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Liouville's theorem for vector-valued functions

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Abstract. It is shown in [2] that any X-valued analytic map on $\mathbb{C} \cup \{\infty\}$ is a constant map in case when X is a strongly galbed Hausdorff space. In [3] this result is generalized to the case when X is a topological linear Hausdorff space, the von Neumann bornology of which is strongly galbed. A new detailed proof for the last result is given in the present paper. Moreover, it is shown that for several topological linear spaces the von Neumann bornology is strongly galbed or pseudogalbed.

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In 1847 Joseph Liouville presented in his lecture the following result (which was published by A. L. Cauchy in 1844 but now is known as Liouville's theorem): *every bounded entire function* $f : \mathbb{C} \to \mathbb{C}$ *is a constant function.* In the theory of Banach algebras the following generalization of this result is used (see, for example, [4, Theorem 3.12]): *if* X *is a complex normed space and* f *a bounded weakly holomorphic* X*-valued map on* $\mathbb{C}$*, then* f *is a constant map.*

In 1947 (see [6, Theorem 1]) Richard Arens generalized this result to the case of a locally convex Hausdorff space X and later on to the case of a topological linear Hausdorff space X the topological dual of which has nonzero elements. It is well-known (see, for example, [10, p. 158]) that topological linear spaces which are not locally convex could not have any nonzero continuous functionals. In this case[1] instead of X-valued holomorphic functions the X-valued analytic functions are used.

In 1973 (see [12, Corollary, p. 56]) Philippe Turpin gave the following generalization of Liouville's theorem: *if* X *is an exponentially galbed Hausdorff space and* f *is an analytic* X*-valued map on*[2] $\mathbb{C}_\infty$*, then* f *is a constant map.* In 2004 (see [2, Theorem 2.1]) Mati Abel generalized this result to the case of strongly galbed Hausdorff space X. Moreover, in 2008 he presented in [3, Theorem 3.1] the following result:

Theorem 1. *Let* X *be a topological linear Hausdorff space over* $\mathbb{C}$*. If the von Neumann bornology* $\mathcal{B}_N$ *of* X *is strongly galbed, then every* X*-valued analytic map on* $\mathbb{C}_\infty$ *is a constant map.*

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[1] In 1966 Lucien Waelbroeck (see [14]) gave conditions for X-valued holomorphic map f on $\mathbb{C}$ to be constant in case of complete pseudoconvex space X, generalizing for it the integral theory for such maps. Unfortunately, his results have been presented mostly without complete proofs. He gave only hints for some parts how to prove.

[2] Here and later on $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

A new detailed proof for this result is given in the present paper. Moreover, it is shown that for several topological linear spaces the von Neumann bornology is strongly galbed.

1 Introduction

1. Let X be a topological linear space over $\mathbb{K}$, the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. By *F-seminorm* on X we mean a map $q : X \to \mathbb{R}^+$ which has the following properties:

(1) $q(\lambda x) \leqslant q(x)$ for each $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$;

(2) $\lim_{n\to\infty} q\big(\frac{1}{n}x\big) = 0$ for each $x \in X$;

(3) $q(x+y) \leqslant q(x) + q(y)$ for each $x, y \in X$.

If from $q(x) = 0$ it follows that $x = \theta_X$ (the zero element of X), then q is an *F-norm* on X. In this case d with $d(x,y) = q(x-y)$ for each $x, y \in X$ defines a metric on X such that $d(x+z, y+z) = d(x,y)$ for each $x, y, z \in X$.

