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Examples of quasitopological groups

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Abstract. In this paper we construct several examples of completely regular submetrizable quasitopological groups with slightly different combinations of properties, in particular, a countable quasitopological group G with countable π-weight, countable tightness, countable δ-character, but not first-countable, and a countable quasitopological group P with countable π-weight, countable tightness, but of uncountable δ-character.

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1 Introduction

All spaces considered below are assumed to be Tychonoff. In terminology and notations we follow [7] and [8]. A space is submetrizable if its topology contains a metrizable topology.

A group G with a topology $\mathcal{T}$ is a semitopological (paratopological, respectively) group if the multiplication is separately continuous (jointly continuous, respectively).

If G is a semitopological and the inverse operation $x \to x^{-1}$ is continuous, then G is said to be a quasitopological group.

Recall that a π-base of a space X is a family β of non-empty open subsets of X such that every open non-empty set U contains some member of β. A π-base of a space X at a point $x \in X$ is a family β of non-empty open subsets of X such that every open neighborhood of x contains at least one element of β.

We will say that the δ-character of a space X at a point $x \in X$ is countable, if there exists a sequence $\gamma = \{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X converging to x.

2 The topologies $\mathcal{T}^*$, $\mathcal{T}^{**}$ and $\mathcal{T}^{\triangle}$ on $\mathbb{R}^2$

Let $\mathbb{R}$ be the usual topological group of reals. Consider the group $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the Euclidean topology $\mathcal{T}_E$.

For any $(x, y) \in G$ we put:

– $V((x,y),r) = \{(x,y)\} \cup \{(u,v) : u \neq x, |u-x| < r, 0 < (v-y)/(u-x) < r\}$, where $0 < r$;

– $W((x,y),r) = \{(x,y)\} \cup \{(u,v) : u \neq x, |u-x| < r, -r < (v-y)/(u-x) < r\}$, where $0 < r$;

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– $\mathcal{S}((x,y),r) = \{U \in \mathcal{T}_E : U, -U + (x,y) = U - (x,y), x < u < x + r\} \subset U \subseteq W((x,y),r)\}$, where $0 < r$;

– $\mathcal{S}(x,y) = \cup\{\mathcal{S}((x,y),r) : 0 < r < \infty\}$.

In particular, $U \in \mathcal{S}(0,0)$ if and only if U is open in $\mathbb{R}^2$, $-U = U$ and $\{(t,0) : 0 < t < r\} \subseteq U \subseteq W((0,0),r)$ for some $r > 0$. In this case, since $U = -U$, we have $\{(t,0) : -r < t < 0\} \subseteq U$ too.

By construction, the sets $V(x,y) \setminus \{(x,y)\}$ and $W(x,y) \setminus \{(x,y)\}$ are open in $\mathbb{R}^2$.

Now, we put $O((x,y),r,U) = V((x,y),r) \cup U$, $\mathcal{B}^*(x,y) = \{O((x,y),r,U) : U \in \mathcal{S}((x,y),r), 0 < r < \infty\}$ and $\mathcal{B}^* = \cup\{\mathcal{B}^*(x,y) : (x,y) \in \mathbb{R}^2\}$.

The family $\mathcal{B}^*$ is an open base of a new topology $\mathcal{T}^*$ on the set $\mathbb{R}^2$. In particular, $(\mathbb{R}^2, \mathcal{T}^*)$ is a submetrizable space, and hence, any compact subset of $(\mathbb{R}^2, \mathcal{T}^*)$ is metrizable.

A sequence $s = \{s_n : n \in \mathbb{N}\}$ of real numbers is called an r-basic sequence if $0 < -s_{n+1} < -s_n < n^{-1}$ and $ns_n > -r$ for each $n \in \mathbb{N}$. Consider an r-basic sequence $s = \{s_n : n \in \mathbb{N}\}$. We construct the continuous function $h_s : [0,1] \to \mathbb{R}$, where $h_s(x) = (s_{n+1} - s_n)((n+1)^{-1} - n^{-1})(x - n^{-1}) + s_n$ for each $x \in [(n+1)^{-1}, n^{-1}]$ and $n \in \mathbb{N}$. We put $D^+((x,y),r,s) = \{\{(u,v) : u - x < r, x + (1+n)^{-1} \leq u < x + n^{-1}, h(x) < v \leq y\} : n \in \mathbb{N}\}$, $D^-((x,y),s) = -D^+((x,y),s)$ and $D((x,y),s) = D^+((x,y),s) \cup D^-((x,y),s)$.

