}$, i. e. $\mathcal{V}_a \subseteq \mathcal{V} \subseteq \mathcal{V}_g$.

We put $F(X) = F(X, \mathcal{V})$ and $F^a(X) = F^a(X, \mathcal{V})$ for any space X.

Fix two words $x_1x_2...x_n$ and $y_1y_2...y_m$, where $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m \in \widetilde{X}$.

If $x_1x_2...x_n$ and $y_1y_2...y_m$ are irreducible, then $[x_1x_2...x_n] = [y_1y_2...y_m]$ if and only if $n = m$ and there exists a bijection $h = \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., m\}$ such that $x_i = y_{h(i)}$ for any $i \leq n$.

The words $x_1x_2...x_n$ and $y_1y_2...y_m$ are called equivalent if $[x_1x_2...x_n] = [y_1y_2...y_m]$.

The words $x_1x_2...x_n$ and $y_1y_2...y_m$ are called strongly equivalent if $[x_1x_2...x_n] = [y_1y_2...y_m]$, $n = m$ and there exists a bijection $h : \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., m\}$ such that $x_i = y_{h(i)}$ for any $i \leq n$.

Let $\mathbb{N}_n = \{1, 2, ..., n\}$ for any $n \in \mathbb{N}$. If $i < j$, then we put $[i, j] = [j, i] = \{k \in \mathbb{N} : i \leq k \leq j\}$. If $i, j \in A \subseteq \mathbb{N}$, then $[i, j]_A = [i, j] \cap A$.

A scheme for an element $b \in F^a(X)$ is a word $x_1x_2...x_n$ and a mapping $s : \mathbb{N}_n \to \mathbb{N}_n$ such that:

1. $b = [x_1x_2...x_n]$;
2. $s(i) \neq i$ and $s(s(i)) = i$ for any $i \leq n$;
3. There exist a word $y_1y_2...y_n$ and a bijection $h : \mathbb{N}_n \longrightarrow \mathbb{N}_n$ such that:

– $b = [y_1y_2...y_n]$ and $y_i = x_{h(i)}$ for any $i \leq n$;

– if $\sigma(i) = h^{-1}(s(h(i)))$ then for any $i, j \in \{1, 2, ..., n\}$ the sets $[i, \sigma(i)]$, $[j, \sigma(j)]$ are either disjoint or one contains the other.

A mapping σ from the definition of the scheme has the following properties:

4. There are no $i, j \in \mathbb{N}_n$ such that $i < j < \sigma(i) < \sigma(j)$.
5. For some $i < n$ we have $\sigma(i) = i + 1$.
6. The mappings s and σ are bijections and involutions without fixed points.
7. The number n is even.

The method of scheme for pseudo-metric and $\mathcal{V} \in \{\mathcal{V}_a, \mathcal{V}_g\}$ are due to the work of M. I. Graev [10]. The problem of the extension of pseudo-metrics on $F^a(X, \mathcal{V})$ for any quasi-variety $\mathcal{V}$ of topological algebras was examined in [6]. In the case $\mathcal{V} \in \{\mathcal{V}_a, \mathcal{V}_g\}$ the notion of the scheme for pseudo-quasi-metrics was defined in [15].

We use the method of scheme from [15], in the general case, for any non-Burnside quasi-variety.

On a space X fix a continuous pseudo-quasi-metric ρ. Assume that $e \in X \subseteq F^a(X)$ and e is the identity of the group $F^a(X)$. For any $x, y \in X$ we put $\rho^*(x^{-1}, y^{-1}) = \rho(y, x)$, $\rho^*(x^{-1}, y) = \rho^*(x^{-1}, e) + \rho^*(e, y)$ and $\rho^*(x, y^{-1}) = \rho^*(y^{-1}, x) = \rho^*(x, e) + \rho^*(e, y^{-1})$. Obviously, ρ^* is a pseudo-quasi-metric on $\widetilde{X}$.

For any $b \in F^a(X)$ we put $N_\rho(b) = inf\{\frac{1}{2}\sum\{\rho^*(x_i^{-1}, x_{s(i)}) : i \leq 2m\} : m \in \mathbb{N}, s : N_m \to N_m$ *is a scheme*, $b = [x_1x_2...x_m]\}$.

As in [15] we say that the word $x_1x_2...x_n$ is almost irreducible if:

– $x_i \in Sup^*([x_1, x_2, ..., x_n])$ for any $i \leq n$;

– any word $y_1y_2...y_n$ which is strongly equivalent with the word $x_1x_2...x_n$ does not contain two consecutive symbols of the form $u^{-1}u$, $u \in \widetilde{X} \setminus \{e\}$.