It is well-known (see, for example, [9, p. 39, Theorem 3]) that the topology of any topological linear spaces coincides with the initial topology defined on by a collection of F-seminorms. A topological linear space (X, τ) topology τ of which has been defined by a F-norm $\| \ \|$ and X is complete with respect to $\| \ \|$ is an *F-space*. Moreover, if X is a *locally pseudoconvex space* (see, [11, p. 4], or [15, p. 4]), then X has a base $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of neighborhoods of zero consisting of balanced ($\mu U_\lambda \subset U_\lambda$ when $|\mu| \leqslant 1$) and pseudoconvex ($U_\lambda + U_\lambda \subset \mu U_\lambda$ for $\mu \geqslant 2$) sets. This base defines a set of numbers $\{k_\lambda : \lambda \in \Lambda\}$ in $(0,1]$ (see, for example, [10, pp. 161–162] or [15, pp. 3–6]) such that

$$U_\lambda + U_\lambda \subset 2^{\frac{1}{k_\lambda}} U_\lambda$$

and

$$\Gamma_{k_\lambda}(U_\lambda) \subset 2^{\frac{1}{k_\lambda}} U_\lambda$$

for each $\lambda \in \Lambda$, where

$$\Gamma_k(U) =$$

$$= \Big\{ \sum_{\nu=1}^{n} \mu_\nu u_\nu : n \in \mathbb{N}, u_1, \dots, u_n \in U \text{ and } \mu_1, \dots, \mu_n \in \mathbb{K} \text{ with } \sum_{\nu=1}^{n} | \mu_\nu |^k \leqslant 1 \Big\}$$

for any subset U of X and $k \in (0,1]$. The set $\Gamma_k(U)$ is the *absolutely k-convex hull* of U in X. A subset $U \subset X$ is *absolutely k-convex* if $U = \Gamma_k(U)$ and is *absolutely pseudoconvex* if $U = \Gamma_k(U)$ for some $k \in (0,1]$. In the case when

$$\inf\{k_\lambda : \lambda \in \Lambda\} = k > 0,$$

X is a *locally k-convex space* and when $k = 1$, then a *locally convex space*.

It is known (see [15, pp. 3–6] or [7, pp. 189 and 195]) that the topology on a locally pseudoconvex space X can be defined by a family $\mathcal{P} = \{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous seminorms (that is, $p_\lambda(\mu a) = \mid \mu \mid^{k_\lambda} p_\lambda(a)$ for each $\lambda \in \Lambda$, $\mu \in \mathbb{K}$ and $a \in E$), where the power of homogeneity $k_\lambda \in (0,1]$ for each $\lambda \in \Lambda$ and every seminorm p_λ is defined by

$$p_\lambda(a) = \inf\{|\,\mu\,|^{k_\lambda} \colon a \in \mu\Gamma_{k_\lambda}(U_\lambda)\}$$

for each $a \in A$.

Let now l be the set of all $\mathbb{K}$-valued sequences (x_n) for which $\sum_{k=0}^{\infty} |x_n| < \infty$, l^0 be the subset of l of sequences with only finite number of nonzero elements and let $l_0 = l \setminus l^0$.

A topological linear space X is a *galbed space* (see [2]) if there exists a sequence (α_n) in l_0 and for every neighbourhood O of zero in X there is another neighbourhood U of zero such that[3]

$$\bigcup_{n\in\mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \alpha_k u_k : u_0, \ldots, u_n \in U \Big\} \subset O.$$

In particular, when

$$\alpha_0 \neq 0 \ \text{ and } \ \alpha = \inf_{n>0} |\alpha_n|^{\frac{1}{n}} > 0, \tag{1}$$

a galbed space X is *strongly galbed* and X is *exponentially galbed* when $\alpha_n = \frac{1}{2^n}$ for each $n \in \mathbb{N}_0$. It is known (see [1, Proposition 2] or [3, Corollary 2.2]) that every locally pseudoconvex space is exponentially galbed (hence strongly galbed too).

2. A *bornology* on a set X is a collection $\mathcal{B}$ of subsets of X which satisfies the following conditions:

(a) $X = \bigcup\limits_{B\in\mathcal{B}} B$;

(b) if $B \in \mathcal{B}$ and $C \subseteq B$, then $C \in \mathcal{B}$;

(c) if $B_1, B_2 \in \mathcal{B}$, then $B_1 \cup B_2 \in \mathcal{B}$.

