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On the number of metrizable group topologies on countable groups

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Abstract. If a countable group G admits a non-discrete metrizable group topology τ_0, then in the group G, there are:
- Continuum of non-discrete metrizable group topologies stronger than τ_0, and any two of these topologies are incomparable;
- Continuum of non-discrete metrizable group topologies stronger than τ_0, and any two of these topologies are comparable.

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1 Introduction

Researches on the possibility of the definition of a Hausdorff, group topologies on countable groups were started in [1]. In this work also a method to define such group topologies on any countable group was given.

Later, in [2] it was proved that any infinite Abelian group admits a non-discrete Hausdorff group topology, and in [3] an example of a countable group which does not admit non-discrete Hausdorff group topologies was constructed.

This article is a continuation of the research in this direction. The main results of this article are Theorems 13 and 14.

2 Basic results

To highlight the main results we need the following well-known result (see [4], p. 203, Proposition 1, and p. 205, Corollary):

Theorem 1. *A set Ω of subsets of a group G is a basis of the filter of neighborhoods of the unity element of a Hausdorff group topology on G if and only if the following conditions are satisfied:*

1) $\bigcap\limits_{V\in\Omega} V = \{e\}$;

2) For any V_1 and $V_2 \in \Omega$, there exists $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \bigcap V_2$;

3) For any $V_1 \in \Omega$, there exists $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$;

4) For any $V_1 \in \Omega$, there exists $V_2 \in \Omega$ such that $V_2^{-1} \subseteq V_1$;

5) For any $V_1 \in \Omega$ and any element $g \in G$, there exists $V_2 \in \Omega$ such that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$.

Remark 2. From Theorem 1 it easily follows that if a countable group G admits a non-discrete group topology τ_0 such that the topological space (G, τ_0) is a Hausdorff space, then the group G admits a non-discrete group topology τ_1 such that the topological space (G, τ_1) is a Hausdorff space, and it has a countable basis of the filter of neighborhoods of the unity element.

Remark 3. From ([5], Theorem 8.1.21) it easily follows that a topology τ of topological group (G, τ) is given by a metric if and only if the topological space (G, τ) is a Hausdorff space, and it has a countable basis of the filter of neighborhoods of the unity element.

Such a topology is called *a metrizable topology*.

Notations 4. If $V_1, V_2, \dots$ and $S_1, S_2, \dots$ are some sequences of non-empty symmetric subsets of a group G, then for each natural number k by induction we define a subset $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ of G as follows: take $F_1\Big(V_1; S_1\Big) = \Big\{g \cdot V_1 \cdot g^{-1} | g \in S_1\Big\} \bigcup V_1 \cdot V_1$ and $F_{k+1} = F_1\Big(V_1 \bigcup F_k(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}); S_1\Big)$.

Proposition 5. *For subsets $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ the following statements are true:*

5.1. *If $e \in V_1$, then $V_1 \subseteq V_1 \cdot V_1 \subseteq F_1(V_1; S_1)$ and $g \cdot V_1 \cdot g^{-1} \subseteq F_1(V_1; S_1)$ for any $g \in S_1$;*

5.2. *If $k \in N$ and the sets S_i and V_i are finite for $1 \le i \le k$, then $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ is a finite symmetric set;*

5.3. *$F_k\Big(\{e\}, \dots, \{e\}; S_1, \dots, S_k\Big) = \{e\}$ for any $k \in N$;*

5.4. *If $U_i \subseteq V_i$ and $T_i \subseteq S_i$ for any $1 \le i \le k$, then $F_k\Big(U_1, \dots, U_k; T_1, \dots, T_k\Big) \subseteq F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$;*

5.5. *If $k, p \in N$ and $e \in V_i$ for all $i \le k$ and $V_{k+j} = \{e\}$ for $1 \le j \le p$, then $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big) = F_{k+p}\Big(V_1, \dots, V_{k+p}; S_1, \dots, S_{k+p}\Big)$;*

5.6. *For $k \ge 2$ the equality $F_k\Big(V_1, \dots, V_k;\ S_1, \dots, S_k\Big) = F_k\Big(V_1 \bigcup F_{k-1}\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big), \dots, V_{k-1} \bigcup F_1(V_k; S_k), V_k; S_1, \dots, S_k\Big)$ is true;*

5.7. *If $e \in V_i$ for any $1 \le i \le k$, then $V_t \subseteq F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ for any $1 \le t \le k$;*

5.8. *If $e \in V_i$ for any $1 \le i \le k$, then $F_{k+1}\Big(V_s, \dots, V_{k+s}; S_s, \dots, S_{k+s}\Big) \subseteq F_{k+s-t+1}\Big(V_t, \dots, V_{k+s}; S_1, \dots, S_{k+s}\Big)$ for any $k, s, t \in N$ and $t \le s$.*

Proof. Statement 5.1 follows easily from the definition of the set $F_1\Big(V_1; S_1\Big)$.

