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Infinitely many maximal primitive positive clones in a diagonalizable algebra

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Abstract. We present a rather simple example of infinitely many maximal primitive positive clones in a diagonalizable algebra, which serve as an algebraic model for the provability propositional logic GL.

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1 Introduction

The present paper deals with clones of operations of a diagonalizable algebra which are closed under definitions by existentially quantified systems of equations. Such clones are called *primitive positive clones* [1] (in [2] they are referred to as *clones acting bicentrally*, and are also called *parametrically closed classes* in [3, 4]). Diagonalizable algebras [5] are known to be algebraic models for the propositional provability logic GL [6].

The proof that there are finitely many primitive positive clones in any k-valued logic was given in [1]. In the case of 2-valued boolean functions, i. e. $card(A) = 2$, A. V. Kuznetsov stated there are 25 primitive positive clones [3], and A. F. Danil'čenco proved there are 2986 primitive positive clones among 3-valued functions [4]. In the present paper we construct a diagonalizable algebra, generated by its least element, which has infinitely many primitive positive clones, moreover, these primitive positive clones are maximal.

2 Definitions and notations

Diagonalizable algebras. A diagonalizable algebra [5] $\mathfrak{D}$ is a boolean algebra $\mathfrak{A} = (A; \&, \vee, \supset, \neg, \mathbb{0}, \mathbb{1})$ with an additional operator Δ satisfying the following relations:

$$\begin{aligned}\Delta(x \supset y) &\leq \Delta x \supset \Delta y,\\ \Delta x &\leq \Delta\Delta x,\\ \Delta(\Delta x \supset x) &= \Delta x,\\ \Delta\mathbb{1} &= \mathbb{1},\end{aligned}$$

where $\mathbb{1}$ is the unit of $\mathfrak{A}$.

We consider the diagonalizable algebra $\mathfrak{M} = (M; \&, \vee, \supset, \neg, \Delta)$ of all infinite binary sequences of the form $\alpha = (\mu_1, \mu_2, \dots)$, $\mu_i \in \{0,1\}$, $i = 1, 2, \dots$. The boolean operations $\&, \vee, \supset, \neg$ over elements of M are defined component-wise, and the operation Δ over element α is defined by the equality $\Delta\alpha = (1, \nu_1, \nu_2, \dots)$, where $\nu_i = \mu_1 \,\&\, \cdots \,\&\, \mu_i$. Let $\mathfrak{M}^*$ be the subalgebra of $\mathfrak{M}$ generated by its zero $\mathbb{0}$ element $(0, 0, \dots)$. Remark the unite $\mathbb{1}$ of the algebra $\mathfrak{M}^*$ is the element $(1, 1, \dots)$.

As usual, we denote by $x \sim y$ and $\Delta^2 x, \dots, \Delta^{n+1}x, \dots$ the corresponding functions $(\neg x \vee y) \,\&\, (\neg y \vee x)$ and $\Delta\Delta x, \dots, \Delta\Delta^n x, \dots$. Denote by $\Box x$ the function $x \,\&\, \Delta x$ and denote by ∇x the function $\Box\neg\Box\neg\Box x$.

Primitive positive clones. The term algebra $\mathcal{T}(\mathfrak{D})$ of $\mathfrak{D}$ is defined as usual, stating from constants $\mathbb{0}, \mathbb{1}$ and variables and using operations $\&, \vee, \supset, \neg, \Delta$. We consider the set $Term$ of all term operations of $\mathfrak{M}^*$, which obviously forms a clone [7].