Now we put $H((x,y),r,s) = V((x,y),r) \cup D((x,y),r,s)$ for each $r > 0$ and each r-basic sequence $s = \{s_n : n \in \mathbb{N}\}$.

Property 2.1. The group $\mathbb{R}^2$ with the topology $\mathcal{T}^*$ is a quasitopological group.

Proof. By construction, $O((0,0),r,U) = -O((x,y),r,U)$, $O((0,0),r,U) + (x,y) = O((x,y),r,U) + (x,y)$ and $U + (x,y) \in \mathcal{S}((x,y),r)$ for all $U \in \mathcal{S}((0,0),r)$ and $0 < r < \infty\}$.

Property 2.2. The family $\mathcal{H}(x,y) = \{H((x,y),r,s)) : 0 < r \leq 1, s$ *is an r-basic sequence*$\}$ is an open base of the space $(\mathbb{R}^2, \mathcal{T}^*)$ at the point (x,y).

Proof. Fix $O((0,0),r,U) = V((0,0),r) \cup U$, where $r > 0$ and $U \in \mathcal{S}((x,y),r)$. Let k be the first natural number for which $1/k < r$. We put $r_1 = 1/k$. The set U is open and the sets $F_n = \{(t,0) : 1/(n+1) \leq t \leq 1/n\}$ are compact in the space $(\mathbb{R}^2, \mathcal{T}_E)$. For each $n \geq k$ we have $F_n \subseteq U$. Hence, there exists $\delta_n > 0$ such that $\{(u,v) : 1/(n+1) \leq u \leq 1/n, -\delta_n < v \leq 0\} \subseteq U$. We can assume that $\delta_{n+1} < \delta_n \leq 1/n$ for each $n \geq k$, $\delta_m < 1/m$ for $i < k$ and $\delta = \{\delta_n : n \in \mathbb{N}\}$ is an r_1-basic sequence. By construction, $H((x,y),r_1) \subseteq O((x,y),r,U)$ and $H((x,y),r_1,\delta) \in \mathcal{T}_1(0,0)$.

Property 2.3. If $r_2 < r_1 \leq 1$, $s = \{s_n : n \in \mathbb{N}\}$ is an r_1-basic sequence and $\delta = \{\delta_n : n \in \mathbb{N}\}$ is an r_2-basic and $\delta_n < r_n$ for each $n \in \mathbb{N}$, then the closure of the set $H((x,y),r_2,\delta)$ in the space $(\mathbb{R}^2, \mathcal{T}^*)$ is a subset of the set $H((x,y),r_1,s)$.

Proof. It is obvious.

Property 2.4. The space $(\mathbb{R}^2, \mathcal{T}^*)$ is completely regular.

Proof. Fix an $r > 0$, an r-basic sequence $s = \{s_n : n \in \mathbb{N}\}$ and the neighborhood $H = H((0,0),r,s)$ of the point $(0,0)$.

Consider the function $f : \mathbb{R}^2 \to [0, 1]$, where:
(1) $f((0, 0)) = 1$ and $f((-x, -y)) = f((x, y))$ for any point $(x, y) \in \mathbb{R}^2$;
(2) $f((0, y)) = 0$ for each $y \in \mathbb{R} \setminus \{0\}$;
(3) if $(x, y) \in \mathbb{R}^2$ and $x \geq r$, then $f((x, y)) = 0$;
(4) if $(x, y) \in \mathbb{R}^2$, $x > 0$ and $y/x \geq r$, then $f((x, y)) = 0$;
(5) if $(x, y) \in \mathbb{R}^2$, $0 < x < r$ and $y/x \leq r$, then $f((x, y)) = r^{-2}x^{-1}(r-x)(rx-y)$;
(6) if $(x, y) \in \mathbb{R}^2$, $n \in \mathbb{N}$, $(n+1)^{-1} \leq x \leq n^{-1}, x < r$ and $y \leq 0$, then $f((0, y)) = 0$ for $y \leq h_s(x)$ and $f((x, y)) = r^{-1}h_s(x)^{-1}(r - x)(h_s(x) - y)$ for $y > h_s(x)$.