Statements 5.2, 5.3 and 5.4 can be easily proved by induction on k, using that the sets S_i and V_i for $i \in N$ are symmetric and the definition of the set $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$.

We prove Statement 5.5 by induction on k.

If $k = 1$, then using Statement 5.3 we get $F_{1+p}\Big(V_1, \{e\}, \dots, \{e\}; S_1, \dots, S_{1+p}\Big) = F_1\Big(V_1 \bigcup F_p(\{e\}, \dots, \{e\}; S_2, \dots, S_{1+p}); S_1\Big) = F_1\Big(V_1 \bigcup \{e\}; S_1\Big) = F_1\Big(V_1; S_1\Big)$ for any $p \in N$.

Assume that the equality is proved for the number k and all $p \in N$. Then

$$F_{k+1+p}\Big(V_1, \dots, V_{k+1}, \{e\}, \dots, \{e\}; S_1, \dots, S_{k+1+p}\Big) =$$

$$F_1\Big(V_1 \bigcup F_{k+p}(V_2, \dots, V_{k+1}, \{e\}, \dots, \{e\}; S_2, \dots, S_{k+1+p}); S_1\Big) =$$

$$F_1\Big(V_1 \bigcup F_k(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}); S_1\Big) =$$

$$F_{k+1}\Big(V_1, V_2, \dots, V_{k+1}; S_1, S_2, \dots, S_{k+1}\Big).$$

Statement 5.5 is proved for the number $k+1$, and hence, Statement 5.5 is proved for any natural number.

We prove Statement 5.6 by induction on k.

If $k = 2$, then $F_2\Big(V_1,\ V_2;\ S_1,\ S_2\Big) = F_1\Big(V_1 \bigcup F_1\Big(V_2;\ S_2\Big);\ S_1\Big) = F_1\Big(V_1 \bigcup F_1\Big(V_2; S_2\Big) \bigcup F_1\Big(V_2; S_2\Big); S_1\Big) = F_2\Big(V_1 \bigcup F_1\Big(V_2; S_2\Big), V_2; S_1, S_2\Big)$.

Assume that the equality holds for the number $k \geq 2$. Then

$$F_{k+1}\Big(V_1, \dots, V_{k+1}; S_1, \dots, S_{k+1}\Big) = F_1\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big); S_1\Big) =$$

$$F_1\Big(\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big)\Big) \bigcup F_k\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big)\Big); S_1\Big) =$$

$$F_1\Big(\ \Big(V_1 \bigcup F_k\Big(V_2 \bigcup F_{k-1}\Big(V_3, \dots, V_{k+1}; S_3, \dots, S_{k+1}\Big)\Big), \dots, V_{k-1} \bigcup$$

$$F_k\Big(V_2 \bigcup F_{k-1}\Big(V_3, \dots, V_{k+1}; S_3, \dots, S_{k+1}\Big), \dots, V_{k-1} \bigcup$$

$$F_1(V_k; S_k), V_k; S_2, \dots, S_k\Big)\Big); S_1\Big) = F_{k+1}\Big(V_1 \bigcup$$

$$F_k\Big(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}\Big), \dots, V_k \bigcup F_1(V_{k+1}; S_{k+1}), V_{k+1}; S_1 \dots S_{k+1}\Big).$$

Statement 5.6 is proved for the number $k+1$, and hence, Statement 5.6 is proved for any integer $k \geq 2$.

We prove Statement 5.7 by induction on k.

If $k = 1$, then $t = 1$. Then, by Proposition 2.1, $F_1\Big(V_1; S_1\Big) \supseteq V_1$.

Assume that the required inclusion is proved for the number k and all $1 \le t \le k$, and let $t \le k + 1$.