If $b \in F^a(X)$ and $2m \geq l(b) \geq 1$, then $b = [x_1, x_2, ..., x_{2m}]$ for some almost irreducible word $x_1x_2...x_{2m}$. The next property of the function N_ρ is important.

Lemma 2 (see [15], Claim 2). *If $b \in F^a(X)$ and $l(b) = n$, then there exist an almost irreducible word $x_1x_2...x_{2m}$ and a scheme $s : \mathbb{N}_m \to \mathbb{N}_m$ such that:*

1. $b = [x_1x_2...x_{2m}]$ and $n \leq 2m \leq 2n$;

2. $2N_\rho(b) = \sum\{\rho^*(x_i^{-1}, x_{s(i)}) : i \leq 2m\}$.

Proof. Obviously, we can assume that $b \neq e$. Let $b = [x_1, x_2...x_{2m}]$ and $s : \mathbb{N}_{2m} \to \mathbb{N}_{2m}$ be a scheme.

Assume that the word $x_1x_2...x_m$ is not almost irreducible. Then we can suppose that there exist $i < 2m$ and $u \in \widetilde{X}$ such that $x_i = u$ and $x_{i+1} = u^{-1}$. If $h(i) = i+1$, then we put $A = \{1, ..., i-1, i+2, ..., 2m\}$ and $\sigma = s|A$. Then σ is a scheme for the element $x = [x_1x_2...x_{i-1}x_{i+2}...x_{2m}]$ and respective word $x_1x_2...x_{i-1}x_{i+2}...x_{2m}$, $|A| = 2m-2$ and $\sum\{\rho^*(x_j^{-1}, x_{\varphi(j)}) : j \in A\} \leq \sum\{\rho^*(x_i^{-1}, x_{h(i)}) : i \in \mathbb{N}_{2m}\}$. If $r = s(i) \neq i+1$ and $t = s(i+1)$, then $A = \{1, ..., i-1, i+2, ..., 2m\}, \sigma(j) = s(j)$ for $j \in \mathbb{N}_{2m} \setminus \{i, i+1, r, t\}$ and $\sigma(r) = t$, $\sigma(t) = r$. Since $\rho^*(x_r^{-1}, x_t) + \rho^*(x_t^{-1}, x_r) \leq \rho^*(x_1^{-1}, u^{-1}) + \rho^*(u^{-1}, x_r) + \rho^*(x_r^{-1}, u) + \rho^*(u, x_1) = \rho^*(x_t^{-1}, x_{i+1}) + \rho^*(x_i^{-1}, x_r) + \rho^*(x_r^{-1}, x_i) + \rho^*(x_{i+1}^{-1}, x_t)$, σ is a scheme and $\sum\{\rho^*(x_j^{-1}, x_{\varphi(j)}) : j \in A\} \leq \sum\{\rho^*(x_i^{-1}, x_{h(i)}) : i \in \mathbb{N}_{2m}\}$. Thus we can assume that the word $x_1x_2...x_{2m}$ is almost irreducible and for any $i \leq 2m$ we have $x_{i+1} \neq x_i^{-1}$. In particular, if $i < 2m$, then $x_i \cdot x_{i+1} \neq e$. In this conditions, the word $x_1x_2...x_{2m}$ is almost irreducible and $2m \leq 2n = 2l(a)$. Since there exists a finite set of almost irreducible words of the length $\leq 2l(b)$ which represents the given element $b \in F^a(X)$, the proof is complete. □

Lemma 3 (see [15], Claim 4). *$N_\rho(x^{-1}y) = \rho(x, y)$ for all $x, y \in X$.*

Proof. Fix $x, y \in X$. If $x = y$, then $x^{-1} \cdot y = y \cdot x^{-1} = e, N_\rho(e) = 0 = \rho(x, y)$. Assume that $x \neq y$. Then $l(x^{-1}y) = 2$ and for the element $b = x^{-1}y$ there exist only the next possible almost irreducible words of the length ≤ 4: $x^{-1}y$, $ex^{-1}ey$, $x^{-1}eye$, $ex^{-1}ye$, $y^{-1}x$, $ey^{-1}ex$, $y^{-1}exe$, $ey^{-1}xe$. If $\mathcal{V} \neq \mathcal{V}_a$, then there exist only the next possible almost irreducible words of the length ≤ 4: $x^{-1}y$, $ex^{-1}ey$, $x^{-1}eye$, $ex^{-1}ye$. The direct calculation permits to obtain $N_\rho(x^{-1}y) = \rho(x, y)$. □

Lemma 4 (see [15], Claim 3). *The function N_ρ has the the next properties:*

1. $N_\rho(e) = 0$ *and* $N_\rho(b) \geq 0$ *for any* $b \in F^a(X)$.
2. $N_\rho(a \cdot b) \leq N_\rho(a) + N_\rho(b)$ *for any* $a, b \in F^a(X)$.
3. $N_\rho(xbx^{-1}) = N_\rho(b)$ *for any* $b, x \in F^a(X)$.