Let us recall that a *primitive positive formula* Φ over a set of operations Σ of $\mathfrak{D}$ is of the form

$$\Phi(x_1, \dots, x_m) = (\exists x_{m+1}) \dots (\exists x_n)((f_1 = g_1) \,\&\, \cdots \,\&\, (f_s = g_s)),$$

where $f_1, g_1, \dots, f_s, g_s \in \mathcal{T}(\mathfrak{D}) \cup Id_A$ and the formula $(f_1 = g_1) \,\&\, \cdots \,\&\, (f_s = g_s)$ contains variables only from $x_1, \dots, x_n$. An *n-ary term operation f of $\mathcal{T}(\mathfrak{D})$ is (primitive positive) definable over* Σ if there is a primitive positive formula $\Phi(x_1, \dots, x_n, y)$ over Σ of $\mathcal{T}(\mathfrak{D})$ such that for any $a_1, \dots, a_n, b \in \mathfrak{D}$ we have $f(a_1, \dots, a_n) = b$ if and only if $\Phi(a_1, \dots, a_n, b)$ on $\mathfrak{D}$ [8]. Denote by $[\Sigma]$ all term operations of $\mathfrak{D}$ which are primitive positive definable over Σ of $\mathfrak{D}$. They say also $[\Sigma]$ is a primitive positive clone on $\mathfrak{D}$ generated by Σ. If $[\Sigma]$ contains $\mathcal{T}(\mathfrak{D})$ then it is referred to as a *complete primitive positive clone on* $\mathfrak{D}$. A primitive positive clone C of $\mathfrak{D}$ is *maximal in* $\mathfrak{D}$ if $\mathcal{T}(\mathfrak{D}) \not\subseteq C$ and for any $f \in \mathcal{T}(\mathfrak{D}) \setminus C$ we have $\mathcal{T}(\mathfrak{D}) \subseteq [C \cup \{f\}]$.

Let $\alpha \in \mathfrak{D}$. They say $f(x_1, \dots, x_n) \in \mathcal{T}(\mathfrak{D})$ conserves the relation $x = \alpha$ on $\mathfrak{D}$ if $f(\alpha, \dots, \alpha) = \alpha$. According to [9] the set of all functions that preserves the relation $x = \alpha$ on an arbitrary k-element set is a primitive positive clone.

3 Preliminary results

We start by presenting some useful properties of the term operations $\Delta, \Box$ and ∇ of $\mathfrak{M}^*$.

Proposition 1. *Let x, y be arbitrary elements of $\mathfrak{M}^*$. Then:*

$$\Box x \geq \Delta\mathbb{0} \text{ if and only if } \nabla x = \mathbb{1} \tag{1}$$

$$\Box x = \mathbb{0} \text{ if and only if } \nabla x = \mathbb{0} \tag{2}$$

$$\text{For any } x, y, \text{ either } \Box x \leq \Box y \text{ or } \Box y \leq \Box x \tag{3}$$

$$\Delta x = \Delta\Box x \tag{4}$$

$$\nabla\mathbb{0} = \mathbb{0},\ \nabla\mathbb{1} = \mathbb{1} \tag{5}$$

$$\Box x \geq \Delta\mathbb{0} \text{ if and only if } \Box\neg x = \mathbb{0} \tag{6}$$

$$\Box x = \mathbb{0} \text{ if and only if } \Box\neg x \geq \Delta\mathbb{0} \tag{7}$$

Proof. The proof is almost obvious by construction of the algebra $\mathfrak{M}^*$. $\square$

Let us mention the following

Remark 1. Any function f of $\mathcal{T}(\mathfrak{D})$ is primitive positive definable on $\mathfrak{D}$ via the system of functions $x \,\&\, y,\ x \vee y,\ x \supset y,\ \neg x, \Delta y$.

Let us consider on $\mathfrak{D}$ the following functions (8) and (9) of $\mathcal{T}(\mathfrak{D})$, denoted by $f_\neg(x,y)$ and $f_\Delta(x,y)$ correspondingly, where $\alpha_i, \xi \in \mathfrak{D}$, $\alpha_i = \neg\Delta^i\mathbb{0}$, where $\xi \neq \alpha_i$ and $\eta \neq \alpha_i$:

$$(\nabla\neg(x \sim y) \,\&\, ((\neg x \sim y) \sim \xi)) \vee (\nabla(x \sim y) \,\&\, \alpha_i), \tag{8}$$

$$(\nabla y \,\&\, ((\Delta x \sim y) \sim \eta)) \vee (\neg\nabla y \,\&\, \alpha_i). \tag{9}$$

Proposition 2. *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$. If $\neg\alpha = \beta$ on $\mathfrak{M}^*$, then*

$$f_\neg(\alpha,\beta) = \xi$$

on $\mathfrak{M}^*$.