If $t > 1$, then considering the induction assumption, we get that

$$F_{k+1}\Big(V_1, \dots, V_{k+1}; S_1, \dots, S_{k+1}\Big) \supseteq F_1\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_{k+1};$$

$$S_2, \dots, S_{k+1}\Big); S_1\Big) \supseteq F_1\Big(V_1 \bigcup V_t; S_1\Big) \supseteq V_1 \bigcup V_t \supseteq V_t.$$

If $t = 1$, then applying Statements 5.4 and 5.3, and the induction assumption, we see that

$$F_{k+1}\Big(V_1, \dots, V_{k+1}; S_1, \dots, S_{k+1}\Big) \supseteq$$

$$F_1\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}\Big); S_1\Big) \supseteq$$

$$F_1\Big(V_1 \bigcup F_k\Big(\{e\}, \dots, \{e\}; S_2, \dots, S_{k+1}\Big); S_1\Big) = F_1\Big(V_1; S_1\Big) \supseteq V_1.$$

By this Statement 5.7 is proved.

We prove Statement 5.8 by induction on the number $s - t$.

If $s - t = 0$, then $t = s$, and hence, $F_{k+1}\Big(V_s, \dots, V_{k+s}; S_s, \dots, S_{k+s}\Big) = F_{k+s-t+1}\Big(V_t, \dots, V_{k+s}; S_t, \dots, S_{k+s}\Big)$.

Assume that the required inclusion is proved for $s - t = n$ and any $k \in N$, and let $s - t = n + 1$. Then, by the inductive assumption and Statement 5.7,

$$F_{k+1}\Big(V_s, \dots, V_{k+s}; S_s, \dots, S_k\Big) \subseteq F_{k+(s-t-1)+1}\Big(V_2, \dots, V_{k+s}; S_2, \dots, S_{k+s}\Big) \subseteq$$

$$V_1 \bigcup F_{k+(s-t-1)+1}\Big(V_2, \dots, V_{k+s}; S_2, \dots, S_{k+s}\Big) \subseteq$$

$$F_1(V_1 \bigcup F_{k+s-t}\Big(V_2, \dots, V_{k+s}; S_2, \dots, S_{k+s}\Big); S_1) =$$

$$F_{k+s-t+1}\Big(V_1, \dots, V_{k+s}; S_1, \dots, S_{k+s}\Big)$$

for all $s, k \in N$.

By this Statement 5.8 is proved, and hence, Proposition 5 is proved. □

Definition 6. Let G be a group and let x be a variable. An expression of the form $g_1 \cdot x^{k_1} \cdot g_2 \cdot x^{k_2} \cdot \ldots \cdot g_s \cdot x^{k_s} \cdot g_{s+1}$, where $g_i \in G$ for $1 \le i \le s+1$ and k_j are integers for $1 \le j \le s$, is called *a word on the variable x over the group G*.

The set of all words on the variable x over the group G will be denoted by $G(x)$.

Remark 7. If we assume that $x^0 = e$, then the set $G(x)$ is a group under the multiplication of words.

Adding, if it is necessary, the unity element of the group in the expression $g_1 \cdot x^{k_1} \cdot g_2 \cdot x^{k_2} \cdot \ldots \cdot g_s \cdot x^{k_s} \cdot g_{s+1}$ we can assume that $k_i \in \{-1, 0, 1\}$.

Definition 8. If $f(x)$ is a word on the variable x over the group G, then an expression of the form $f(x) = g$, where $g \in G$, is called *an equation over a group* G.

Definition 9. An element b of a group G is called *a root of the equation* $f(x) = g$ over the group G if $f(b) = g$.

Notations 10. Let G be a countable group, and let $G = \big\{e, g_1^{\pm 1}, g_2^{\pm 1}, \dots\big\}$ be a numbering of elements of the group G (this numbering will follow throughout the article).

For each natural number k, we put $S_k = \big\{g_1^{\pm 1}, g_2^{\pm 1}, \dots, g_k^{\pm 1}\big\}$, for each pair of natural numbers (i, j) we define subsets $V_{(i,j)}$ and $S_{(i,j)}$ of the group G, and for each triple of natural numbers (i, j, k) such that $1 \le k \le j$ we define the set $\Phi_{(i,j,k)}(x)$ of the equations on the variable x over the group G as follows: $V_{(1,j)} = \{e\}$, $S_{(1,j)} = S_j$, and $\Phi_{(1,j,k)}(x) = \big\{x = c \mid c \in S_k\big\}$ for all $j, k \in N$ and $k \le j$.