By construction, $f((0, 0)) = 1$ and $\mathbb{R}^2 \setminus H = f^{-1}(0)$. Moreover, if $Z = \mathbb{R}^2 \setminus \{(0, y) : y \in \mathbb{R}\}$ is a a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$, the function $f|Z : Z \to [0, 1]$ is continuous on Z. From this fact, the condition $H \subseteq W((x, y), r)$ and the construction (5) it follows that the function f is continuous on the space $(\mathbb{R}^2, \mathcal{T}^*)$. Hence, the space $(\mathbb{R}^2, \mathcal{T}^*)$ is completely regular.

The family $\mathcal{B}^\triangle = \{W((x, y), r) : (x, y) \in \mathbb{R}^2, r > 0\}$ is an open base of the topology $\mathcal{T}^\triangle$ on $\mathbb{R}^2$.

Property 2.5. The group $\mathbb{R}^2$ with the topology $\mathcal{T}^\triangle$ satisfies the following conditions:

1. It is a completely regular quasitopological group.
2. It is a first countable space with a countable π-base.
3. It is a not normal space and has the Baire property.
4. It is submetrizable and Dieudonné complete.
5. It is not a topological group.

Denote by $\mathcal{T}^{**}$ the topology on the space $\mathbb{R}^2$ generated by the open base $\mathcal{B}^{**} = \{U \cup \{(x, y)\} : (x, y) \in \mathbb{R}^2, U \in \mathcal{S}(x, y)\}$. By construction, $\mathcal{T}_E \subseteq \mathcal{T}^\triangle \subset \mathcal{T}^* \subset \mathcal{T}^{**}$. In particular, $(\mathbb{R}^2, \mathcal{T}^{**})$ is a submetrizable space and any compact subset of $(\mathbb{R}^2, \mathcal{T}^{**})$ is metrizable. Consider $Z = \mathbb{R}^2 \setminus \{(0, y) : y \in \mathbb{R}\}$ as a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$.

Property 2.6. The group $\mathbb{R}^2$ with the topology $\mathcal{T}^{**}$ is a quasitopological group.

Proof. By construction, if $U \in \mathcal{T}(0, 0)$, then $U = -U$ and $U + (x, y) \in \mathcal{T}(x, y)$.

Property 2.7. The space $(\mathbb{R}^2, \mathcal{T}^{**})$ is completely regular. *Proof.* Fix $U \in \mathcal{S}(0, 0)$.

The set U is open in X and $F = Z \cap \{(x, 0) : -r \leq x \leq r\} \subseteq U$ for some $r > 0$. Since the set F is closed in Z and the space Z is metrizable, there exists a continuous function $g : Z \to [0, 1]$ such that $X \setminus U = g^{-1}(0)$ and $F = g^{-1}(1)$. Put $f((0, 0)) = 1$, $f((0, y)) = 0$ for any $y \neq 0$ and $f((x, y)) = g((x, y))$ for any $(x, y) \in Z$. By definition of the topology $\mathcal{T}^{**}$, the function f is continuous on G, $f((0, 0)) = 1$ and $G \setminus (U \cup \{(0, 0)\}) = f^{-1}(0)$.

3 Some subgroups of the group $(\mathbb{R}^2, \mathcal{T}^*)$

Fix two dense subgroups A and B of the topological group $\mathbb{R}$ in the Euclidean topology.

Put $G = A \times B$. We will consider G as a subspace and subgroup of the quasitopological group $(\mathbb{R}^2, \mathcal{T}^*)$.

Property 3.1. G is a quasitopological group.

Proof. Use Property 2.1.

Property 3.2. The space G is completely regular, not first-countable.

Proof. The space G is completely regular, by Property 2.4.

Fix an infinite sequence $\{s^k = \{s_{kn} : n \in \mathbb{N}\} : k \in \mathbb{N}\}$ of r_n-basic sequences. For each $n \in \mathbb{N}$ fix a number s_n such that $max\{-n^{-1}, s_{nn}\} < s_n < 0$. Then $s = \{s_n : n \in \mathbb{N}\}$ is a 1-basic sequence. Obviously $(G \cap H((0,0), r_n, s^n)) \setminus H((0,0), 1, s) \neq \emptyset$ for each $n \in \mathbb{N}$. Thus, the space G is not first-countable.

Property 3.3. If $indA = indB = 0$, then $indG = 0$.

