BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Numbers 2(72)–3(73), 2013, Pages 53–60
ISSN 1024–7696

Functional compactifications of T_0-spaces and bitopological structures

Mitrofan M. Choban, Laurenţiu I. Calmuţchi

Abstract. We study the compactifcation of T_0-spaces generated by families of special continuous mappings into a given standard space E. In this context we have introduced the notions of E-thin and E-rough g-compactifications. The maximal E-thin and E-rough g-compactifications are constructed.

Mathematics subject classification: 54D35, 54B10, 54C20, 54D40.
Keywords and phrases: Compactifcation, ring of sets, bitopological space.

1 Introduction

In functional analysis and related areas of mathematics different dual pairs of topologies are used. A *bitopological structure* on a set X is called a pair of topologies $\{\mathcal{T}, \mathcal{T}^d\}$ on X. In this case $(X, \mathcal{T}, \mathcal{T}^d)$ is a bitopological space. The general concept of a bitopological structure was introduced by J. C. Kelly [7] and applied in distinct domains by many authors (see [8, 9]).

If $(X, \mathcal{T}, \mathcal{T}^d)$ is a bitopological space, then we put $\mathcal{T}' = max\{\mathcal{T}, \mathcal{T}^d\}$ and say that $\mathcal{T}$ is the initial topology, $\mathcal{T}^d$ is the dual topology and $\mathcal{T}'$ is the final topology. In many constructions the final topology is a Hausdorff topology.

Example 1. Let X be a set and $\mathcal{Q} = \{\rho_\alpha : \alpha \in A\}$ be a family of functions on $X \times X$ with the next properties:
– $sup\{\rho_\alpha(x, y) + \rho_\alpha(y, x) : \alpha \in A\} = 0$ if and only if $x = y$;
– $\rho_\alpha(x, y) + \rho_\alpha(y, z) \geq \rho_\alpha(x, z)$ for all $x, y, z \in X$ and $\alpha \in A$.

Then we say that $\mathcal{Q}$ is a family of pseudo-quasimetrics on X. We put $V(x, \rho_\alpha, r) = \{y \in X : \rho_\alpha(x, y) < r\}$ for all $x \in X$, $\alpha \in A$ and $r > 0$. The intersections of finite elements of the family $\{V(x, \rho_\alpha, r) : x \in X, \alpha \in A, r > 0\}$ form a base of the topology $\mathcal{T}(\mathcal{Q})$ on X. The functions $\mathcal{Q}^d = \{\rho^d_\alpha(x, y) = \rho_\alpha(y, x) : \alpha \in A\}$ form the dual family of pseudo-quasimetrics on X and the dual topology $\mathcal{T}(\mathcal{Q}^d)$. The functions $\mathcal{Q}^s = \{\rho^s_\alpha(x, y) = 2^{-1}(\rho_\alpha(x, y) + \rho_\alpha(y, x)) : \alpha \in A\}$ form the final family of pseudo-metrics on X and the final topology $\mathcal{T}(\mathcal{Q}^s) = sup\{\mathcal{T}(\mathcal{Q}), \mathcal{T}(\mathcal{Q}^d)\} = \mathcal{T}(\mathcal{Q} \cup \mathcal{Q}^d)$. Then $(X, \mathcal{T}(\mathcal{Q}), \mathcal{T}(\mathcal{Q}^d))$ is a bitopological space with the initial topology $\mathcal{T}(\mathcal{Q})$, the dual topology $\mathcal{T}(\mathcal{Q}^d)$ and the completely regular final topology $\mathcal{T}(\mathcal{Q}^s)$.

The first examples of bitopological spaces were constructed in this way [7–9]. In many cases the family $\mathcal{Q}$ is a singleton set, i. e. is a quasimetric on X.

For our aim the initial and the final topologies on X are important. From this point of view, in the present article we suppose that any bitopological structure $\{\mathcal{T}, \mathcal{T}'\}$ on X has the following properties:

– $\mathcal{T} \subseteq \mathcal{T}'$;

– any compact subspace of the space $(X, \mathcal{T}')$ is Hausdorff and closed;

– the space $(X, \mathcal{T})$ is a T_0-space.