Assume that the sets $V_{(i,j)}$, $S_{(i,j)}$ and $\Phi_{(i,j,k)}(x)$ for $i \le p$ and all $j, k \in N$ and $k \le j$ are defined for a natural number p.

If $p + 1$ is even, then we take:

$V_{(p+1,j)} = \{e\}$ for any $j \ge p + 1$;

$V_{(p+1,j)} = V_{(p,j)} \bigcup \{g, g^{-1}\}$, where g is an element of the set $G \setminus \bigcup\limits_{s=1}^{j} S_{(p,j)}$ [1] for any $j < p + 1$;

$\Phi_{(p+1,j,k)}(x) = \Phi_{(p,j,k)}(x)$ for all $k < j \in N$;

$S_{(p+1,j)} = \big\{g \in G \mid g \textit{ is a root of an equation from } \bigcup\limits_{k=1}^{j} \Phi_{(p+1,j,k)}\big\}$ for all $j \in N$.

If $p + 1$ is odd, then we take:

$V_{(p+1,j)} = \{e\}$ for $j \ge p + 1$;

$V_{(p+1,j)} = F_{p+1-j}\big(V_{(p,j+1)}, \dots, V_{(p,p+1)}; S_{j+1}, \dots, S_{p+1}\big) \bigcup V_{(p,j)}$ for $j < p + 1$;

$\Phi_{(p+1,j,j)}(x) = \big\{x = g \mid g \in S_j\big\}$ for all $j \in N$ and $\Phi_{(p+1,j,k)}(x) = \Big\{f(x) = g \mid f(x) \in F_{j-k}\big(V_{(p+1,k+1)}, \dots, V_{(p+1,j-1)}, V_{(p,j)} \bigcup \{x, x^{-1}\};\ S_{k+1}, \dots, S_j\big)$ and $g \in S_k\Big\}$ for all $k < j \in N$;

$S_{(p+1,j)} = S_{(p,j)}$ for any $j \in N$.

So, we have identified the subsets $V_{(i,j)}$ and $S_{(i,j)}$ of the group G for each pair of natural numbers (i, j) and the set $\Phi_{(i,j,k)}(x)$ of equations over a group G for each triple of natural numbers (i, j, k), such that $1 \le k \le j$, respectively.

Theorem 11. *If a countable group G admits a non-discrete Hausdorff group topology τ, then for any finite set $M = \big\{f_1(x) = a_1, \dots, f_m(x) = a_m\big\}$ of equations over the group G for which the unity element e of the group G is not a root of any of these equations, in the topological group (G, τ) there exists a neighborhood W of the unity element such that its any element is not a root of any of these equations.*

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[1] If $G \setminus \bigcup\limits_{s=1}^{j} S_{(p,j)} = \emptyset$, then we take $V_{(p+1,j)} = V_{(p,j)}$.

Proof. For each positive integer $1 \leq i \leq m$ of the mapping $f_i : (G, \tau) \to (G, \tau)$ is a continuous mapping. Since the topological group is a Hausdorff space, then the set $\{g\}$ is a closed set in the topological group (G, τ) for any element $g \in G$. Then $V_i = G \setminus f_i^{-1}(a_i)$ is an open set, and $e \in V_i$. If $V = \bigcap\limits_{j=1}^{m} V_j$, then V is a neighborhood of the unity element and $a_i \notin f_i(V)$ for any $1 \leq i \leq m$, and hence any element from V is not a root of any equation $f_i(x) = a_i$ for any $1 \leq i \leq m$.

By this the theorem is proved. $\square$

Proposition 12. (see the example 3.6.18 in [5]) *There exists a set $\tilde{\mathbb{N}}$ of cardinality continuum of infinite subsets of the set $\mathbb{N}$ of natural numbers such that $A \bigcap B$ is a finite set for any distinct $A, B \in \tilde{\mathbb{N}}$*

Theorem 13. *If a countable group G admits a non-discrete metrizable group topology τ_0, then G admits continuum of non-discrete metrizable group topologies stronger than τ_0, and any two of these topologies are incomparable.*

Proof. Let $G = \Big\{e, g_1^{\pm 1}, \dots\Big\}$ be a numbering of elements of the group G and $S_n = \Big\{g_1^{\pm 1}, \dots, g_n^{\pm 1}\Big\}$ for any $n \in N$. There exists a countable basis $\Big\{V_1, V_2, \dots\Big\}$ of the filter of neighborhoods of the unity element in the topological group (G, τ_0) such that $V_k^{-1} = V_k$, $V_k \bigcap S_k = \emptyset$ and $g \cdot V_{k+1} \cdot g^{-1} \subseteq V_k$ for any $g \in S_k$, $k \in N$.