Proof. Assume that $indA = indB = 0$. Fix $r > 0$, $U \in \mathcal{S}((0,0), r)$ and $O((0,0), r, U) = V((0,0), r) \cup U$. Let $G^+ = \{(x,y) \in G : x > 0\}$ be a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$, $F = \{(x,y) \in G^+ : 2x \leq r, 0 \leq 2y/x \leq r\}$ and $H = O((0,0), r, U)$. Then G^+ is a separable metrizable space, $dimG^+ = 0$, the set H is open in G^+, the set F is closed in G^+ and $F \subseteq H$. Thus there exists an open-and-closed subset H_1 of the space G^+ such that $F \subseteq H_1 \subseteq H$. Then the set $H_2 = H_1 \cup (-H_1) \cup \{(0,0)\}$ is an open-and-closed subset of the space G such that $(0,0) \in H_2 \subseteq H$.

Property 3.4. G is a space with a countable π-base.

Proof. If $\mathcal{L}$ is a base of $(\mathbb{R}^2, \mathcal{T}_E)$, then $\{U \cap G : U \in \mathcal{L}\}$ is a π-base of G.

Property 3.5. G is not a topological group.

Proof. Any topological group with a countable π-base is metrizable (see [7]). Property 3.2 completes the proof.

Property 3.6. Any point of G has a countable δ-character in G.

Proof. The family $\{\{(u,v) \in G : u^2 + v^2 < 2^{-n}, 0 < v < 2^{-n}u\} : n \in \mathbb{N}\}$ is a strong π-base of the space G at the point $(0,0)$.

Property 3.7. If $(a,b) \in G$, then:

1. The subspace $\{a\} \times B$ of G is discrete.
2. The subspace $A \times \{b\}$ of G is separable, metrizable and a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$.

Property 3.8. If the set B is countable, then the space G is Lindelöf and has a countable network. Moreover, if the groups A and B are countable, the the group G is countable too.

Proof. Clearly, G is a union of a countable family of separable metrizable subspaces. Hence, G has a countable network.

Property 3.9. If $B = \mathbb{R}$, then the space G is not normal.

Proof. The proof is similar to the proof for the Niemytski plane ([8], Example 1.5.9).

Property 3.10. The tightness of the space G is countable.

Proof. Let $M \subseteq \{(x,y) : x > 0, y < 0\}$ and $(0,0) \in cl_G M$. We put $K = (cl_{(\mathbb{R}^2, \mathcal{T}_E)} M \cap \{(x,0) \in G : x \in \mathbb{R}\}) \setminus \{(0,0)\}$. We have two possible cases.

Case 1. $(0,0) \notin cl_{(\mathbb{R}^2, \mathcal{T}_E)} K$.

There exists $k \in \mathbb{N}$ such that $\{(x,y) : x \leq k^{-1}\} \cap K = \infty$. Fix $0 < r < (2k)^{-1}$. and $O > s_i > -(2i)^{-1}r$ for each $i < k$. Since the sets $F_n = \{(u,0) : (n+1)^{-1} \leq u \leq n^{-1}\}$ are compact, there exists a sequence $\{s_n : n \geq k\}$ such that $s_k < -(2n)^{-1}r \leq s_n < s_{n+1} < 0$ and $M \cap \{(u,v) : u - x < r, x + (1+n)^{-1} \leq u < x + n^{-1}, h_s(x) < v \leq 0\} = \emptyset$ for each $n \geq k$.

The sequence $s = \{s_n : n \in \mathbb{N}\}$ is an r-basic sequence, $M \cap D^+((x,y), r, s) = M \cap H((x,y), r, s) = \emptyset$. Thus, $(0,0) \notin cl_G M$. Hence, Case 1 is impossible.

Case 2. $(0,0) \in cl_{(\mathbb{R}^2, \mathcal{T}_E)} K$.

For each $n \in \mathbb{N}$ fix a point $(a_n, 0) \in K$ such that $0 < a_n < 2^{-n}$. Since $(a_n, 0) \in K$, there exists a sequence $\{(a_{nm}, b_{nm}) \in M : m \in \mathbb{N}\}$ such that $|a_{nm} - a_n| - b_{nm} < 2^{-n-m}$ for each $m \in \mathbb{N}$. By construction, the set $\{(a_{nm}, b_{nm}) : n, m \in \mathbb{N}\}$ is countable, $L \subseteq M$ and $(0,0) \in cl_G L$. The proof is complete.

A space X is Dieudonné complete if there exists a complete uniformity on the space X, i.e the universal uniformity on X is complete [8].