In this case we say that $\mathcal{T}$ is the initial or weak topology on X and $\mathcal{T}'$ is the final or strong topology on X. For any subset A of X consider two closures: the initial closure $clA = cl_{(X,\mathcal{T})}A$ and the strong closure $s - clA = cl_{(X,\mathcal{T}')}A$.

We use the terminology from [5, 6].

Definition 1. *A g-compactification of a space X is a pair (Y, f), where Y is a compact T_0-space, $f : X \to Y$ is a continuous mapping, the set $f(X)$ is dense in Y. If the set $\{y\}$ is closed in Y for any point $y \in Y \setminus f(X)$, then (Y, f) is called a g-compactification of a space X with a T_1-remainder. If f is an embedding, then we say that Y is a compactification of X and consider that $X \subseteq Y$, where $f(x) = x$ for any $x \in X$.*

Let (Y, f) and (Z, g) be two g-compactifications of the space X. We consider that $(Y, f) \leq (Z, g)$ if there exists a continuous mapping $\varphi : Z \longrightarrow Y$ such that $f = \varphi \circ g$, i.e. $f(x) = \varphi(g(x))$ for each $x \in X$. If $(Y, f) \leq (Z, g)$ and $(X, f) \leq (Y, g)$, then we say that g-compactifications (Y, f) and (Z, g) are equivalent. If φ is a homeomorphism of Z onto Y, then we say that the g-compactifications (Y, f) and (Z, g) are identical. We identify the identical g-compactifications.

The class of all compactifications of a given non-empty space is not a set (see [3]).

A family $\mathcal{L}$ of subsets of a space X is called a *WS-ring* if $\mathcal{L}$ is a family of closed subsets of X, $X \in \mathcal{L}$, $\emptyset \in \mathcal{L}$ and $F \cap H, F \cup H \in \mathcal{L}$ for any $F, H \in \mathcal{L}$.

For any family $\mathcal{L}$ of closed subsets of a space X denote by $r\mathcal{L}$ the minimal WS-ring of sets containing $\mathcal{L}$.

Fix a family $\mathcal{L}$ of closed subsets of X. Let $\mathcal{L}' = \{X\} \cup \mathcal{L}$. Denote by $M(\mathcal{L}, X)$ the family of all $\mathcal{L}'$-ultrafilters $\xi \in \mathcal{L}$. We put $\xi_{\mathcal{L}}(x) = \{H \in \mathcal{L}' : x \in H\}$. Let $\omega_{\mathcal{L}}X = M(\mathcal{L}, X) \cup \{\xi_{\mathcal{L}} : x \in X\}$.

Consider the mapping $\omega_{\mathcal{L}} : X \to \omega_{\mathcal{L}}X$, where $\omega_{\mathcal{L}}(x) = \xi_{\mathcal{L}}(x)$ for any $x \in X$. On $\omega_{\mathcal{L}}X$ consider the topology generated by the closed semibase $\omega\mathcal{L} = \{< H >= \{\xi \in \omega_{\mathcal{L}}X : H \in \xi\} : H \in \mathcal{L}'\}$. If $\mathcal{L}$ is a WS-ring, then $\omega\mathcal{L}$ is a closed base.

The pair $(\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ is a g-compactification of the space X with a T_1-remainder.

If $\mathcal{L}$ is a closed base of the space X, then $(\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ is a compactification of the space X with a T_1-remainder.

By virtue of the following theorem, it is sufficient to consider the g-compactifications $\omega_{\mathcal{L}}X$ for WS-rings $\mathcal{L}$.

Theorem 1 (see [3]). *$\omega_{r\mathcal{L}}X = \omega_{\mathcal{L}}X$.*

Definition 2. *A g-compactification (Y, f) of a space X is called a Wallman-Shanin g-compactification of the space X if $(X, f) = (\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ for some WS-ring $\mathcal{L}$.*

If $\mathcal{L}$ is the family of all closed subsets of a space X, then $\omega X = \omega_{\mathcal{L}} X$ is the *Wallman compactification* of the space X and $\omega_X : X \longrightarrow \omega X$ is the identical mapping (see [3, 5]).