Proof. Since $\neg\alpha = \beta$ we get $\alpha \sim \beta = \mathbb{0}, \neg(\alpha \sim \beta) = \mathbb{1}$ and by (5) we have

$$\nabla(\alpha \sim \beta) = \mathbb{0},\ \nabla\neg(\alpha \sim \beta) = \mathbb{1},$$

which implies

$$f_\neg(\alpha,\beta) = (\mathbb{1} \,\&\, (\mathbb{1} \sim \xi)) \vee (\mathbb{0} \,\&\, \alpha_i) = \xi.$$

$\square$

Proposition 3. *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$. If $\neg\alpha \neq \beta$ on $\mathfrak{M}^*$, then*

$$f_\neg(\alpha,\beta) \neq \xi$$

on $\mathfrak{M}^*$.

Proof. Since $\neg\alpha \neq \beta$ we get $\neg\alpha \sim \beta \neq \mathbb{1}, \alpha \sim \beta \neq \mathbb{0}$. We distinguish two cases: 1) $\Box(\alpha \sim \beta) = \mathbb{0}$, and 2) $\Box(\alpha \sim \beta) \geq \Delta\mathbb{0}$.

In the case 1) by (7), (1) and (2) we get $\Box\neg(\alpha \sim \beta) \geq \Delta\mathbb{0}$, $\nabla\neg(\alpha \sim \beta) = \mathbb{1}$, and $\nabla(\alpha \sim \beta) = \mathbb{0}$, which implies

$$\begin{aligned} f_\neg(\alpha,\beta) &= (\nabla\neg(\alpha \sim \beta) \,\&\, ((\neg\alpha \sim \beta) \sim \xi)) \vee (\nabla(\alpha \sim \beta) \,\&\, \alpha_i \\ &= (\mathbb{1} \,\&\, ((\neg\alpha \sim \beta) \sim \xi) \vee (\mathbb{0} \,\&\, \alpha_i) = (\neg\alpha \sim \beta) \sim \xi \neq \xi, \end{aligned}$$

Thus the first case has already been examined.

Now consider the second case, when $\Box x \geq \Delta\mathbb{0}$. Again, since $\neg\alpha \neq \beta$ by (1), (2) and (6) we obtain $\Box\neg(\alpha \sim \beta) = \mathbb{0}$, $\nabla\neg(\alpha \sim \beta) = \mathbb{0}$, $\nabla(\alpha \sim \beta) = \mathbb{1}$. Then,

$$\begin{aligned} f_\neg(\alpha,\beta) &= (\nabla\neg(\alpha \sim \beta) \,\&\, ((\neg\alpha \sim \beta) \,\&\, \xi)) \vee (\nabla(\alpha \sim \beta) \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, ((\neg\alpha \sim \beta) \,\&\, \xi)) \vee (\mathbb{1} \,\&\, \alpha_i) = \alpha_i \neq \xi. \end{aligned}$$

$\square$

Proposition 4. *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$ be such that $\Delta\alpha = \beta$. Then*

$$f_\Delta(\alpha, \beta) = \eta.$$

Proof. Since $\Delta\alpha \geq \mathbb{0}$ and $\Delta\alpha = \beta$ we have $\Box\beta \geq \Delta\mathbb{0}$, $\Delta\alpha \sim \beta = \mathbb{1}$ and by (1) we get $\nabla\beta = \mathbb{1}, \neg\nabla\beta = \mathbb{0}$. These ones imply the following relations:

$$\begin{aligned} f_\Delta(\alpha, \beta) &= (\nabla\beta \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\neg\nabla\beta \,\&\, \alpha_i) \\ &= (\mathbb{1} \,\&\, (\mathbb{1} \sim \eta)) \vee (\mathbb{0} \,\&\, \alpha_i) = \mathbb{1} \sim \eta = \eta. \end{aligned}$$

□

Proposition 5. *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$ be such that $\Delta\alpha \neq \beta$. Then*

$$f_\Delta(\alpha, \beta) \neq \eta.$$

Proof. We consider 2 cases: 1) $\Box\beta = \mathbb{0}$, and 2) $\Box\beta \geq \Delta\mathbb{0}$.

