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Minimal m-handle decomposition of three-dimensional handlebodies

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Abstract. For the 3-dimensional handlebody we build an m-handle decomposition with minimal number of handles and prove a criterion of minimality. It is proved that two functions can be connected by a path in the m-function space without inner critical points on the solid torus if they have the same number of critical points of each index.

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Let M be a three-dimensional handlebody, i. e. a closed bounded domain in the Euclidean space whose boundary is a smooth closed surface $F = \partial M$. In this paper, we consider m-functions without inner critical points on M. For such functions, the restriction of the function on the boundary is a Morse function. The index of critical points of the Morse function is defined as the index of the quadratic form of Hesse (Hessian matrix of this form consists of the second partial derivatives at the critical point). In addition, the direction of the gradient field is given by the sign ($\epsilon = \pm 1$) at the critical point. The index of a critical point of an m-function is the pair (the index of the restriction to the boundary, the number of ϵ). $\epsilon = -1$ if the gradient field is directed to inside of the manifold and $\epsilon = +1$ if it is directed outside. Note that, similar to a Morse function on a closed manifold, m-functions exist and form an open set in the space of all functions.

V. Sharko [1] and S. Maksymenko [2] proved that two Morse functions can be connected by a path in the space of Morse functions on a closed two-dimensional manifold if and only if the functions have the same number of critical points of each index.

Topological properties of the m-functions and the m-handle decomposition were investigated in [3, 4]. In [5] using m-handles the authors give a criterion of the existence of a path between two m-functions on the three-dimensional body without inner critical points.

The aim of this work is to construct an m-handle decomposition of the handlebody with a minimal number of m-handles of each index, to study the conditions when the decomposition is minimal, and to apply the minimal handle decompositions for the homotopy classification of m-functions without inner critical points on the solid torus.

1 m-handle decompositions

Let us start with the handle decomposition of a closed surface F. A handle of index λ is the product $H^\lambda = D^\lambda \times D^{2-\lambda}$. The curve $\partial D^\lambda \times D^{2-\lambda}$ is called a gluing curve, and $D^\lambda \times \partial D^{2-\lambda}$ is called an inner curve. Thus, the gluing curve is 1) $\emptyset$ for a handle of index 0, 2) a pair of segments for a handle of index 1, and 3) a circle for a handle of index 2. It is known from Morse Theory that if a function $g : F \to \mathbb{R}$ has one critical point on the interval $[y, z]$ in the inner segment and only one critical point of index λ takes this value, then $g^{-1}(z) \cong g^{-1}(y) \cup_\varphi H^\lambda$ is obtained from $g^{-1}(y)$ by attaching a handle of index λ for some embedding $\varphi : \partial D^\lambda \times D^{2-\lambda} \to \partial g^{-1}(y)$.

m-handles can be obtained from ordinary handles by the multiplication with the interval $[0, 1]$. Denote them H^λ_+ and H^λ_-. Thus, $H^\lambda_+ \cong H^\lambda_- \cong D^\lambda \times D^{2-\lambda} \times [0, 1]$.

The boundary ∂H^λ_- of a handle of index $(\lambda, -1)$ is divided into three parts:

1) the outside region $D^\lambda \times D^{2-\lambda} \times 0$,

2) the attaching region $\partial D^\lambda \times D^{2-\lambda} \times [0, 1]$,

3) the inside region $D^\lambda \times \partial D^{2-\lambda} \times [0, 1] \cup D^\lambda \times D^{2-\lambda} \times 1$.

The boundary ∂H^λ_+ of a handle of index $(\lambda, +1)$ is divided into two parts:

1) the outside region $D^\lambda \times D^{2-\lambda} \times 1$,

2) the attaching region $\partial(D^\lambda \times D^{2-\lambda}) \times [0, 1] \cup D^\lambda \times D^{2-\lambda} \times 0$.

As a result of m-handle attaching, the boundary consists of inside and outside regions. Their common boundary is called a corner of the manifold. The attaching region of next handles is embedded in the inside region. Moreover, $\partial D^\lambda \times D^{2-\lambda} \times 0$ is embedded in the corner for handles of index $(\lambda, -1)$ and $\partial D^\lambda \times D^{2-\lambda} \times 1$ is embedded in the corner for handles of index $(\lambda, +1)$. Thus, outside regions of m-handles give a handle decomposition of the surface F. Moreover, the union of attaching regions is equal to the union of inside regions.

Like regular handle decompositions, one can perform the following operations with m-handles:

1. A permutation of handles – if two handles are disjoint, they can be attached in any order.

2. An isotopy of the attaching map of handles, if one of $(1, \pm 1)$ handle slides over another $(1, \pm 1)$ handle, in this case we say that it is added to this handle.

3. A reduction of pairs of additional handles – if a handle of index $(1, -1)$ intersects a $(0, -1)$- or $(2, 1)$-handle along a 2-disk, then the pair of handles can be reduced (we can build another handle decomposition without these two handles). Similarly, a pair consisting of a $(1, +1)$-handle and a $(0, +1)$- or a $(2, +1)$-handle whose intersection is an interval can be reduced. The inverse operation to the reduction is the introduction of pairs of additional handles.

Note that m-handles will be additional if they have the same sign of ϵ and that additional handles are on the edge of their limits.

A criterion for homotopy equivalence of functions was proved in [5]:

Theorem 1. *Two functions on the three-dimensional handlebody are homotopy equivalent if and only if they have the same number of handles for each index and the m-handle decomposition of one manifold can be obtained from the m-handle decomposition of another one using isotopy, permutations, additions, reductions and the introduction of pairs of additional handles.*

Our next task will be for an arbitrary handle decomposition using operations 1) − 3) to build a minimal handle decomposition and investigate its topological properties.