If X is a linear space over $\mathbb{K}$, a bornology $\mathcal{B}$ on X is called a *linear* or *vector bornology* if the following conditions are satisfied:

(d) if $B_1, B_2 \in \mathcal{B}$, then $B_1 + B_2 \in \mathcal{B}$;

(e) if $B \in \mathcal{B}$ and $\lambda \in \mathbb{K}$, then $\lambda B \in \mathcal{B}$;

(f) $\bigcup\limits_{|\lambda|\leqslant 1} \lambda B \in \mathcal{B}$ for every $B \in \mathcal{B}$.

A linear bornology $\mathcal{B}$ on a linear space X is *convex* if $\Gamma_1(U) \in \mathcal{B}$ for every $U \in \mathcal{B}$ and *pseudoconvex* if there exists a number $k \in (0,1]$ such that $\Gamma_k(U) \in \mathcal{B}$ for every

[3] Here and later on $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

$U \in \mathcal{B}$. Moreover, a bornology $\mathcal{B}$ on a linear space X over $\mathbb{K}$ is a *galbed bornology* (see [3]) if there is a sequence (α_n) in l_0 such that

$$S((\alpha_n), B) = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \alpha_k b_k : b_0, \ldots, b_n \in B \Big\} \in \mathcal{B} \tag{2}$$

for all $B \in \mathcal{B}$. In particular, when (α_n) satisfies the condition (1), $\mathcal{B}$ is a *strongly galbed bornology* on X, and when $\alpha_n = \frac{1}{2^n}$ for each $n \in \mathbb{N}$, $\mathcal{B}$ is *an exponentially galbed bornology* on X (see [5]). Moreover, we shall say that a bornology $\mathcal{B}$ is *pseudogalbed* if for every $B \in \mathcal{B}$ there exists a sequence $(\alpha_n) \in l_0$ such that $S((\alpha_n), B) \in \mathcal{B}$. In particular, when (α_n) satisfies the condition (1), we shall say that the bornology $\mathcal{B}$ is *strongly pseudogalbed.*

3. Let X be a topological linear space over $\mathbb{C}$. An X-valued map f on $\mathbb{C}_\infty$ is *analytic at* $\lambda_0 \in \mathbb{C}$ if there exists a number $\varepsilon > 0$ and a sequence (x_n) in X such that

$$f(\lambda_0 + \lambda) = \sum_{k=0}^{\infty} x_k \lambda^k$$

whenever $|\lambda| < \varepsilon$, and is *analytic at* ∞ if there exists a number $R > 0$ and a sequence (y_k) in X such that

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{y_k}{\lambda^k}$$

whenever $|\lambda| > R$.

If X is a topological linear space, then the set of all bounded sets forms a linear bornology which is called the *von Neumann bornology* on X or the *bornology on X defined by the topology* of X.

2 Topological linear spaces with strongly galbed and pseudogalbed von Neumann bornology

First we describe these topological linear spaces, the von Neumann bornology $\mathcal{B}_N$ of which is strongly galbed[4].

Proposition 1 (see [3])**.** *The von Neimann bornology of any strongly galbed space is strongly galbed.*

Proof. Let X be a strongly galbed space. Then there exists a sequence $(\alpha_n) \in l_0$ which satisfies the condition (1), and for every neighbourhood O of zero in X there is another neighbourhood U of zero such that $S((\alpha_n), U) \subseteq O$. Moreover, for any $B \in \mathcal{B}_N$ there is a number $\mu_B > 0$ such that $B \subseteq \mu_B U$. Since

$$S((\alpha_n), B) \subseteq S((\alpha_n), \mu_B U) \subseteq \mu_B O,$$

then $S((\alpha_n), B) \in \mathcal{B}_N$ for every $B \in \mathcal{B}_X$. Hence $\mathcal{B}_N$ is strongly galbed. $\square$

[4]Proposition 1 is proved in [3]. A modified proof for this result is given here.