Proof. Assertions (1) and (2) are obvious. Let $b_1b_2...b_{2m}$ be an almost irreducible word, $b = [b_1b_2..., b_{2m}]$, $s : \mathbb{N}_{2m} \to \mathbb{N}_{2m}$ be a scheme and $2\mathbb{N}_\rho(b) = \sum\{\rho^*(b_i^{-1}, b_{s(i)}) : i \in \mathbb{N}_{2m}\}$. Fix the irreducible word $x_1x_2...x_k$. Put $x = [x_1, x_2, ..., x_k]$, $y_{2m+1}y_{2m+2}...y_{2m+k} = x_1x_2...x_k$, $y_{2m+k+1}...y_{2m+2k} = x_k^{-1}$, $A = \{1, 2, ..., 2m, ..., 2m+2k\}$, $\varphi(i) = s(i)$ for $i \leq 2m$ and $\varphi(2m+i) = 2m+2k-i+1$ for $i \leq k$.

Let $y_1, y_2, ..., y_{2m} = b_1, b_2, ..., b_{2m}$. Then φ is a scheme on A for the element $x^{-1}bx$, $x_{-1}bx = [y_{2m+k+1}...y_{2m+2k}y_1...y_{2m}y_{2m+1}...y_{2m+k}]$ and $\sum\{\rho^*(y_j^{-1}, y_{h(j)}) : j \in A\} = \sum\{\rho^*(b_i^{-1}, b_{h(j)}) : i \in \mathbb{N}_m\}$. Hence $N_\rho(x^{-1}bx) \leq \mathbb{N}_\rho(b)$ and $N_\rho(b) = M_\rho((xx^{-1})b(xx^{-1})) \leq \mathbb{N}_\rho(x^{-1}bx)$. The property (3) is proved. □

Lemma 5. *The function $d(x,y) = N_\rho(x^{-1}y)$ is an invariant pseudo-quasi-metric on $F^a(X)$. Moreover, $d(x,y) \leq \widehat{\rho}(x,y)$ and $N_\rho(b) \leq \widehat{\rho}(e,b)$, where $\widehat{\rho}$ is the maximal extension of ρ on $F^a(X)$, for any $x, y, b \in F^a(X)$.*

Proof. Really, $d(xa, xb) = N_\rho(a^{-1}x^{-1}xb) = N_\rho(a^{-1}b) = d(a,b)$ and $d(ax, bx) = N_\rho(x^{-1}a^{-1}bx) = N_\rho(a^{-1}b) = d(a,b)$. Since $d(x,y) = \rho(x,y)$ for $x, y \in X$, we have $d(x,y) \leq \widehat{\rho}(x,y)$ for all $x, y \in F^a(X)$. The proof is complete. □

Proposition 11. *Let $r > 0$ and X be a linear ordered space with the topology generated by the quasi-metric $\rho(x,y) = r$ if $x < y$ and $\rho(x,y) = 0$ if $y \leq x$. Then the maximal extension d of ρ on $F^a(X,e)$ is a quasi-metric for any point $e \in X$.*

Proof. We can assume that $r = 1$.

Let $e \in X$ and $e_1 \notin X$. We put $Y = X \cup \{e_1\}$, $\rho(e_1, e_1) = 0$ and $\rho(x, e_1) = \rho(e_1, x) = 1$ for any $x \in X$. Then (X, ρ) is a quasi-metric subspace of the quasi-metric space (Y, ρ). On $F^a(Y) = F^a(Y, \mathcal{V}, e_1)$ consider the function $\mathbb{N}_\rho(y)$. □

Claim 1. *If $b \in F^a(Y) \setminus \{e_1\}$, then $\mathbb{N}_\rho(b) + \mathbb{N}_\rho(b^{-1}) \neq 0$.*