The compactifications of the Wallman-Shanin type were introduced by N. A. Shanin [10] and studied by many authors (see [1–4, 11–13]). There exist Hausdorff compactifications of discrete spaces which are not Wallman–Shanin compactifications [13].

2 Functional compactifications

A space E with the topology $\mathcal{T}$ is called *a standard space* if it has the next properties:

– E is a commutative additive topological semigroup with the zero element $0 \in E$;

– there exist a point $1 \in E$ and an open subset U of E such that $0 \in U$ and $1 \notin U$;

– on E a topology $\mathcal{T}'$ is given such that the pair of topologies $\{\mathcal{T}, \mathcal{T}'\}$ is a bitopological structure on E.

In particular, $\mathcal{T} \subseteq \mathcal{T}'$ and any compact subspace of the space $(E, \mathcal{T}')$ is Hausdorff and closed.

Fix a standard space E. Let E be the set E with the initial topology $\mathcal{T}$ and E_s be the set E with the final topology $\mathcal{T}'$. Denote by $C_b(X, E)$ the space of all continuous mappings f of a space X into the space $(E, \mathcal{T})$ for which the set $s - clf(X)$ is a compact subset of the space $(E, \mathcal{T}')$. Since $\mathcal{T} \subseteq \mathcal{T}'$, we consider that $C_b(X, E_s) \subseteq C_b(X, E)$.

We say that a space X is *E-regular* if for each closed subset B of X and any point $x_0 \in X \setminus B$ there exists a mapping $g \in C_b(X, E)$ such that $f(x_0) \notin cl_E g(B)$. A space X is *E-completely regular* if for each closed subset B of X and any point $x_0 \in X \setminus B$ there exists a mapping $g \in C_b(X, E_s)$ such that $f(x_0) \notin cl_E g(B)$ (in this case $f(x_0) \notin cl_{E_s} g(B)$ too).

If the space X is E-completely regular, then the space X is a Tychonoff space. Really, $E_f = s - clf(X)$ is a Hausdorff compact subspace of the space E_s and X is a subspace of the Hausdorff compact space $\Pi\{E_f : f \in C_b(X, E_s)\}$.

Fix a non-empty space X.

Any non-empty set $\mathcal{F} \subseteq C_b(X, E)$ generates two mappings $l_{\mathcal{F}} : X \longrightarrow E_s^{\mathcal{F}}$ and $e_{(\mathcal{F},X)} : X \longrightarrow E^{\mathcal{F}}$, where $e_{\mathcal{F}}(x) = l_{\mathcal{F}}(x) = (f(x) : f \in \mathcal{F})$ for any point $x \in X$, and the identical mapping $i_{\mathcal{F}} : E_s^{\mathcal{F}} \longrightarrow E^{\mathcal{F}}$. Now we put $e_{\mathcal{F}} = e_{(\mathcal{F},X)}$.

Consider the family $B_{\mathcal{F}} = \{f^{-1}(H) : S \setminus H \in \mathcal{T}\}$ of closed subsets of X, the compact space $r_{\mathcal{F}} X$ which is the closure of the set $e_{\mathcal{F}}(X)$ in $E_s^{\mathcal{F}}$, the compact space $c_{\mathcal{F}} X$ which is the closure of the set $l_{\mathcal{F}}(X)$ in $E^{\mathcal{F}}$ and the compact space $s_{\mathcal{F}} X = i_{\mathcal{F}}(r_{\mathcal{F}} X)$. Let $(\omega_{\mathcal{F}} X, \omega_{\mathcal{F}}) = (\omega_{B_{\mathcal{F}}} X, \omega_{B_{\mathcal{F}}})$.

The pairs $(c_{\mathcal{F}} X, e_{\mathcal{F}})$ are called the *E-rough functional g-compactifications* of the space X. The pairs $(s_{\mathcal{F}} X, e_{\mathcal{F}})$ are called the *E-thin functional g-compactifications* of the space X.