By induction on k one can easily prove that $F_k\Big(V_{i+1}, \dots, V_{i+k}; S_{i+1}, \dots, S_{i+k}\Big) \subseteq V_i$ for all $i, k \in N$.

Further proof of the theorem will be realized in several steps.

Step I. Construction of an auxiliary sequence of elements and a sequence of natural numbers.

By induction, we construct a sequence $k_1, k_2, \dots$ of natural numbers such that $k_i \geq i$ for all $i \in N$, and a sequence $h_1, h_2, \dots$ of elements of the set $G \backslash \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ and $h_i \notin F_k\big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{i-1}, h_{i-1}^{-1}\}, \{e\}, \{e, h_{i+1}, h_{i+1}^{-1}\}, \dots, \{e, h_k, h_k^{-1}\}; S_1, \dots, S_k\big)$ for any integers $1 \leq i < k$.

We take $k_1 = 1$, and as h_1 we take an arbitrary element of the set $V_1 \backslash \{e\}$.

Suppose that we have already defined natural numbers $k_1, k_2, \dots, k_n$ such that $k_i \geq i$ and elements $h_1, h_2, \dots, h_n$ from the set $G \backslash \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ and $h_i \notin F_n\big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{i-1}, h_{i-1}^{-1}\}, \{e\}, \{e, h_{i+1}, h_{i+1}^{-1}\}, \dots, \{e, h_n, h_n^{-1}\}; S_1, \dots, S_n\big)$ for any $i \in N$, $1 \leq i < n$ and $h_n \notin F_{n-1}\Big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{n-1}, h_{n-1}^{-1}\}; S_1, \dots, S_{n-1}\Big)$.

For any $i \in N$, $i < n+1$ we consider the set $\Omega_{(n+1,\ i)}(x) = F_{n+1}\Big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{i-1}, h_{i-1}^{-1}\}, \{e\}, \{e, h_{i+1}, h_{i+1}^{-1}\}, \dots, \{e, h_n, h_n^{-1}\}, \{x, x^{-1}\}; S_1, \dots, S_{n+1}\Big)$ of words on the variable x over the group G and the set of equations $\Phi'_{n+1}(x) = \bigcup\limits_{i=1}^{n} \Big\{f(x) = g \mid f(x) \in \Omega_{(n+1,i)}, g \in \{h_i, h_i^{-1}\}\Big\}$ over the group G.

Since (see Statement 5.5) $F_{n+1}\Big(\{e,\ h_1,\ h_1^{-1}\},\dots,\{e,\ h_{i-1},\ h_{i-1}^{-1}\},\{e\},$ $\{e,\ h_{i+1},\ h_{i+1}^{-1}\},\ \dots\ ,\{e,\ h_n,\ h_n^{-1}\},\ \{e\};\ S_1,\dots,S_{n+1}\Big) = F_n\Big(\{e,\ h_1,\ h_1^{-1}\},\dots,$ $\{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots,\{e,h_n,h_n^{-1}\};S_1,\dots,S_n\Big)$, and by the induction assumption, $h_i \notin F_n\Big(\{e,h_1,h_1^{-1}\},\dots,\{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots,$ $\{e,h_n,h_n^{-1}\};S_1,\dots,S_n\Big)$, then $f(e) \notin \{h_i,h_i^{-1}\}$ for any $i \le n$ and for any word $f(x)$ of the set $\Omega_{(n+1,i)}(x)$. Hence, the unity element e of the group G is not a root of any equation of the set $\Phi'_{n+1}(x)$.

So, we have proved that $\Phi'_{n+1}(x)$ is a finite set of equations over the group G and the unity element e of the group G is not a root of any equation of the set $\Phi'_{n+1}(x)$.

Since the topology τ_0 is a non-discrete Hausdorff group topology, then by Theorem 11, the topological group (G,τ) has a neighborhood W of the unity element such that any its element is not a root of any equation of the set $\Phi'_{n+1}(x)$.