Proof. Assume that $\mathbb{N}_\rho(b) + \mathbb{N}_\rho(b^{-1}) = 0$, $b = [b_1b_2...b_n]$ and the word $b_1b_2...b_n$ is irreducible. Then $b_1, b_2, ..., b_n \in \widetilde{X} \subseteq \widetilde{Y}$. Since $\mathbb{N}_\rho(b) = 0$, there exists an almost irreducible word $x_1x_2...x_{2m}$ of the minimal length and a scheme $s : \mathbb{N}_{2m} \to \mathbb{N}_{2m}$ such that $b = [x_1x_2...x_{2m}]$ and $\sum\{\rho^*(x_i^{-1}, x_{s(i)}) : i \leq 2m\} = 0$. From the minimality of the length of the word $x_1x_2...x_n$ it follows that $x_i \neq e_1$ for any $i \leq 2m$. Really, if $x_i = e_1$, then $x_{s(i)} \neq e_1$ and $\rho^*(x_{s(i)}^{-1}, x_i) = 1$, a contradiction. Thus the words $x_1x_2...x_{2m}$ and $b_1b_2...b_n$ are equivalent, $n = 2m$ is an even number and s is a scheme on $\mathbb{N}_{2m}$ for the element b. We can assume that $x_i = b_i$ for any $i \leq n$. Therefore $\sum\{\rho^*(b_i, b_{s(i)}) : i \leq n\} = 0$. We can assume that for any $i, j \in \{1, 2, ..., n\}$ the sets $[i, s(i)]$, $[j, s(j)]$ are either disjoint or one contains the other.

Since $\mathbb{N}_\rho(b^{-1}) = 0$ and $b = [b_n^{-1}...b_2^{-1}b_1^{-1}]$, there exists a scheme $q : \mathbb{N}_n \to \mathbb{N}_n$ such that $\sum\{\rho^*(b_i, b_{q(i)}^{-1}) : i \leq n\} = 0$. Since the word $b_1b_2...b_n$ is irreducible, $b_{i+1} \neq b_i^{-1}$ for any $i < n$.

We affirm that $s = q$. Assume that $i_1 \in \mathbb{N}$ and $s(i_1) \neq q(i_1)$. Put $j_1 = s(i_1)$. Since $s(j_1) \neq q(j_1)$ and $\rho^*(x_i^{-1}, x_{j_1}) = \rho^*(j_i^{-1}, x_{i_1}) = 0$, $X \cap \{x_{i_1}, x_{j_1}\} \neq \emptyset$ and $X^{-1}\{x_{i_1}, x_{j_1}\} \neq \emptyset$. We can assume that $x_{i_1} \in X$ and $x_{j_1} \in X^{-1}$. For any $k \geq 1$ we put $i_{k+1} = q(j_k)$ and $j_{k+1} = s(i_{k+1})$. For any $k < 1$ we have $x_{j_k} \neq x_k \neq x_{i_{k+1}}$, $\rho^*(x_k^{-1}, x_{k_k}) = \rho^*(x_{j_k}^{-1}, x_{i_k}) = \rho^*(x_{j_k}, x_{i_{k+1}}^{-1}) = \rho^*(x_{j_{k+1}}, x_{j_k}^{-1})$. Let k be the first number for which $\{i_{k+1}, j_{k+1}\} \cap \{i_1, j_1, i_2, j_2, ..., i_k, j_k\} \neq \emptyset$. Suppose that $i_{k+1} = q(j_k) \in \{i_1, j_1, ..., i_k, j_k\}$. If $i_{k+1} = i_p$ for some $p \leq k$, then $j_k = q(i_{k+1}) = q(i_p) = j_{p-1}$, a contradiction. If $i_{k+1} = i_p$ for some $p \leq k$, then $j_k = s(i_{k+1}) = h(j_p) = i_{p+1}$. Since $i_k \neq j_k$, we have $p + 1 < k$ and $j_k \in \{i_{p+1}, j_{p+1}\}$, a contradiction. Now suppose that $j_{k+1} \in \{i_1, j_1, ..., i_k, j_k\}$. If $j_{k+1} = i_p$ for some $p \leq k$, then $i_{k+1} = s(j_{k+1}) = s(i_p) = j_p$, a contradiction. If $j_{k+1} = j_p$ for some $p \leq k$, then $i_{k+1} = s(i_{k+1}) = s(j_p) = i_p$, a contradiction. Therefore $q(i) = s(i)$ for any $i \leq n$. There exists $i < n$ such that $h(i) = q(i) = i+1$. Let $x_i \in X$. Then $x_{i+1} \in X^{-1}$. Since $\rho^*(x_{i+1}^{-1}, x_i) = \rho^*(x_{h(i)}^{-1}) = 0$, we have $x_{i+1}^{-1} \leq x_i$. Since $\rho^*(x_i, x_{i+1}^{-1}) = \rho^*(x_i, x_{q(i)}^{-1}) = 0$, we have $x_i \leq x_{i+1}^{-1}$. Hence $x_i = x_{i+1}^{-1}$ and $x_i \cdot x_{i+1} = e$, a contradiction with the condition of irreducibility of the word $b_1b_2...b_n$. Claim 1 is proved. □