2 Minimal m-handle decomposition

In the beginning, from an arbitrary handle decomposition we construct a decomposition with minimal number of handles for each index. Since the boundary of a manifold is connected, then for each (0, ± 1)-handle, except the first one, there exists an additional (1, ± 1)-handle. If they are of the same sign, then this pair of handles can be reduced. Two $(0,-1)$-handles can not be connected by a $(1,+1)$-handle. However, a $(0,+1)$-handle can be connected by a $(1,-1)$-handle with other handles. In this case, this pair of handles is equivalent to a simple 1-handle on a 3-manifold. Similarly, $(2,\pm1)$-handles, exept one $(2,+1)$-handle, can be connected by $(1,\pm1)$-handles. Pairs of the same sign are reduced, and the pair of $(1,+1)$- and $(2,-1)$-handles is equivalent to a simple 2-handle. If the pair of $(0,+1)$- and $(1,-1)$-handles forms a simple 1-handle, the unglued part of the border of the $(0,+1)$-handle in the inside region is on the border of the surface. Then every $(1,+1)$-handle, with attached at least one end to the boundary components, can be made additional $(0,1)$-handles. We do the same in the case of a simple 2-handle. Thus, we construct an m-handle decomposition with one $(0,-1)$-handle, without $(0,+1)$- and $(2,-1)$-handles and one $(2,+1)$-handle. Obviously, this decomposition has no additional pairs of handles. A handle decomposition that does not contain pairs of additional handles or handles which can be made additional after an isotopy, is called minimal.

Theorem 2. *A handle decomposition is minimal if and only if it contains by one* $(0,-1)$- *and* $(2,+1)$-*handles and no* $(0,+1)$- *and* $(2,-1)$-*handles.*

Proof. Necessity. It follows from the previous discussion that if a handle decomposition has more than one $(0,-1)$- or $(2,+1)$-handle or has $(0,+1)$- and $(2,-1)$-handles, all these handles can be reduced. At the same time as the boundary of a manifolds is compact and the restriction of any function on the boundary has a minimum point (of index 0) and a maximum point (of index 2), so the corresponding handle decomposition has handles of index 0 and 2 and the m-handle decomposition has handles of indexes $(0,-1)$ and $(2,+1)$.

Sufficiency. Let an m-handle decomposition have by one $(0,-1)$- and $(2,+1)$-handles and no $(0,+1)$- and $(2,-1)$-handles. Since $(0,-1)$- and $(2,+1)$-handles can not be reduced, they are not additional for other handles. The remaining $(1,\pm1)$-handles can not be reduced because they can not have additional handles. Thus, the handle decomposition is minimal. □

3 m-functions on the solid torus

On the solid torus we fix a parallel u, which is a curve on the boundary that defines the generators of the fundamental group of the solid torus. We also fix a meridian v which is a curve on the boundary that intersects transversally one parallel at one point and is the boundary of a 2-disk on the solid torus. We fix the orientation of these curves.

Theorem 3. *Two m-functions without inner critical points on the solid torus can be connected by a m-function space without inner critical points if they have the same number of critical points of each index.*

Proof. Necessity follows from Theorem 1.

Sufficiency. Let the functions have the same number of points of each index. Construct from them a minimal m-handle decomposition. Theorem 2 implies that such a decomposition has four m-handles whose indexes are $(0,-1)$, $(1,-1)$, $(1,1)$ and $(2,+1)$.

Consider the union of $(0,1)$- and $(1,-1)$-handles for the first function. Let L be the intersection of the boundary of their union with the inside region of the union of $(0,1)$- and $(1,1)$-handles. Two components of the boundary ∂L are homotopic to the meridian u in the solid torus. Let w_1 be one of the two components. We choose its orientation to be parallel to the meridian. Then $[w_1] = [u] + n_1[v]$ in the one-dimensional homology group of the torus. For the second function, by analogy, we have $[w_2] = [u] + n_2[v]$.

Since L is homeomorphic to a cylinder $S^1 \times [0,1]$, and the attaching points of $(1,+1)$-handle are on different bases of the cylinder (because after removing attaching area of this handle from L it should remain a two-dimensional disk) an isotopy of the attaching point of a $(1,1)$-handle can ensure that the intersection of the $(1,-1)$- and $(1,+1)$-handles is empty.

Then we change the order of attaching handles so that a $(1,+1)$-handle be the first attached one. The inside region of the union of $(0,-1)$- and $(1,1)$-handles is homeomorphic to two 2-disks the boundaries of which γ_1 and γ_2 are homotopic to the meridian v on the torus. Slipping one of the two attaching points of $(1,-1)$-handle $n_2 - n_1$ times in one of two directions along γ_1 and γ_2, achieve that $[w_1] = [w_2]$. We have that the curves of w_1 and w_2 are isotopic. Then the m-handle decompositions for two functions are isotopic, too. Applying Theorem 1 completes the proof. □

4 Conclusion

The m-handle decomposition expansion with the minimum number of handles has been built and a criterion of minimality has been proved for the m-handle decomposition of the 3-dimensional handlebody. This construction allowed us to prove that two functions can be connected by a path in the m-function space without inner critical points on the solid torus if and only if they have the same number of critical points of each index.

The authors expect that the minimal m-handle decomposition can be used for homotopy classification of m-functions on other handlebodies. However, in this case one may have a lot of non-isotopic minimal m-handle decompositions.

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