By construction, $(c_{\mathcal{F}}X, e_{\mathcal{F}}) \leq (s_{\mathcal{F}}X, e_{\mathcal{F}})$ and $s_{\mathcal{F}}X$ is a dense subspace of the space $c_{\mathcal{F}}X$.

If $\mathcal{F} = C_b(X, E)$, then we put $(R_E X, e_E) = (c_{\mathcal{F}}X, e_{\mathcal{F}})$ and $(\beta_E X, e_E) = (s_{\mathcal{F}}X, e_{\mathcal{F}})$.

Theorem 2. *Let $\emptyset \neq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq C_b(X, E)$. Then $(c_{\mathcal{F}_1}X, e_{\mathcal{F}_1}) \leq c_{\mathcal{F}_2}X, e_{\mathcal{F}_2})$ and $(s_{\mathcal{F}_1}X, e_{\mathcal{F}_1}) \leq s_{\mathcal{F}_2}X, e_{\mathcal{F}_2})$.*

Proof. Let $p : E^{\mathcal{F}_2} \longrightarrow E^{\mathcal{F}_1}$ and $q : E_s^{\mathcal{F}_2} \longrightarrow E_s^{\mathcal{F}_1}$ be the natural projections. Then $p(c_{\mathcal{F}_2}X) \subseteq c_{\mathcal{F}_1}X$ and $q(r_{\mathcal{F}_2}X) = r_{\mathcal{F}_1}X$. These facts complete the proof. □

Theorem 3. *Let (Y, f) be a compactification of a space X and $\mathcal{F} \subseteq \{g \circ f : g \in C_b(Y, E)\}$. Then:*

1. $(c_{\mathcal{F}}X, e_{\mathcal{F}}) \leq (Y, f))$.

2. If $\mathcal{F} = \{g \circ f : g \in C_b(Y, E)\}$ and the space Y is E-completely regular, then $(s_{\mathcal{F}}X, e_{\mathcal{F}}) \geq (Y, f))$.

3. If $\mathcal{H} = \{g \circ f : g \in C_b(Y, E_s)\}$ and the space Y is E-completely regular, then $(s_{\mathcal{H}}X, e_{\mathcal{F}}) = (Y, f))$.

Proof. By virtue of Theorem 2, we can assume that $\mathcal{F} = \{g \circ f : g \in C_b(Y, E)\}$. Then there exists a continuous mapping $h : Y \longrightarrow E^{\mathcal{F}}$ such that $e_{\mathcal{F}} = h \circ f : X \longrightarrow E^{\mathcal{F}}$. Thus $h(Y) \subseteq c_{\mathcal{F}}X$. The assertion 1 is proved. □

If the space Y is E-completely regular, then we put $H = \{g \circ f : g \in C_b(Y, E_s)\}$. Obviously, $H \subseteq F$ and h is an embedding.

In this case $l_{\mathcal{H}}(X) = i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(X)) \subseteq i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(h(Y)))$ and $i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(h(Y))$ is a compact set. Then $Y = r_{\mathcal{H}}X = i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(h(Y)))$ and $(s_{\mathcal{F}}X, e_{\mathcal{F}}) \geq (s_{\mathcal{H}}X, e_{\mathcal{H}}) \geq (Y, f)$. The proof is complete.

Remark 1. The pair $(R_E X, e_E)$ is the unique maximal element of the set of g-compactifications $\{(c_{\mathcal{F}}X, e_{\mathcal{F}}) : \mathcal{F} \subseteq C_b(X, E)\}$.

Remark 2. The pair $(\beta_E X, e_E)$ is the unique maximal element of the set of g-compactifications $\{(s_{\mathcal{F}}X, e_{\mathcal{F}}) : F \subseteq C_b(X, E)\} \cup \{(c_{\mathcal{F}}X, e_{\mathcal{F}}) : \mathcal{F} \subseteq C_b(X, E)\}$.