Claim 2. *On $F^a(Y, e_1)$ there exists an invariant quasi-metric such that:*

- *$d_1(x, y) = \rho(x, y)$ for any $x, y \in X$;*
- *$d_1(x, y) \in \{0, 1\}$ for any $x, y \in F^a(Y, e_1)$.*

Proof. Let $d_2(x, y) = N_\rho(x^{-1}y)$ for all $x, y \in F^a(Y, e_1)$. By construction, $d_2(x, y) \geq 1$ provided $d_2(x, y) > 0$. From this fact and the Claim 1 it follows that $d_1(x, y) = min\{1, d_2(x, y)\}$ is the desired quasi-metric. □

Claim 3. *Let $e \in X \subseteq Y$. Then on $F^a(Y, e)$ there exists a quasi-metric ρ_1 such that:*

- *$\rho_1(x, y) = \rho(x, y)$ for any $x, y \in X$;*
- *$\rho_1(x, y) \in \{0, 1\}$ for any $x, y \in F^a(Y, e)$.*

Proof. Let d_1 be quasi-metric with the properties from Claim 2. In the proof of Proposition 4 it was established that there exists an isomorphism $\varphi : F^a(X, e) \to F^a(Y, e_1)$ such that $\varphi(x) = xe^{-1}$ for any $x \in Y$. We put $\rho_1(x, y) = d_1(\varphi(x), \varphi(y))$ for any $x, y \in F^a(Y, e)$. Since the quasi-metric d_1 is continuous on the space $(F^a(Y, e_1), T(y, e_1))$, the quasi-metric ρ_1 is continuous on the space $(F^a(Y, e), T(Y, e))$. For any $x, y \in Y$ we have $\rho_1(x, y) \in \{0, 1\}$. Let $x, y \in X$ and $x < y$. Then $\rho(y, x) = 0$, $\rho(x, y) = 1$ and $1 \in \{\rho_1(x, y), \rho(y, x)\}$. Since the quasi-metric ρ_1 is continuous, we have $\rho_1(x, y) = 1 = \rho(x, y)$ and $\rho_1(y, x) = 0 = \rho(y, x)$. Claim 3 is proved. □

Since $F^a(Y, e)$ is a subgroup of the group $F^a(Y, e)$, the proof is complete.

Corollary 5. *For any $n \in \mathbb{N}$ the maximal extension d_n of the quasi-metric ρ_n on $F^a(E_n)$ is a quasi-metric.*

Corollary 6. *The maximal extension d_∞ of the quasi-metric ρ_∞ on $F^a(E_\infty)$ is a quasi-metric.*

From Corollary 5 it follows

Corollary 7. *For any pointed T_0-space X the free group $F(X)$ is abstract free and $e_X : X \to F(X)$ is an embedding. Moreover on $F(X)$ there exists a topology T which is generated by some family of almost invariant pseudo-quasi-metrics and e_X is an embedding of X into $(F(X), T)$.*

9 On quasi-varities of paratopological groups

Let $\mathcal{S}$ be a set of properties of paratopological groups, any paratopological group with invariant base has the properties $\mathcal{S}$, $\mathcal{W}$ be a non-trivial $\mathcal{S}$-complete quasi-variety of paratopoligical groups. Denote by $\mathcal{V}$ the $\mathcal{S}$-complete variety of paratopoligical groups generated by the quasi-variety $\mathcal{W}$. We say that $\mathcal{W}$ is a Burnside quasi-variety if $\mathcal{V}$ is a Burnside variety.

A quasi-variety $\mathcal{W}$ is a non-Burnside quasi-variety if and only if $\mathbb{Z} \in \mathcal{W}$.

The next assertions affirm that the free objects of spaces in quasi-varieties are the same as in varieties.

Proposition 12. *For any pointed space X:*

1. There exists an isomorphism $\varphi : F^a(X, \mathcal{V}) \longrightarrow F^a(X, \mathcal{W})$ such that $\varphi(x) = x$ for any $x \in X$.

2. There exists a topological isomorphism $\varphi : F(X, \mathcal{V}) \longrightarrow F(X, \mathcal{W})$ such that $\psi(e_{(X,\mathcal{V})}(x)) = e_{(X,\mathcal{W})}(x)$ for any $x \in X$.

Proof. The assertion 1 is obvious.

Fix a space X. Let $(F(X, \mathcal{V}), e_{(X,\mathcal{V})})$ be the free object of the space X in the class $\mathcal{V}$ and $(F(X, \mathcal{W}), e_{(X,\mathcal{W})})$ be the free object of the space X in the class $\mathcal{W}$. There exists a continuous homomorphism $\varphi : F(X, \mathcal{V}) \longrightarrow F(X, \mathcal{W})$ such that $\varphi(x) = x$ for any $x \in X$.