The finiteness of the set $F_n(\{e,h_1,h_1^{-1}\},\dots,\{e,h_n,h_n^{-1}\};S_1,\dots,S_n)$ and the fact that τ_0 is a Hausdorff topology imply that there exists a number $n+1 < k_{n+1} \in N$ such that $W_{k_{n+1}} \subseteq W$ and

$$F_n(\{e,h_1,h_1^{-1}\},\dots,\{e,h_n,h_n^{-1}\};S_1,\dots,S_n)\bigcap W_{k_{n+1}} = \{e\}.$$

We take as h_{n+1} any element of the set $W_{k_{n+1}}\setminus\{e\}$.

We show that these conditions are statisfied for numbers $k_1,k_2,\dots,k_{n+1}$ and for elements $h_1,h_2,\dots,h_{n+1}$ of the group G.

Since $h_{n+1} \in W_{k_{n+1}}\setminus\{e\}$, then $h_{n+1} \notin F_n\Big(\{e,\ h_1,\ h_1^{-1}\},\dots,\{e,h_n,h_n^{-1}\};$ $S_1,\dots,S_n\Big)$. Moreover, by the inductive assumption, $h_i \notin F_n\Big(\{e,h_1,h_1^{-1}\},\dots,$ $\{e,\ h_{i-1},\ h_{i-1}^{-1}\},\ \{e\},\ \{e,\ h_{i+1},\ h_{i+1}^{-1}\},\ \dots\ ,\ \{e,\ h_n,\ h_n^{-1}\};\ S_1,\ \dots\ ,\ S_n\Big) =$ $F_{n+1}\Big(\{e,\ h_1,\ h_1^{-1}\},\ \dots,\{e,\ h_{i-1},\ h_{i-1}^{-1}\},\ \{e\},\ \{e,\ h_{i+1},\ h_{i+1}^{-1}\},\dots,\{e,\ h_n,\ h_n^{-1}\},\{e\};$ $S_1,\dots,S_{n+1}\Big)$.

Since the element h_{n+1} is not a root of any equation of the set $\Phi'_{n+1}(x) = \bigcup\limits_{j=1}^{n}\Big\{f(x) = g \mid f(x) \in \Omega'_{(n+1,j)}, g \in \{h_j,h_j^{-1}\}\Big\}$, then $h_i \notin F_{n+1}\Big(\{e,h_1,h_1^{-1}\},\dots,$ $\{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots,\{e,h_{n+1},h_{n+1}^{-1}\};S_1,\dots,S_{n+1}\Big)$.

Thus, we have constructed the sequence of natural numbers $k_1,k_2,\dots$ and the sequence $h_1,h_2,\dots$ of elements of the group G such that $k_i \ge i$, $\{e,h_i,h_i^{-1}\} \subseteq W_{k_i}$ for any $i \in N$ and $h_i \notin F_k\big(\{e,h_1,h_1^{-1}\},\dots,\{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots,$ $\{e,h_k,h_k^{-1}\};S_1,\dots,S_k\big)$ for any natural numbers $1 \le i < k$.

Step II. Construction of a metrizable group topology $\tau(A)$ for any infinite set A of of natural numbers.

For any natural number i we consider the set $U_{i,A} = \{e\}$ if $i \notin A$, and $U_{i,A} = \{e,h_i,h_i^{-1}\}$ if $i \in A$, and for any pair (i,j) of natural numbers we consider the set $U_{(i,j),A} = F_j(U_{i+1,A}, \dots, U_{i+j,A}; S_{i+1}, \dots, S_{i+j})$. We will show that for the sets $U_{(i,j),A}$ the following inclusions are true:

1. From Statements 5.3 and 5.4 it follows that $e \in U_{(i,j),A}$ for any $i, j \in N$.
2. From Statement 5.5 it follows that $U_{(k,j),A} \subseteq U_{(k,n),A}$ for any $j \leq n$.
3. From Statement 5.8 it follows that $U_{(i,j),A} \subseteq U_{(k,j),A}$ for any $k \leq i$.
4. From Statement 5.2 it follows that $U_{(i,j),A}$ is a symmetric set, i. e. $\Big(U_{(i,j),A}\Big)^{-1} = U_{(i,j),A}$ for any $i, j \in N$.
5. By induction on j, we prove that $U_{(i+1,j),A} \cdot U_{(i+1,j),A} \subseteq U_{(i,j),A}$ and $g \cdot U_{(i+1,j),A} \cdot g^{-1} \subseteq U_{(i,j),A}$ for any $i, j \in N$, $j > 1$ and $g \in S_{i+1}$.