We say that a space X is an *E-extensible* (respectively, *a strong E-extensible*) *space* if for each mapping $f \in C_b(X, E)$ there exists a (respectively, exists a unique) mapping $\omega f \in C_b(\omega X, E)$ such that $f = \omega f | X$.

Theorem 4. *Let $\emptyset \neq \mathcal{F} \subseteq C_b(X, E)$, $(\omega_{\mathcal{F}}X, \omega_{\mathcal{F}}) \leq (\omega X, \omega_X)$ and X is an E-extensible space. Then $(c_{\mathcal{F}}X, e_{\mathcal{F}}) \leq (\omega_{\mathcal{F}}X, \omega_{\mathcal{F}})$.*

Proof. By definition, $e_{\mathcal{F}}(x) = (f(x) : f \in \mathcal{F}) \in E^{\mathcal{F}}$ and $c_{\mathcal{F}}X$ is the closure of the set $e_{\mathcal{F}}(X)$ in $E^{\mathcal{F}}$. Fix $f \in \mathcal{F}$ and the continuous extension $\omega f : \omega X \to E$ of f.

We put $\Omega = \{\Pi\{U_f : f \in \mathcal{F}\} : U_f$ *is open in* E *and the set* $\{f : U_f \neq E\}$ *is finite*$\}$. By construction, Ω is the standard open base of the space $E^{\mathcal{F}}$. Moreover, $U \cap V \in \Omega$ for all $U, V \in \Omega$. If $\mathcal{L} = \{X \setminus \omega_{\mathcal{F}}^{-1}(H) : H \in \Omega\}$, then $\omega_{\mathcal{F}}X = \omega_{\mathcal{L}}X$.

Consider the continuous mapping $\psi : \omega X \to E^{\mathcal{F}}$, where $\psi(z) = (\omega f(z) : f \in \mathcal{F})$ for each $z \in \omega X$. Obviously, $\psi | X = e_{\mathcal{F}}$.

There exists a continuous mapping $\varphi : \omega X \longrightarrow \omega_{\mathcal{F}} X$ such that $\varphi(x) = \omega_{\mathcal{F}}(x)$ for each $x \in X$. In this case, for $< H > = \{\xi \in \omega_{\mathcal{L}} X : H \in \xi\}$ we have $\varphi^{-1}(< H >) = cl_{\omega X}(H)$ for each $H \in \mathcal{L}$. If $z \in= \omega_{\mathcal{F}} X \setminus = \omega_{\mathcal{L}}(X)$, then $\psi(\varphi^{-1}(z))$ is a singleton set and we put $h(z) = \psi(\varphi^{-1}(z))$. The mapping $h : \omega_{\mathcal{L}}(X) \longrightarrow (c_{\mathcal{F}} X$ is continuous and $h(\omega_{\mathcal{L}}(x) = e_{\mathcal{F}}(x)$ for all $x \in X$. The proof is complete. □

Remark 3. $(R_E X, e_E) \leq \omega X$ for any E-extensible space X and each standard space E.

Remark 4. Let Y be a non-empty subspace of a space X, $\mathcal{H} \subseteq C_b(X, E)$ and $\mathcal{F} = \{g|Y : g \in \mathcal{H}\}$. Then:

1. $\mathcal{F} \subseteq C_b(Y, E)$.
2. $(s_{\mathcal{F}} Y \subseteq cl_{s_{\mathcal{H}} X} e_{\mathcal{H}}(Y) \subseteq c_{\mathcal{F}} Y = cl_{c_{\mathcal{H}} X} e_{\mathcal{H}}(Y) \subseteq c_{\mathcal{H}} X$.
3. $e_{(\mathcal{F}, Y)} = e_{(\mathcal{F}, X)}|X$.

Theorem 5. *Let $f : X \longrightarrow Y$ be a continuous mapping of a space X into a space Y. Then there exist a continuous mapping $\omega f : R_E X \longrightarrow R_E Y$ and a unique continuous mapping $\beta f : \beta_E X \longrightarrow \beta_E Y$ such that $\beta f = \omega f|\beta_E X$ and $\beta f \circ e_{(E,X)} = e_{(E,Y)} \circ f$.*

Proof. By virtue of Remark 4, we can assume that $Y = f(X)$. In this case:

1. For any E-thin compactification (Z, φ) of the space Y the pair $(Z, \varphi \circ f)$ is a E-thin compactification of the space X. Thus we can consider that any E-thin compactification (Z, φ) of the space Y is a E-thin compactification of the space X. Then $(\beta_E Y, e_{(E,Y)}) \leq (\beta_E X, e_{(E,X)})$.