Case 1. $\mathbb{Z} \notin \mathcal{W}$.

In this case $\mathcal{W} \subseteq \mathcal{B}_n$ for some $n \in \mathbb{N}$. By virtue of Proposition 3, $\mathcal{W}$ is a quasi-variety of topological groups. For quasi-varieties of topological groups the assertions of Proposition 12 are known (see [5, 8]).

Case 2. $\mathbb{Z} \in \mathcal{W}$.

In this case the variety $\mathcal{V}$ is not a Burnside variety. Then, by virtue of Corollary 7, the free objects $F(X, \mathcal{V})$, $F(X, \mathcal{W})$ are abstract free and the mappings $e_{(X,\mathcal{V})}$ and $e_{(X,\mathcal{W})}$ are embeddings. The proof is complete. □

Theorem 2. *If $\mathbb{Z} \in \mathcal{W}$, then for any pointed space X we have:*

1. The free topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping e_X is an embedding.

2. If ρ is a continuous pseudo-quasi-metric on the space X, then:

(2a) the maximal extension $\widehat{\rho}$ of the pseudo-quasi-metric ρ on $F(X, \mathcal{W})$ is a continuous invariant pseudo-quasi-metric;

(2b) $\rho(x,y) = \widehat{\rho}(e_X(x), e_X(y))$ *for all* $x, y \in X$*;*

(2c) if $x, y \in F(X, \mathcal{W})$ *and* ρ *is a quasi-metric on* $Sup^*(x) \cup Sup^*(y)$*, then* $\widehat{\rho}(x,y) + \widehat{\rho}(y,x) > 0$*;*

(2d) if ρ *is a quasi-metric, then* $\widehat{\rho}$ *is a quasi-metric too.*

Proof. We can assume that $e_X(x) = x$ for any $x \in X$ and $X \subseteq (F(X, \mathcal{W})$.

On $(F(X, \mathcal{W})$ consider the function $N_\rho(b)$ and the pseudo-quasi-metric $d(x,y) = N_\rho(x^{-1}y)$. By virtue of Lemma 5, d is an invariant pseudo-quasi-metric. From Lemma 3 it follows that $d(x,y) = N_\rho(x^{-1}y) = \rho(x,y)$ for all $x, y \in X$. Thus $d(x,y) \leq \widehat{\rho}(x,y)$ for all $x, y \in F(X, \mathcal{W})$ and $d(x,y) = \widehat{\rho}(x,y)$ for all $x, y \in X$.

Let $x, y \in F(X, \mathcal{W})$ and ρ be a quasi-metric on $Z = Sup^*(x) \cup Sup^*(y)$. We put $b = x^{-1}y$. Then $Sup^*(b) \subseteq Z$. Let $r = min\{\rho(u,v) : u, v \in Z, \rho(u,v) > 0\}$. Since the space Z is finite, we have $r > 0$ and there exists an ordering on Z such that $\rho(u,v) > 0$ provided $u < v$. We have $Z \subseteq F^a(Z, \mathcal{W}) \subseteq F(X, \mathcal{W})$. By virtue of Proposition 11, $N_\rho(c) + N_\rho(c^{-1}) > 0$ for each $c \in F^a(Z, \mathcal{W})$. Since $b \in F^a(Z, \mathcal{W})$, $0 < N_\rho(b) + N_\rho(b^{-1}) = d(x,y) + d(y,x) \leq \widehat{\rho}(x,y) + \widehat{\rho}(y,x)$. The assertions 1, (2a), (2b) and (2c) are proved. The assertion (2d) follows from the assertion (2c). The proof is complete. □

10 Free groups of quasi-uniform spaces

A quasi-uniformity on a set X is a family $\mathcal{U}$ of entourages of the diagonal $\Delta(X) = \{(x,x) : x \in X\}$ and a family $\mathcal{P}$ of the pseudo-quasi-metrics on X, which satisfies the following conditions:

(QU1) If $V \in \mathcal{U}$ and $V \subseteq W \subseteq X \times X$, then $V \in \mathcal{U}$.

(QU2) If $V, W \in \mathcal{U}$, then $V \cap W \in \mathcal{U}$.

(QU3) If $V \in \mathcal{U}$, then there exist $\rho \in \mathcal{P}$ and $r > 0$ such that $\{(x,y) \in X \times X : \rho(x,y) < r\} \subseteq V$.

(QU4) $\{(x,y) \in X \times X : \rho(x,y) < r\} \in \mathcal{U}$ for all $\rho \in \mathcal{P}$ and $r > 0$.