Suppose $\Box\beta = \mathbb{0}$. In view of (2) we have $\nabla\beta = \mathbb{0}$ and $\neg\nabla\beta = \mathbb{1}$. Subsequently,

$$\begin{aligned} f_\Delta(\alpha, \beta) &= (\nabla\beta \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\neg\nabla\beta \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\mathbb{1} \,\&\, \alpha_i) = 0 \vee \alpha_i = \alpha_i \neq \eta. \end{aligned}$$

Suppose now $\Box\beta \geq \Delta\mathbb{0}$. Let us note $\Delta\alpha \sim \beta \neq \mathbb{1}$. Then considering (1) we get

$$\begin{aligned} f_\Delta(\alpha, \beta) &= (\nabla\beta \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\neg\nabla\beta \,\&\, \alpha_i) \\ &= (\mathbb{1} \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\mathbb{0} \,\&\, \alpha_i) = (\Delta\alpha \sim \beta) \sim \eta \neq \eta. \end{aligned}$$

□

Proposition 6. *Let arbitrary $\alpha \in \mathfrak{M}^*$. Then*

$$f_\neg(\alpha, \alpha) = \alpha_i.$$

Proof. Let us calculate $f_\neg(\alpha, \alpha)$. By (5) we obtain immediately:

$$\begin{aligned} f_\neg(\alpha, \alpha) &= (\nabla\neg(\alpha \sim \alpha) \,\&\, ((\neg\alpha \sim \alpha) \,\&\, \xi)) \vee (\nabla(\alpha \sim \alpha) \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, (\mathbb{0} \,\&\, \xi)) \vee (\mathbb{1} \,\&\, \alpha_i) = \alpha_i. \end{aligned}$$

□

Proposition 7. *Let arbitrary $\alpha \in \mathfrak{M}^*$ and $\Box\alpha = \mathbb{0}$. Then*

$$f_\Delta(\alpha, \alpha) = \alpha_i.$$

Proof. Taking into account (2) we have

$$\begin{aligned} f_\Delta(\alpha, \alpha) &= (\nabla\alpha \,\&\, ((\Delta\alpha \sim \alpha) \sim \eta)) \vee (\neg\nabla\alpha \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, ((\Delta\alpha \sim \alpha) \sim \eta)) \vee (\mathbb{1} \,\&\, \alpha_i) = \mathbb{0} \vee \alpha_i = \alpha_i. \end{aligned}$$

□

4 Important properties of some primitive positive clones

Consider an arbitrary value i, $i = 1, 2, \dots$. Let K_i be the primitive positive clone of $\mathfrak{M}^*$ consisting of all functions of $\mathfrak{M}^*$ which preserve the relation $x = \neg\Delta^i\mathbb{0}$ on $\mathfrak{M}^*$. For example, K_1 is defined by the relation $x = (0, 1, 1, 1, \dots)$.

Remark 2. The functions $\Box x, x \,\&\, y,\ x \vee y,\ \neg\Delta^i\mathbb{0} \in K_i$, and $\neg x, \Delta x \notin K_i$.

Remark 3. Since K_i is a primitive positive clone it follows from the above statement the functions $\neg x$ and Δx are not primitive positive definable via functions of K_i on $\mathfrak{M}^*$, so $\mathcal{T}(\mathfrak{M}^*) \nsubseteq K_i$ and thus the clone K_i is not complete in $\mathfrak{M}^*$.

Remark 4. By Propositions 6 and 7 we have the earlier defined functions $f_\neg(x, y)$ and $f_\Delta(x, y)$ are in K_i.