2. For any E-rough compactification (Z, φ) of the space Y the pair $(Z, \varphi \circ f)$ is a E-rough compactification of the space X. Thus we can consider that any E-rough compactification (Z, φ) of the space Y is a E-rough compactification of the space X. Then $(R_E Y, e_{(E,Y)}) \leq (R_E X, e_{(E,X)})$.

The proof is complete. □

Example 2. Let E_1 be an infinite countable set $0 \notin E_1$ and $E = E_1 \cup \{0\}$. Consider that $0 + x = x + 0 = 0$ for each $x \in E$ and $x + y = x$ for all $x, y \in E_1$. On E consider the topology $\mathcal{T} = \{E, \emptyset\} \cup \{E \setminus F : F \textit{ is a finite set}\}$ and the topology $\mathcal{T}' = \mathcal{T} \cup \{H \subseteq E_1\}$. Then $(E, \mathcal{T}')$ is the Alexandroff one-point compactification of the discrete space E_1. Let $X = \{r_1, r_2, ...\}$ be the space of all rational numbers in the usual topology. The space X is metrizable and $\omega X = \beta X$ is the Stone-Čech compactification of the space X. Fix a countable subset $A = \{a_1, a_2, ...\}$ of E_1 and we suppose that $a_n \neq a_m$ for $n \neq m$. Then the mapping $g : X \longrightarrow E$, where $g(r_n) = a_n$, is continuous. Since the space E is countable, the mapping g is not continuous extendable on ωX. Thus the space X is not E-extensible. If $\mathcal{F} = \{g\}$, then $(c_{\mathcal{F}} X, e_{\mathcal{F}}) = (E, g)$ and $s_{\mathcal{F}} X = \{0\} \cup A$. In particular, $(R_E X, e_E) \not\leq \omega X$.

Example 3. Let E_1 be an infinite set $0 \notin E_1$ and $E = E_1 \cup \{0\}$. Consider that $0 + x = x + 0 = 0$ for each $x \in E$ and $x + y = x$ for all $x, y \in E_1$. On E consider the topology $\mathcal{T} = \{E, \emptyset\} \cup \{E \setminus F : F \textit{ is a finite set}\}$ and the topology $\mathcal{T}' = \mathcal{T} \cup \{H \subseteq E_1\}$. Then $(E, \mathcal{T}')$ is the Alexandroff one-point compactification of the discrete space E_1. Assume that the cardinality $|E| \geq exp(exp(\aleph_0))$. Let $X = \{r_1, r_2, ...\}$ be the space of all rational numbers in the usual topology. Obviously $|\omega X| \leq |E|$. Thus the space X is E-extensible and not strong E-extensible. For each mapping $f \in C_b(X,E)$ we fix a mapping $\omega f : \omega X \longrightarrow E$ such that $\omega f(x) = f(x)$ for $x \in X$ and $\omega f(y) \neq \omega f(z)$ for distinct points $y, z \in \omega X \setminus X$. Then ωf is a continuous extension of f. There exist many extensions of this kind. Hence $(R_E X, e_E) \leq \omega X$. Since the space X is countable, $(\beta_E X, e_E) \not\leq \omega X$.

3 Examples

For any space X with a topology $\mathcal{T}$ denote by X_h the set X with the topology generated by the open semibase $\mathcal{T} \cup \{X\setminus : U \subseteq X, U \textit{ is an open compact subset}\}$.

A space X is called a spectral space if the space X_h is compact and on X there exists an open base $\mathcal{B}$ of open compact subsets and $U \cap V \in \mathcal{B}$ for all $U, V \in \mathcal{B}$ [3].