Corollary 1. *The von Neumann bornology of every exponentially galbed apace is strongly galbed.*

Proposition 2. *The von Neumann bornology of any metrizable topological linear space is pseudogalbed.*

Proof. Let X be a metrizable topological linear space. Then X has a countable base $\mathcal{L}_X = \{O_n : n \in \mathbb{N}_0\}$ of balanced neighbourhoods of zero. We can assume that $O_{n+1} + 0_{n+1} \subseteq O_n$ for each $n \in \mathbb{N}_0$ (the addition in X is continuous). Let O be an arbitrary neighbourhood of zero in X. Then there is a number $n_0 \in \mathbb{N}_0$ such that $O_{n_0} \subseteq O$ and

$$\bigcup_{n \geqslant n_0} \sum_{k=n_0}^{n} O_{k+1} \subseteq O_{n_0},$$

because

$$O_{n_0+1} + \cdots + O_{n+1} \subseteq O_{n_0+1} + \cdots + O_n + O_n \subseteq O_{n_0+1} + \cdots O_{n-1} + O_{n-1} \subseteq$$

$$\subseteq \cdots \subseteq O_{n_0+1} + O_{n_0+1} \subseteq O_{n_0}$$

for each $n \geqslant n_0$.

Let $B \in \mathcal{B}_N$ be a balanced set. Then for each $k \in \mathbb{N}_0$ there exists a number $\mu_k = \mu_k(B) > 1$ such that $B \subseteq \mu_k O_{n_0+k+1}$. Here $\mu_k \leqslant \mu_{k+1}$ because $O_{n+1} \subseteq O_n$ for each $n \in \mathbb{N}_0$. Put

$$\alpha_n = \frac{1}{\max\{\mu_n, \mu_1^n\}}$$

for each $n \in \mathbb{N}_0$. Then $|\alpha_n| \leqslant \frac{1}{\mu_1^n}$ for each $n \in \mathbb{N}_0$. Hence $(\alpha_n) \in l_0$. Since

$$\sum_{k=0}^{n} \alpha_k b_k \in \sum_{k=0}^{n} \Big(\frac{\mu_k}{\max\{\mu_k, \mu_1^k\}} O_{n_0+k+1} \Big) \subseteq \sum_{k=n_0}^{n_0+n} O_{k+1}$$

$$\subset \bigcup_{n \geqslant n_0} \sum_{k=n_0}^{n} O_{k+1} \subseteq O_{n_0} \subseteq O$$

for each $n \geqslant 0$ and each choice of elements $b_0, b_1, \ldots, b_n \in B$, then

$$S((\alpha_n), B) = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \alpha_k b_k : b_0, \ldots, b_n \in B \Big\} \subseteq O.$$

Hence, $S((\alpha_n), B) \in \mathcal{B}_N$, because of which $\mathcal{B}_N$ is pseudogalbed. □

Corollary 2. *The von Neumann bornology of every F-space is pseudogalbed.*

3 Proof of Theorem 1

Now we give a new and detailed proof for Theorem 1.

Proof. Let X be a topological linear Hausdorff space and f an X-valued analytic map on $\mathbb{C}_\infty$. We can assume that X is complete, otherwise we consider X as a dense subset in $\tilde{X}$, the completion of X, and f as $\tilde{X}$-valued analytic map on $\mathbb{C}_\infty$.

Let first $\lambda_0 \in \mathbb{C}$. Then there is a number $r > 0$ and a sequence (x_n) in X such that

$$f(\lambda_0 + \lambda) = \sum_{k=0}^{\infty} x_k \lambda^k$$

whenever $|\lambda| < r$. By assumption, the von Neumann bornology $\mathcal{B}_N$ of X is strongly galbed. Therefore there exists a sequence $(\alpha_n) \in l_0$ with $\alpha < 1$ such that (2) holds for any $B \in \mathcal{B}_N$. Take $r_0 \in (0, \frac{r}{\alpha})$. Then the series

$$\sum_{k=0}^{\infty} x_k (\alpha r_0)^k$$

converges in X. Therefore the sequence $(x_n(\alpha r_0)^n)$ tends to zero in X. Hence, the set $\big\{x_n(\alpha r_0)^n : n \in \mathbb{N}_0\big\}$ is bounded in X. Let $U_{\lambda_0} = \{\lambda_0 + \lambda : |\lambda| < \alpha^2 r_0\}$ and

$$X_{\lambda_0} = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} x_k (\alpha r_0)^k t_k : (t_k) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k \Big\}.$$