Property 3.11. The space G is Dieudonné complete.

Proof. Any submetrizable space is Dieudonné complete.

Property 3.12. If the space $A \times B$ has the Baire property, then the space G has the Baire property too.

Proof. Any dense open subset of G contains a dense open subset of the space $A \times B$ and any dense subset of $A \times B$ is dense in G too.

Property 3.13. If bG is a Hausdorff compactification of the space G, then the remainder $bG \setminus G$ is not Lindelöf and is not pseudocompact.

Proof. A space X is of countable type if every compact subset of X is contained in a compact subset of countable character. M. Henriksen and J. R. Isbel [9] have proved that a space X is of countable type if and only if any remainder of X is Lindelöf. The character of any non-empty compact subset of G in G is uncountable. Therefore, the remainders of G are not Lindelöf.

Since the δ-character of the space G in G is countable at some point, then any remainder of G is not pseudocompact (see [3]).

4 Some subgroups of the group $(\mathbb{R}^2, \mathfrak{T}^{**})$

Fix two dense subgroups A and B of the topological group $\mathbb{R}$ in the Euclidean topology.

Denote $P = A \times B$. We consider P as a subspace and subgroup of the quasitopological group $(\mathbb{R}^2, \mathfrak{T}^{**})$.

Property 4.1. G is a quasitopological group.

Proof. Use Property 2.6.

Property 4.2. The space P is completely regular and the δ-character of P is not countable.

Proof. From Property 2.7 it follows that the space P is completely regular. If the space P has countable δ-character at the $(0,0)$, then there exists a sequence $S =$

$\{(a_n, b_n) \in P : n \in \mathbb{N}\}$ such that $a_n \cdot b_n \neq 0$ for each $n \in \mathbb{N}$ and $\{(0,0)\} = cl_P S \setminus S$. Then the set $Z \setminus S$ is open in Z and $\{(0,0)\} \cup (Z \setminus S)$ is open in P, a contradiction.

Property 4.3. If $indA = indB = 0$, then $indP = 0$.

Proof. The proof is similar to the proof of Property 3.3.

Property 4.4. P is a space with a countable π-base.

Proof. If $\mathcal{L}$ is a base of $(\mathbb{R}^2, \mathcal{T}_E)$, then $\{U \cap P : U \in \mathcal{L}\}$ is a π-base of P.

Property 4.5. P is not a topological group.

Proof. Any topological group with a countable π-base is metrizable (see [7]). Property 4.4 completes the proof.

Property 4.6. If $(a, b) \in P$, then:

1. The subspace $\{a\} \times B$ of P is discrete.
2. The subspace $A \times \{b\}$ of P is separable, metrizable and a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$.

Property 4.7. If the set B is countable, then the space P is Lindelöf and has a countable network. Moreover, if the groups A and B are countable, the the group P is countable too.

Proof. It is similar to the proof of Property 3.8.

Property 4.8. If $B = \mathbb{R}$, then the space P is not normal.

Proof. The proof is as for the Niemytski plane ([8], Example 1.5.9).

Property 4.9. The tightness of the space P is countable.

Proof. It is similar to the proof of Property 3.10.

Property 4.10. The space P is Dieudonné complete.

Proof. Any submetrizable space is Dieudonné complete.

Property 4.11. If the space $A \times B$ has the Baire property, then the space P has the Baire property too.

Proof. Any dense open subset of P contains a dense open subset of the space $A \times B$ and any dense subset of $A \times B$ is dense in P too. The proof is complete.

5 General construction

Let E be a metrizable additive commutative topological group without isolated points, $dimE = 0$ and in E there exists an infinite sequence $\{c_n : n \in \mathbb{N}\}$ of distinct points of E such that $lim_{n \to \infty} c_n = 0$, where 0 is the neutral element of E. Fix a sequence $\{O_n : n \in \mathbb{N}\}$ of open-and-closed subsets of the space E such that:

– $(O_n \cup (-O_n)) \cap (O_m \cup (-O_m)) = \emptyset$ for $n, m \in \mathbb{N}$ and $n \neq m$;

– if U is open in E and $0 \in U$, then there exists $n \in \mathbb{N}$ such that $O_m \subseteq U$ for all $m \geq n$.