(QU5) If $\rho_1, \rho_2 \in \mathcal{P}$, then there exists $\rho \in \mathcal{P}$ such that $max\{\rho_1(x,y), \rho_2(x,y)\} \leq \rho(x,y)$ for all $x, y \in X$.

(QU6) If $x, y \in X$ and $x \neq y$, then $\rho(x,y) + \rho(y,x) > 0$ for some $\rho \in \mathcal{P}$.

Obviously, the quasi-uniformity $\mathcal{U}$ is generated by a family of pseudo-quasi-metrics $\mathcal{P}$.

Fix a non-trivial I_p-complete quasi-variety $\mathcal{W}$ of paratopological groups.

Let $G \in \mathcal{W}$. Denote by $QP(G)$ the family of all continuous pseudo-quasi-metrics on the space G, $LQP(G) = \{d \in QP(G) : d \textit{ is left invariant}\}$, $RQP(G) = \{d \in QP(G) : d \textit{ is right invariant}\}$ and $IQP(G) = LQP(G) \cap RQP(G)$.

The pseudo-quasi-metrics $LQP(G)$ generate the left quasi-uniformity $\mathcal{U}_l$ on G and the pseudo-quasi-metrics $RQP(G)$ generate the left quasi-uniformity $\mathcal{U}_r$ on G. These quasi-uniformities generate the topology of the space G. If G is a paratopological group with the invariant base at the identity e, then $\mathcal{U}_l = \mathcal{U}_r$.

Assume that $\mathcal{W}$ is not a Burnside quasi-variety. Fix a quasi-uniformity pointed space $(X, \mathcal{U})$ generated by the pseudo-quasi-metrics $\mathcal{P}$. For any $\rho \in \mathcal{P}$ denote by $\widehat{\rho}$

its maximal extension on $F(X, \mathcal{W})$. We put $\widehat{\mathcal{P}} = \{\widehat{\rho} : \rho \in \mathcal{P}\}$. The family $\widehat{\mathcal{P}}$ generates an invariant quasi-uniformity on $F(X, \mathcal{W})$.

11 Free quasitopological groups

A class $\mathcal{V}$ of quasitopological groups is called *a C-complete quasi-variety* of quasitopological groups if:

(QF1) the class $\mathcal{V}$ is multiplicative;

(QF2) if $G \in \mathcal{V}$ and A is a subgroup of G, then $A \in \mathcal{V}$;

(QF3) every space $G \in \mathcal{V}$ is a T_0-space;

(QF4) if $G \in \mathcal{V}$, $\mathcal{T}$ is a compact T_0-topology on G and $(G, \mathcal{T})$ is a quasitopological group, then $(G, \mathcal{T}) \in \mathcal{V}$.

Lemma 6. *Let G be a quasitopological group. If G is a T_0-space, then G is a T_1-space.*
Proof. It is obvious. □

On any set X there exists the profinite topology $\mathcal{T}_{pf}(X) = \{X, \emptyset\} \cup \{X \setminus F : F$ *is a finite set*$\}$. The space $(X, \mathcal{T}_{pf}(X))$ is a compact T_1-space.

Lemma 7. *Let G be a group. Then $(G, \mathcal{T}_{pf}(G))$ is a quasitopological group.*

Proof. It is obvious. □

Theorem 3. *Let $\mathcal{W}$ be a non-trivial C-complete quasi-variety of quasitopological groups. For any T_1-space X the free group $F(X, \mathcal{W})$ is abstract free and $e_X : X \to F(X, \mathcal{W})$ is an embedding.*

Proof. For any infinite cardinal τ we fix a group $G_\tau \in \mathcal{W}$ of the cardinality τ with the profinite topology $\mathcal{T}_{pf}(G_\tau)$. Further we fix an infinite group $G_0 \in \mathcal{W}$. Let F be a non-empty closed subset of the space X and $b \notin F$.

Case 1. The set $X \setminus F$ is finite.

In this case the sets F and $X \setminus F$ are open-and-closed. There exists a mapping $g : X \longrightarrow G_0$ such that $g(p_X) = e$, $g(F)$ and $g(X \setminus F)$ are singletons and $g^{-1}(g(F)) = F$. Then the mapping g is continuous and $g(b) \notin cl_{G_0} g(F) = g(F)$.

Case 2. The set $X \setminus F$ is infinite.

Let $\tau = |X \setminus F|$. There exists a mapping $g : X \longrightarrow G$ such that $g(X) = G_\tau$, $g^{-1}(g(F)) = F$, $g(F)$ is a singleton $g(p_X) = e$ and $g(x) \neq g(y)$ for distinct points $x, y \in X \setminus F$. Since X is a T_1-space, the mapping g is continuous and $g(b) \notin cl_{G_0} g(F) = g(F)$.