Then X_{λ_0} is an absolutely convex and bounded set in A. Indeed, if $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leqslant 1$ and $x, y \in X_{\lambda_0}$, then there exists $n_1, n_2 \in \mathbb{N}_0$ and $t_0^x, \dots, t_{n_1}^x$ and $t_0^y, \dots, t_{n_2}^y$ such that $|t_k^x| \leqslant \alpha^k$ and $|t_k^y| \leqslant \alpha^k$ for each k,

$$x = \sum_{k=0}^{n_1} x_k (\alpha r_0)^k t_k^x$$

and

$$y = \sum_{k=0}^{n_2} x_k (\alpha r_0)^k t_k^y.$$

If $n_1 > n_2$, then we put

$$t_{n_2+1}^y = \dots = t_{n_1}^y = 0$$

(otherwise we act similarly), then

$$\lambda x + \mu y = \sum_{k=0}^{n_1} x_k (\alpha r_0)^k (\lambda t_k^x + \mu t_k^y) \in X_{\lambda_0},$$

because

$$|\lambda t_k^x + \mu t_k^y| \leqslant |\lambda||t_k^x| + |\mu||t_k^y| \leqslant \alpha^k(|\lambda| + |\mu|) \leqslant \alpha^k$$

for each k. Thus X_{λ_0} is an absolutely convex set.

To show that X_{λ_0} is bounded, let O be an arbitrary balanced neighbourdhood of zero in X. Because $(x_n(\alpha r_0)^n)$ is a bounded sequence in X, there is a number $\rho > 0$ such that $x_n(\alpha r_0)^n \in \rho O$ for each $n \in \mathbb{N}_0$. Therefore

$$x_n(\alpha r_0)^n \frac{t_n}{\alpha_n} = x_n(\alpha r_0)^n \frac{t_n}{\alpha^n} \frac{\alpha^n}{\alpha_n} \in \rho\Big(\frac{t_n}{\alpha^n}\frac{\alpha^n}{\alpha_n} O\Big) \subset \rho O$$

for all $n \in \mathbb{N}_0$ and all (t_n) with $\frac{|t_n|}{\alpha^n} \leqslant 1$ for every n, because $\frac{\alpha^n}{|\alpha_n|} \leqslant 1$ and O is balanced. Hence, the set

$$B = \Big\{x_n(\alpha r_0)^n \frac{t_n}{\alpha_n} : n \in \mathbb{N}_0, (t_n) \ \text{ is a sequence with } |t_n| \leqslant \alpha^n \text{ for each } n\Big\} \in \mathcal{B}_N.$$

Thus, $X_{\lambda_0} \subset S((\alpha_n), B) \in \mathcal{B}_N$, because the von Neumann bornology $\mathcal{B}_N$ is strongly galbed. Moreover, it is easy to see that (S_n), where

$$S_n = \sum_{k=0}^{n} x_k(\alpha r_0)^k t_k$$

for each $n \in \mathbb{N}_0$ and fixed sequence (t_n) with $|t_n| \leqslant \alpha^n$ for each n, is a Cauchy sequence in X. To show this, let O be an arbitrary neighbourhood of zero in X and $m \in \mathbb{N}$ a fixed number. Then there exists a balanced neighbourhood O_1 of zero in X such that

$$\underbrace{O_1 + \cdots + O_1}_{m \text{ summands}} \subset O$$

and a positive number ρ such that $x_n(\alpha r_0)^n \subset \rho O_1$ for all $n \in \mathbb{N}_0$ because the sequence $(x_n(\alpha r_0)^n)$ is bounded. Since $\alpha < 1$, then the sequence (α^n) vanishes. Hence, there is a number $n_0 \in \mathbb{N}_0$ such that $\alpha^n < \frac{1}{\rho}$ whenever $n > n_0$. Since