Definition 3. *A g-compactification (Y, f) of a space X is called a spectral g-compactification of the space X if Y is a spectral space and the set $f(X)$ is dense in the space Y_h.*

Example 4. Denote by $\mathbb{F}$ the set $\{0, 1\}$ by the initial topology $\mathcal{T} = \{\emptyset, \{0\}, \mathbb{F}\}$ and by the final discrete topology $\mathcal{T}' = \{\emptyset, \{0\}, \{1\}, \mathbb{F}\}$. On F consider the additive operation $0 + 0 = 0$ and $0 + 1 = 1 + 0 = 1 + 1 = 1$. Then $(\mathbb{F}, \mathcal{T}, \mathcal{T}')$ is a standard space. Any T_0-space is $\mathbb{F}$-regular and $\mathbb{F}$-extensible. A space X is a $\mathbb{F}$-completely regular space if and only if $indX = 0$, i. e. X has a family of open-and-closed sets which form an open base. In this case any zero-dimensional g-compactification (Y, f) of a T_0-space X is a $\mathbb{F}$-thin g-compactification. A g-compactification (Y, f) of a space X is a $\mathbb{F}$-thin g-compactification if and only if the g-compactification (Y, f) is a spectral g-compactification. If the space X is not discrete, then the maximal $\mathbb{F}$-thin compactification $\beta_{\mathbb{F}} X$ is not completely regular. If $\mathcal{H} \subseteq C_b(X, \mathbb{F})$, $x_0 \in X$, $g_0 \in \mathcal{H}$, $g_0(X) = \{0\}$, $f(x_0) = 0$ for any $f\mathcal{H}$ and $e_{\mathcal{F}} : X \longrightarrow F^{\mathcal{H}}$ is an embedding of X, then the $\mathbb{F}$-rough compactification $c_{\mathcal{H}} X$ is not $\mathbb{F}$-thin. In this case $s_{\mathcal{H}} X \neq c_{\mathcal{H}} X = \mathbb{F}^{\mathcal{H}}$.

Example 5. Denote by $\mathbb{D}$ the set $\{0, 1\}$ by the initial and final discrete topologies $\mathcal{T} = \mathcal{T}' = \{\emptyset, \{0\}, \{1\}, F\}$. On F consider the additive operation $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$. Then $(\mathbb{D}, \mathcal{T}, \mathcal{T}')$ is a standard space. A space X is a D-regular space if and only if $indX = 0$, i. e. X has a family of open-and-closed sets which form an open base. A g-compactification (Y, f) of a space X is a $\mathbb{D}$-thin g-compactification if and only if the g-compactification (Y, f) is zero-dimensional. Any $\mathbb{D}$-rough g-compactification is $\mathbb{D}$-thin.

Example 6. Denote by $\mathbb{R}$ the space of reals in the usual topology $\mathcal{T}'$ and by $\mathbb{R}_u$ the space of reals in the topology $\mathcal{T}$ generated by the open base $\{(-\infty, t) : t \in \mathbb{R}\}$. Then $(\mathbb{R}_u, \mathcal{T}, \mathcal{T}')$ is a standard space with the initial topology $\mathcal{T}$ and the final topology $\mathcal{T}'$. Any T_0-space is $\mathbb{R}_u$-regular space and $\mathbb{R}_u$-extensible. A space X is a completely regular space if and only if X is a $\mathbb{R}_u$-completely regular space. In this case any Hausdorff g-compactification (Y, f) of a T_0-space X is a $\mathbb{R}_u$-thin

g-compactification. Any $\mathbb{F}$-thin g-compactification is $\mathbb{R}_u$-thin. If (Y, f) is a Hausdorff g-compactification of a T_0-space X and $indY > 0$, then (Y, f) is a $\mathbb{R}_u$-thin and not spectral g-compactification of the space X.