Therefore the mapping $e_X : X \to F(X, \mathcal{W})$ is an embedding. Thus we can assume that $e = p_X \in X \subseteq F(X, \mathcal{W})$.

Assume that $e = p_X \in X \subseteq F^a(X, \mathcal{W})$. On $F(X, \mathcal{W})$ we consider the profinite topology $\mathcal{T}_{pf}(F^a(X, \mathcal{W}))$. Then the mapping $a_X : X \longrightarrow F^a(X, \mathcal{W})$ is a continuous injection. Therefore there exists a continuous homomorphism $\psi : F(X, \mathcal{W}) \to F^a(X, \mathcal{W})$ such that $\psi(x) = x$ for any $x \in X$. Hence ψ is an isomorphism. The proof is complete. □

12 Free left topological groups

A group G with topology is called a *left* (respectively, *right*) topological group if the left translation $L_a(x) = ax$ (respectively, the right translation $R_a(x) = xa$) is continuous for any $a \in G$.

A class $\mathcal{V}$ of left topological groups is called *an LI-complete quasi-variety* of left topological groups if:

(LF1) the class $\mathcal{V}$ is multiplicative;

(LF2) if $G \in \mathcal{V}$ and A is a subgroup of G, then $A \in \mathcal{V}$;

(LF3) every space $G \in \mathcal{V}$ is a T_0-space;

(LF4) if $G \in \mathcal{V}$, $\mathcal{T}$ is a compact T_0-topology on G and $(G, \mathcal{T})$ is a left topological group, then $(G, \mathcal{T}) \in \mathcal{V}$;

(SF5) if $G \in \mathcal{V}$, $\mathcal{T}$ is a T_0-topology on G and $(G, \mathcal{T})$ is a paratopological group with an invariant base, then $(G, \mathcal{T}) \in \mathcal{V}$.

From Theorem 2 it follows

Corollary 8. *Let $\mathcal{W}$ be a non-trivial LI-complete quasi-variety of left topological groups and $\mathbb{Z} \in \mathcal{W}$. Then for any pointed space X the free left topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping e_X is an embedding.*

From Theorem 3 it follows

Corollary 9. *Let $\mathcal{W}$ be a non-trivial LI-complete quasi-variety of left topological groups, $n \in \mathbb{N}$ and $x^n = e$ for any $x \in G$ and $G \in \mathcal{W}$. Then for any pointed T_1-space X the free left topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping e_X is an embedding.*

The following assertion completes Corollary 9.

Lemma 8. *Let G be a left topological group and for any $x \in G$ there exists $n(x) \in \mathbb{N}$ such that $x^{n(x)} = e$. Then G is a T_1-space.*

Proof. Any finite T_0-space contains a closed one-point subset. Thus any finite left topological group is a T_1-space. By conditions, any point $a \in G$ is contained in a finite subgroup $G(a) = \{a^i : 0 \leq i \leq n(a)\}$. Thus $\{e\}$ is a closed subset of the group G and G is a T_1-space. □

Remark 3. The similar assertions are true for classes of right topological groups.

13 Free semitopological groups

A class $\mathcal{V}$ of semitopological groups is called *a CI-complete quasi-variety* of semitopological groups if:

(SF1) the class $\mathcal{V}$ is multiplicative;

(SF2) if $G \in \mathcal{V}$ and A is a subgroup of G, then $A \in \mathcal{V}$;

(SF3) every space $G \in \mathcal{V}$ is a T_0-space;

(SF4) if $G \in \mathcal{V}$, $\mathcal{T}$ is a compact T_0-topology on G and $(G, \mathcal{T})$ is a quasitopological group, then $(G, \mathcal{T}) \in \mathcal{V}$;

(SF5) if $G \in \mathcal{V}$, $\mathcal{T}$ is a T_0-topology on G and $(G, \mathcal{T})$ is a paratopological group with an invariant base, then $(G, \mathcal{T}) \in \mathcal{V}$.

From Theorem 2 it follows

Corollary 10. *Let $\mathcal{W}$ be a non-trivial CI-complete quasi-variety of semitopological groups and $\mathbb{Z} \in \mathcal{W}$. Then for any pointed space X the free topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping e_X is an embedding.*

From Theorem 3 it follows

Corollary 11. *Let $\mathcal{W}$ be a non-trivial CI-complete quasi-variety of semitopological groups, $n \in \mathbb{N}$ and $x^n = e$ for any $x \in G$ and $G \in \mathcal{W}$. Then for any pointed T_1-space X the free topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping e_X is an embedding.*

Lemma 6 completes Corollary 11.

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