$$S_{n+m} - S_n = \sum_{k=n+1}^{n+m} x_k(\alpha r_0)^k t_k \in \rho O_1 t_{n+1} + \cdots + \rho O_1 t_{n+m} \subset \underbrace{O_1 + \cdots + O_1}_{m \text{ summands}} \subset O$$

whenever $n > n_0$ for every fixed $m \in \mathbb{N}_0$, then (S_n) is a Cauchy sequence in X. Hence, (S_n) converges in X. Therefore

$$\sum_{k=0}^{\infty} x_k(\alpha r_0)^k t_k \in X$$

for every fixed (t_n) such that $|t_n| \leqslant \alpha^n$ for each n. It is easy to show that the closure K_{λ_0} of the set X_{λ_0} in X is a closed, bounded and absolutely convex subset of X. Therefore (see, for example,[8, pp. 8–9]), the linear hull A_{λ_0} in X, generated by K_{λ_0}, is a normed space with respect to the norm[5] p_{λ_0} defined by

$$p_{\lambda_0}(a) = \inf\{\lambda > 0 : a \in \lambda K_{\lambda_0}\}$$

[5]Here p_{λ_0} is a norm on A_{λ_0} because K_{λ_0} is bounded.

for each $a \in A_{\lambda_0}$. Taking this into account, we have

$$f(\lambda_0 + \lambda) = \sum_{k=0}^{\infty} x_k (\alpha r_0)^k \Big(\frac{\lambda}{\alpha r_0}\Big)^k \in$$

$$\in \Big\{ \sum_{k=0}^{\infty} x_k (\alpha r_0)^k t_k : (t_n) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k \Big\} \subset K_{\lambda_0} \subset A_{\lambda_0}$$

whenever $|\lambda| < \alpha^2 r_0$. Consequently, for any point $\lambda \in \mathbb{C}$ there is an open neighbourhood U_λ of λ and a normed subspace A_λ of X such that the restriction $f|_{U_\lambda}$ of f to U_λ has values in A_λ.

Since f is also analytic at ∞, then there is a sequence (z_n) in X and a number $R > 0$ such that

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{z_k}{\lambda^k}$$

whenever $|\lambda| > R$. Let $R_0 \in (\alpha R, \infty)$. Then the series

$$\sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k}$$

converges in X. Therefore the sequence $\big(\frac{z_n \alpha^n}{R_0^n}\big)$ is bounded in X.

Let $U_\infty = \{\lambda : |\lambda| > \frac{R_0}{\alpha^2}\}$ and

$$X_\infty = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \frac{z_k \alpha^k}{R_0^k} t_k : (t_k) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k \Big\}.$$

Then X_∞ is an absolutely convex and bounded set in X. Indeed, if $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leqslant 1$ and $x, y \in X_\infty$, then there exist $n_1, n_2 \in \mathbb{N}_0$ and $t_0^x, \dots, t_{n_1}^x$ and $t_0^y, \dots, t_{n_2}^y$ such that $|t_k^x| \leqslant \alpha^k$ and $|t_k^y| \leqslant \alpha^k$ for each k,

$$x = \sum_{k=0}^{n_1} \frac{z_k \alpha^k}{R_0^k} t_k^x$$

and

$$y = \sum_{k=0}^{n_2} \frac{z_k \alpha^k}{R_0^k} t_k^y.$$

If $n_1 > n_2$, we put again

$$t_{n_2+1}^y = \dots = t_{n_1}^y = 0$$

(otherwise we act similarly). Therefore

$$\lambda x + \mu y = \sum_{k=0}^{n_1} \frac{z_k \alpha^k}{R_0^k} (\lambda t_k^x + \mu t_k^y) \in X_\infty,$$

because

$$|\lambda t_k^x + \mu t_k^y| \leqslant |\lambda||t_k^x| + |\mu||t_k^y| \leqslant \alpha^k(|\lambda| + |\mu|) \leqslant \alpha^k$$

for each k.