In fact, if $j = 2$, then, applying in succession the definition of the sets $U_{(i,j),A}$, Statements 5.1, 3.4 and 3.6, we obtain:

$$U_{(i+1,2),A} \cdot U_{(i+1,2),A} =$$

$$F_1(U_{i+2,A}; S_{i+2}) \cdot F_1(U_{i+2,A}; S_{i+2}) \subseteq F_1(F_1(U_{i+2,A}; S_{i+2}); S_{i+1}) \subseteq$$

$$F_1(U_{i+1,A} \bigcup F_1(U_{i+2,A}; S_{i+2}); S_{i+1}) = F_2(U_{i+1,A}, U_{i+2,A}; S_{i+1}, S_{i+2}) = U_{(i,2),A} =$$

$$U_{(i,j),A} \ \text{ and } \ g \cdot U_{(i+1,2),A} \cdot g^{-1} = g \cdot F_1\big(U_{i+2,A}; S_{i+2}\big) \cdot g^{-1} \subseteq$$

$$F_1\big(F_1(U_{i+2,A}; S_{i+2}); S_{i+1}\big) \subseteq F_1\Big(U_{i+1,A} \bigcup F_1\big(U_{i+2,A}; S_{i+2}\big); S_{i+1}\Big) =$$

$$F_2\big(U_{i+1,A}, U_{i+2,A}; S_{i+1}, S_{i+2}\big) = U_{(i,2),A}$$

for any $i \in N$.

Assume that the required inclusion is proved for $j = n \geq 2$ and any $i \in N$.

Then

$$U_{(i+1,i+n+1),A} \cdot U_{(i+1,i+n+1),A} = F_n\Big(U_{i+2,A}, \dots, U_{i+n+1,A};$$

$$S_{i+2}, \dots, S_{i+n+1}\Big) \cdot F_n\Big(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}\Big) \subseteq$$

$$F_1\Big(F_n\big(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}\big); S_{i+1}\Big) \subseteq$$

$$F_1(U_{i+1,A} \bigcup F_n(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}); S_{i+1}) =$$

$$F_{n+1}(U_{i+1,A}, \dots, U_{i+n+1,A}; S_{i+1}, \dots, S_{i+n+1}) = U_{(i,n+1),A}$$

and

$$g \cdot U_{(i+1,i+n+1),A} \cdot g^{-1} = g \cdot F_n(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}) \cdot g^{-1} \subseteq$$

$$F_1(F_n(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}); S_{i+1}) \subseteq$$

$$F_1\Big(U_{i+1,A} \bigcup F_n\big(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}\big); S_{i+1}\Big) =$$

$$F_{n+1}\Big(U_{i+1,A}, \dots, U_{i+n+1,A}; S_{i+1}, \dots, S_{i+n+1}\Big) = U_{(i,n+1),A}.$$

So, we have proved that $U_{(i+1,j),A} \cdot U_{(i+1,j),A} \subseteq U_{(i,j),A}$ and $g \cdot U_{(i+1,j),A} \cdot g^{-1} \subseteq U_{(i,j),A}$ for any $i, j \in N$, $j > 1$ and $g \in S_{i+1}$.

Using the inclusions 1–5 proven above, one can prove that the set $\Big\{\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(i,j),A} \mid i \in N\Big\}$ satisfies the conditions of Theorem 1, and hence, this set is a basis of the filter of neighborhoods of the unity element for a metrizable group topology $\tau(A)$ in the group G.

Step III. Construction of the continuum of group topologies.

For any subset $A \in \tilde{\mathbb{N}}$ (for definition of the set $\tilde{\mathbb{N}}$, see Proposition 12) we consider the group topology $\tau(A)$, constructed in the proof of this Theorem, step II.

Since the set $\tilde{\mathbb{N}}$ has the cardinality of the continuum, then to complete the proof, it remains to show that for any sets $A, B \in \tilde{\mathbb{N}}$ the topologies $\tau(A)$ and $\tau(B)$ are incomparable.

Suppose the contrary, for definiteness assume that $\tau(A) \le \tau(B)$.