Lemma 1. *Suppose an arbitrary $f(x_1, \dots, x_k) \in \mathcal{T}(\mathfrak{M}^*)$ and $f \notin K_i$. Then the functions Δx and $\neg x$ are primitive positive definable via functions of $K_i \cup \{f(x_1, \dots, x_k)\}$.*

Proof. Let us note since $f \notin K_i$ we have $f(\neg\Delta^i\mathbb{0}, \dots, \neg\Delta^i\mathbb{0}) \neq \neg\Delta^i\mathbb{0}$. Now consider the next term operations $f'_\neg$ and f'_Δ defined by terms (10) and (11):

$$(\nabla\neg(x \sim y) \,\&\, ((\neg x \sim y) \sim f(\neg\Delta^i\mathbb{0}, \dots, \neg\Delta^i\mathbb{0}))) \vee (\nabla(x \sim y) \,\&\, \neg\Delta^i\mathbb{0}) \tag{10}$$

$$(\nabla y \,\&\, ((\Delta x \sim y) \sim f(\neg\Delta^i\mathbb{0}, \dots, \neg\Delta^i\mathbb{0}))) \vee (\neg\nabla y \,\&\, \neg\Delta^i\mathbb{0}) \tag{11}$$

and examine the primitive positive formulas containing only functions from $K_i \cup \{f\}$:

$$(f'_\neg(x, y) = f(\neg\Delta^i\mathbb{0}, \dots, \neg\Delta^i\mathbb{0})) \text{ and } (f'_\Delta(x, y) = f(\neg\Delta^i\mathbb{0}, \dots, \neg\Delta^i\mathbb{0})).$$

Let us note by Propositions 2 and 3 we have $(\neg x = y)$ if and only if $(f'_\neg(x, y) = f(\neg\Delta^i\mathbb{0}, \dots, \neg\Delta^i\mathbb{0}))$ and according to Propositions 4 and 5 we get $(\Delta x = y)$ if and only if $(f'_\Delta(x, y) = f(\neg\Delta^i\mathbb{0}, \dots, \neg\Delta^i\mathbb{0}))$.

Lemma is proved. □

5 Main result

Theorem 1. *There are infinitely many maximal primitive positive clones in the diagonalizable algebra $\mathfrak{M}^*$.*

Proof. The theorem is based on the example of an infinite family of maximal primitive positive clones presented below.

Example 1. The classes K_1, $K_2, \dots$ of term operations of $\mathcal{T}(\mathfrak{M}^*)$, which preserve on algebra $\mathfrak{M}^*$ the corresponding relations $x = \neg\Delta\mathbb{0}$, $x = \neg\Delta^2\mathbb{0}, \dots$, constitute a numerable collection of maximal primitive positive clones in $\mathfrak{M}^*$.

Really, it is known [9] that these classes of functions represent primitive positive clones. According to Remark 3 each clone K_i is not complete in $\mathfrak{M}^*$. In virtue of Lemma 1 these primitive positive clones are maximal. It remains to show these clones are different. The last thing is obvious since

$$\neg\Delta^j\mathbb{0} \in K_j \text{ and } \neg\Delta^j\mathbb{0} \notin K_i, \text{ when } i \neq j.$$

The theorem is proved. □

6 Conclusions

We can consider the logic $L\mathfrak{M}^*$ of $\mathfrak{M}^*$, which happens to be an extension of the propositional provability logic GL, and consider primitive positive classes of formulas $M_1, M_2, \ldots$ of the propositional provability calculus of GL preserving on $\mathfrak{M}^*$ the corresponding relations $x = \neg\Delta\mathbb{0}, x = \neg\Delta^2\mathbb{0}, \ldots$.

Theorem 2. *The classes of formulas $M_1, M_2, \ldots$ constitute an infinite collection of primitive positive classes of formulas in the extension $L\mathfrak{M}^*$ of the propositional provability logic GL.*

Proof. The statement of the theorem is just another formulation of the Theorem 1 above in terms of formulas of the calculus of GL, which follows the usual terminology of [3]. □

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