Let O be again an arbitrary balanced neighbourhood of zero in X. Because $\left(z_n \frac{\alpha^n}{R_0^n}\right)$ is a bounded sequence in X, there is a number $\pi > 0$ such that $z_n \frac{\alpha^n}{R_0^n} \in \pi O$ for each $n \in \mathbb{N}_0$. Therefore

$$\frac{z_n \alpha^n}{R_0^n} \frac{t_n}{\alpha_n} = \frac{z_n \alpha^n}{R_0^n} \frac{t_n}{\alpha^n} \frac{\alpha^n}{\alpha_n} \in \pi\Big(\frac{t_n}{\alpha^n} \frac{\alpha^n}{\alpha_n} O\Big) \subset \pi O$$

for all $n \in \mathbb{N}_0$ and all (t_n) with $\frac{|t_n|}{\alpha^n} \leqslant 1$ for each n, because $\frac{\alpha^n}{|\alpha_n|} \leqslant 1$ and O is balanced. Hence, the set

$$B' = \Big\{ \frac{z_n \alpha^n}{R_0^n} \frac{t_n}{\alpha_n} : n \in \mathbb{N}_0, (t_n) \text{ is a sequence with } |t_n| \leqslant \alpha^n \text{ for each } n \Big\} \in \mathcal{B}_N.$$

Hence, $X_\infty \subset S((\alpha_n), B') \in \mathcal{B}_N$ because the von Neumann bornology $\mathcal{B}_N$ is strongly galbed. Thus, the closure K_∞ of the set X_∞ in X is a closed, bounded and absolutely convex subset of X. Therefore (similarly as above) the linear hull A_∞ in X, gnerated by K_∞, is a normed space with respect to the norm p_∞, defined by

$$p_\infty(a) = \inf\{\lambda > 0 : a \in \lambda K_\infty\}$$

for each $a \in A_\infty$. The same way as in the first part of the proof,

$$\sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k} t_k \in X$$

for every fixed (t_n) such that $|t_n| \leqslant \alpha^n$ for each n. Since

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k} \Big(\frac{R_0}{\alpha\lambda}\Big)^k \in$$

$$\in \Big\{ \sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k} t_k : (t_k) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k \Big\} \subset K_\infty \subset A_\infty$$

whenever $|\lambda| > \frac{R_0}{\alpha^2}$, there is an open neighbourhood U_∞ of ∞ and a normed subspace A_∞ of X such that the restriction $f|_{U_\infty}$ of f to U_∞ has values in A_∞.

Now $\{U_\lambda : \lambda \in \mathbb{C}\}$ and U_∞ form an open cover of $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ is compact, there are numbers $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that

$$\mathbb{C}_\infty = U_\infty \cup \Big(\bigcup_{k=1}^{n} U_{\lambda_k}\Big).$$

Therefore

$$f(\mathbb{C}_\infty) = f(U_\infty) \bigcup \Big(\bigcup_{k=1}^{n} f(U_{\lambda_k})\Big) \subset A_1 = A_\infty \bigcup \Big(\bigcup_{k=1}^{n} A_{\lambda_k}\Big) \subset A_0,$$

where A_0 is the linear hull of A_1. Without loss of generality we can assume that every element

$$x = \lambda_1 x_1 + \cdots + \lambda_m x_m \in A_0$$

has been presented in the form

$$x = a_1 + \cdots + a_n + a_{n+1},$$

where $a_k \in A_{\lambda_k}$ for each $k \in \{1, 2, \ldots, n\}$ and $a_{n+1} \in A_\infty$, denoting by a_1 the zero element if none of elements $\lambda_1 x_1, \ldots, \lambda_m x_m$ does not belong to A_{λ_1} or the sum of all elements from $\lambda_1 x_1, \ldots, \lambda_m x_m$ which belong to A_{λ_1}; by a_2 the zero element if none of remainder elements from $\lambda_1 x_1, \ldots, \lambda_m x_m$ does not belong to A_{λ_2} or the sum of all remainder elements from $\lambda_1 x_1, \ldots, \lambda_m x_m$ which belong to A_{λ_2} and so on.