Fix an open base $\{U_n : n \in \mathbb{N}\}$ of the space E at the point 0. We can assume that $O_{n+1} \subseteq U_{n+1} \subseteq U_{n+1} + U_{n+1} \subseteq U_n = -U_n$ and U_n is open-and-closed in E for each $n \in \mathbb{N}$.

In $E \times E$ consider the family $\mathcal{B}_1 = \{V : V$ *is open-and-closed in* $E \times E \setminus \{0\} \times E, U = -U$, *there exists* $m \in \mathbb{N}$ *such that* $\cup\{(O_{2n-1} \times U_n) \cup ((U_m \setminus \{0\}) \times \{0\}) : n \in \mathbb{N}, n \geq m\} \subseteq U\}$ and the family $\mathcal{B}_2 = \{V : V$ *is open-and-closed in* $E \times E \setminus \{0\} \times E, U = -U$, *there exists* $m \in \mathbb{N}$ *such that* $(U_m \setminus \{0\}) \times \{0\} \subseteq U\}$.

The family $\mathcal{B}^\circ = \{\{z\} \cup (U + z) : z \in E \times E, U \in \mathcal{B}_1\}$ is a base of the topology $\mathcal{T}^\circ$ on $E \times E$ and the family $\mathcal{B}^{\circ\circ} = \{\{z\} \cup (U + z) : z \in E \times E, U \in \mathcal{B}_2\}$ is a base of the topology $\mathcal{T}^{\circ\circ}$ on $E \times E$. The sets from $\mathcal{B}^\circ$ are open-and-closed in $(E \times E, \mathcal{T}^\circ)$ and the sets from $\mathcal{B}^{\circ\circ}$ are open-and-closed in $(E \times E, \mathcal{T}^{\circ\circ})$. Thus the spaces $(E \times E, \mathcal{T}^\circ)$ and $(E \times E, \mathcal{T}^{\circ\circ})$ are zero-dimensional and completely regular. By construction, $\mathcal{T}^\circ \subseteq \mathcal{T}^{\circ\circ})$.

Fix two subgroups A and B without isolated points of the topological group E. Assume that $\{c_n : n \in \mathbb{N}\} \subseteq cl_E A$

We consider C the set $A \times B$ as a subspace of the space $(E \times E, \mathcal{T}^\circ)$ and D the set $A \times B$ as a subspace of the space $(E \times E, \mathcal{T}^{\circ\circ})$.

Property 5.1. The group C with the topology $\mathcal{T}^\circ|C$ and the group D with the topology $\mathcal{T}^{\circ\circ}|D$ satisfy the following conditions:

1. Are completely regular zero-dimensional qvasitopological groups.
2. C is a space of the countable δ-character and D is a space of the countable π-character.
3. The space C is not first-countable and the δ-character of D is uncountable.
4. The tightnesses of C and D are countable.
5. If the space $A \times B$ has the Baire property, then C and D have the Baire property, too.
6. Are submetrizable, Dieudonné complete and with σ-discrete π-bases.
7. The π-weights of C and D are equal with the weight of the space E.
8. Are not topological groups.
9. Any remainder of C is not Lindelöf and it is not pseudocompact, and any remainder of D is pseudocompact and not Lindelöf.
10. If the space B is σ-discrete, then C and D are paracompact F_σ-metrizable spaces. In particular, C and D are paracompact σ-spaces.

Proof. The proofs of the properties of C are similar to the proof of Properties 3.1–3.13 and the proofs of the properties of D are similar to the proof of Properties 4.1–4.12.

6 Open Problems

In [9] M. Henriksen and J. R. Isbel have proved that a space X is of countable type if and only if any remainder of X is Lindelöf. In [3] Arhangel'skii proved that any remainder of a topological group is either pseudocomact or Lindelöf. Various properties of remainders have been studied in [2–6]. Examples constructed in this paper motivate the following open questions:

Problem 6.1. Is it true that there exists a completely regular sequential (Fréchet-Urysohn) quasitopological group with countable δ-character, but not first-countable?

Problem 6.2. Is it true that there exists a completely regular bisequential non-first-countable quasitopological group with a first-countable remainder?

Problem 6.3. Is it true that there exists a completely regular quasitopological group G with countable π-character, but without countable δ-character and such that G, in addition, satisfies at least one of one of the following properties:

1) G is sequential;
2) G is Fréchet-Urysohn;
3) any remainder of G is not Lindelöf and it is not pseudocompact;
4) G has a first-countable remainder in some compactification.

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