Example 7. Denote by $\mathbb{R}$ the space of reals in the usual topology $\mathcal{T}' = \mathcal{T}$. Then $(\mathbb{R}, \mathcal{T}, \mathcal{T}')$ is a standard space with the initial topology $\mathcal{T}$ and the final topology $\mathcal{T}'$. A space X is a completely regular space if and only if X is a $\mathbb{R}$-regular space. In this case only the Hausdorff g-compactifications (Y, f) of a T_0-space X are $\mathbb{R}$-thin. Any $\mathbb{R}$-rough g-compactification is $\mathbb{R}$-thin.

From the above examples it follows that the notions of thinness and roughness depend on the standard space E and its initial and final topologies.

4 General case

In the present section we suppose that the bitopological structure $\{\mathcal{T}, \mathcal{T}'\}$ on a given standard space E has the following property: $(\mathbb{F}, \mathcal{T})$ is a subspace of the space $(E, \mathcal{T})$.

Theorem 6. *Any $\mathbb{F}$-thin g-compactification $(s_{\mathcal{H}}X, e_{\mathcal{H}})$ of a space X is an E-thin g-compactification of X.*

Proof. If $\mathcal{H} \subseteq C_b(X, \mathbb{F})$, then $\mathcal{H} \subseteq C_b(X, E)$. Obviously, $\mathbb{F}^{\mathcal{H}} \subseteq E^{\mathcal{H}}$, $\mathbb{D}^{\mathcal{H}} = \mathbb{F}^{\mathcal{H}}_s \subseteq E^{\mathcal{H}}_s$, $cl_{\mathbb{D}^{\mathcal{H}}}(l_{\mathcal{H}}(X)) = cl_{E^{\mathcal{H}}_s}(l_{\mathcal{H}}(X))$ and $cl_{\mathbb{F}^{\mathcal{H}}}(e_{\mathcal{H}}(X)) \subseteq cl_{E^{\mathcal{H}}}(e_{\mathcal{H}}(X))$. The proof is complete. $\square$

References

[1] Alo R. A., Shapiro H. L. *Wallman compact and realcompact spaces*. Contribs. Extens. Theory Topol. Struct., Proceed. Sympos. Berlin 1967, Berlin, 1969, 9–14.

[2] Arhangel'skii A. V., Choban M. M. *Remainders of rectifiable spaces*. Topology and Appl., 2010, **157**, 789–799.

[3] Choban M. M., Calmuţchi L. I. *Some problems of the theory of compactifications of topological spaces*. ROMAI Journal, 2006, **2:1**, 25–51.

[4] Choban M. M., Calmuţchi L. I. *On Wallman compactifications of T_0-spaces and related questions*. Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2011, No. 2(66), 102–111.

[5] Engelking R. *General Topology*. PWN, Warszava, 1977.

[6] Gillman L., Jerison M. *Rings of continuous functions*. Springer-Verlag, New York, 1976.

[7] Kelly J. C. *Bitopological spaces*. Proc. London Math. Soc., 1963, **13(3)**, 71—-89.

[8] Kopperman R. *Asymmetry and duality in topology*. Topology Appl., 1995, **66(1)**, 1–39.

[9] Salbany S. *Bitopological spaces, compactifications and completions*. Department of Mathematics, University of Cape Town, Cape Town, 1974.

[10] Shanin N. A. *On special extension of topological spaces*. Dokl. AN SSSR, 1943, **38**, 7–11 (in Rusian).

[11] Shapiro L. B. *Reduction of the basic problem on bicompact extension of the Wallman type*. Soviet Math. Dokl., 1974, **15**, 1020–1023 (Russian original: Dokl. AN SSSR, 1974, **217**, 38–41).

[12] Steiner E. F. *Wallman spaces and compactifications*. Fund. Math., 1968, **61**, 295–304.

[13] Uljanov V. M.. *Solution of a basic problem on compactification of Wallman type*. Soviet Math. Dokl., 1977, **18**, 567–571 (Russian original: Dokl. AN SSSR, 1977, **233**, 1056–1059).

Mitrofan M. Choban, Laurenţiu I. Calmuţchi
Department of Mathematics
Tiraspol State University
MD-2069, Chişinău
Moldova

E-mail: *mmchoban@gmail.com; lcamutchi@gmail.com*

Received March 9, 2013