Now, for every $x \in A_0$ let

$$N(x) = \{\lambda \in \{\lambda_1, \ldots, \lambda_n, \infty\} : x \in A_\lambda\}$$

and let p be the map on A_0, defined by

$$p(x) = \sum_{k=1}^{n+1} \max_{\lambda \in N(a_k)} p_\lambda(a_k)$$

for every $x = a_1 + \cdots + a_{n+1} \in A_0$. It is easy to check that p is a norm on A_0. Hence f maps $\mathbb{C}_\infty$ into the normed space A_0. Now, it is easy to show that $\varphi \circ f$ is a $\mathbb{C}$-valued analytic function on $\mathbb{C}_\infty$ for each continuous linear functional φ on A_0. Hence $\varphi \circ f$ is a constant function by the classical Liouville's Theorem. Since continuous linear functionals separate the points of any normed space, then f is a constant map. □

Now, by Theorem 1, Propositions 1 and Corollaries 1, we have the result of Ph. Turpin (see [12]).

Corollary 3. *If X is an exponentially galbed (in particular a locally pseudoconvex) space, then every X-valued analytic map on $\mathbb{C}_\infty$ is a constant map.*

4 Application

Using the classical Liouville's Theorem, it is easy to prove the Gelfand-Mazur Theorem, that is, *every complex normed division algebra is topologically isomorphic to* $\mathbb{C}$. This result has many generalizations to the case of locally convex and locally pseudoconvex division algebras. Next we give a characterization of complex topological division algebras.

Theorem 2. *A complex Hausdorff division algebra[6] A is topologically isomorphic to $\mathbb{C}$ if and only if*

[6]We assume here that the multiplication in topological algebras is separately continuous.

a) *every element of A is bounded*[7]*;*
b) *the von Neumann bornology of A is strongly galbed.*

Proof. Let A be topologically isomorphic to $\mathbb{C}$. Then every element of A has the form λe_A, where $\lambda \in \mathbb{C}$ and e_A is the unit element of A and every bounded set in A is in the form Ke_A, where K is a bounded set in $\mathbb{C}$. Therefore, every element of A is bounded. To show that the von Neumann bornology of A is strongly galbed, let $(\alpha_n) \in l_0$ be such that the condition (1) holds, and let $L = \sum_k |\alpha_k|$, $M > 0$ and $K_M = \{\lambda \in \mathbb{C} : |\lambda| < M\}$. Moreover, let B be an arbitrary bounded set in A. Then there is a number $M > 0$ such that $B = K_M e_A$. Since

$$\sum_{k=0}^{n} \alpha_k \mu_k e_A = \Big(\sum_{k=0}^{n} \alpha_k \mu_k\Big) e_A$$

for each n and $\mu_1, \ldots, \mu_n \in K_M$ and

$$\Big|\sum_{k=0}^{n} \alpha_k \mu_k\Big| \leqslant \sum_{k=0}^{n} |\alpha_k||\mu_k| \leqslant M \sum_{k=0}^{\infty} |\alpha_k| = ML,$$

then

$$\bigcup_{n\in\mathbb{N}_0} \Big\{\sum_{k=0}^{n} \alpha_k \mu_k e_A : \mu_1, \ldots, \mu_n \in K_M\Big\} \subseteq K_{ML} e_A.$$

Hence, the von Neumann bornology of A is strongly galbed.

Let now A be a complex Hausdorff division algebra. Then (see [3, proof of Proposition 5.1]) A is topologically isomorphic to $\mathbb{C}$ by Theorem 1. □

Now by Proposition 1, Corollary 1 and Theorem 2 we have

Corollary 4. *Every complex strongly galbed (in particular, exponentially galbed) division algebra is topologically isomorphic to $\mathbb{C}$ if and only if every element in A is bounded.*

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