Let $n \in A$. Since $\tau(A) \le \tau(B)$ and $\hat{U}_{n,A}$ is a neighborhood of the unity element in the topological group $(G, \tau(A))$, then there exists a natural number $k \in B$ such that $\hat{U}_{k,B} \subseteq \hat{U}_{n,A}$, and since $A \bigcap B$ is a finite set, then there exists a natural number $s \in B \backslash A$, such that $s > k$ and $s > n$. Then

$$h_s \in F_{k-s}\Big(U_{k+1,B}, \dots, U_{s,B}; S_{k+1}, \dots, S_s\Big) \subseteq \hat{U}_{k,B} \subseteq \hat{U}_{n,A}.$$

From the construction of the elements h_i (see step I of this proof) we have $h_s \notin F_t\Big(\{e,\ h_1,\ h_1^{-1}\}, \dots, \{e,\ h_{s-1},\ h_{s-1}^{-1}\},\ \{e\},\ \{e, h_{s+1},\ h_{s+1}^{-1}\},\ \{e,\ h_{s+t},\ h_{s+t}^{-1}\}; S_1, \dots, S_{s+t}\Big)$ for any $t \in N$.

Since $s \notin A$, then $U_{s,A} = \{e\}$, and hence, $h_s \notin F_t\Big(U_{n+1,A}, \dots, U_{n+t,A}; S_{n+1}, \dots, S_{n+t}\Big) = U_{(n,t),A}$ for any $t \in N$. Then $h_s \notin \bigcup\limits_{t=1}^{\infty} U_{(n,t),A} = \hat{U}_{n,A}$.

We have arrived at a contradiction, so the topologies $\tau(A)$ and $\tau(B)$ are incomparable.

By this the theorem is proved. $\square$

Theorem 14. *Let a countable group G admit a non-discrete metrizable group topology τ_0, then there exists the continuum of non-discrete metrizable group topologies on G stronger than τ_0, and any two of these topologies are comparable.*

Proof. Let P be the set of all prime numbers, let $\mathbb{Q}$ be the set of all rational numbers, and let $\mathbb{R}$ be the set of all real numbers. Then there exists a bijection $\xi : \mathbb{Q} \to P$.

For each positive real number $r \in \mathbb{R}$ we consider the set $A_r = \xi(\{q \in \mathbb{Q} \mid r \le q\})$ of prime numbers, and let $\tau(A_r)$ be group topology on the group G, constructed in the proof of Theorem 13, step II.

We will show that the set $\{\tau(A_r) \mid r \in R\}$ is the required set of group topologies.

Since the set $\left\{\hat{U}_i(A_r) \mid i \in N\right\}$ is a basis of the filter of neighborhoods of the unity element for the group topology $\tau(A_r)$, then the topological group $\big(G, \tau(A_r)\big)$ has a countable basis of the filter of neighborhoods of the unity element.

We show that for any distinct real numbers $r, r' \in \mathbb{R}$ the topologies $\tau(A_r)$ and $\tau(A_{r'})$ are different and comparable.

In fact, if $r < r'$, then $A_r \backslash A_{r'}$ is an infinite set. Then, similarly as in the proof of Theorem 13, step III we show that $\tau(A_r) \neq \tau(A_{r'})$, and hence, the set $\{\tau(A_r) \mid r \in R\}$ has the cardinality of the continuum.

To finish the proof of the Theorem it remains to show that any two topologies from the set $\{\tau(A_r) \mid r \in \mathbb{R}\}$ are comparable.

Let $r, r' \in R$ and suppose (for definiteness) that $r < r'$. Since

$$A_{r'} = \xi(\{q \in \mathbb{Q} \mid r' \leq q\} \subseteq \xi(\{q \in \mathbb{Q} \mid r \leq q\} = A_r,$$

then (see the definition of the sets $U_{(i,j),A}$ in the proof of Theorem 13 step II) $U_{(i,j),A_{r'}} \subseteq U_{(i,j),A_r}$ for any $i, j \in \mathbb{N}$. Then $\hat{U}_{n,A_{r'}} \subseteq \hat{U}_{n,A_r}$ for any $n \in N$, and , the sets $\{U_{n,A_{r'}} \mid n \in \mathbb{N}\}$ and $\{U_{n,A_r} \mid n \in \mathbb{N}\}$ are basis of the filter of neighborhoods of the unity element in topological groups $(G, \tau(A_{r'}))$ and $(G, \tau(A_r))$, respectively. As any group topology is determined by the basis of the filter of neighborhoods of the unity element, then $\tau(A_r) < \tau(A_{r'})$.

By this the theorem is proved. □

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