

## Applications of algebraic methods in solving the center-focus problem

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**Abstract.** The nonlinear differential system  $\dot{x} = \sum_{i=0}^{\ell} P_{m_i}(x, y)$ ,  $\dot{y} = \sum_{i=0}^{\ell} Q_{m_i}(x, y)$  is considered, where  $P_{m_i}$  and  $Q_{m_i}$  are homogeneous polynomials of degree  $m_i \geq 1$  in  $x$  and  $y$ ,  $m_0 = 1$ . The set  $\{1, m_i\}_{i=1}^{\ell}$  consists of a finite number ( $l < \infty$ ) of distinct integer numbers. It is shown that the maximal number of algebraically independent focal quantities that take part in solving the center-focus problem for the given differential system with  $m_0 = 1$ , having at the origin of coordinates a singular point of the second type (center or focus), does not exceed  $\varrho = 2(\sum_{i=1}^{\ell} m_i + \ell) + 3$ . We make an assumption that the number  $\omega$  of essential conditions for center which solve the center-focus problem for this differential system does not exceed  $\varrho$ , i. e.  $\omega \leq \varrho$ .

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### Table of contents

1. Introduction
2. Graded algebras of comitants of the system (1)
3. Krull dimension for Sibirsky graded algebras
4. Hilbert series for Sibirsky graded algebras  $S_{1, m_1, m_2, \dots, m_{\ell}}$  and  $SI_{1, m_1, m_2, \dots, m_{\ell}}$
5. Lie algebras of operators admitted by polynomial differential systems
6. An invariant variety in the center-focus problem of the system  $s(1, m_1, m_2, \dots, m_{\ell})$
7. Null focal pseudo-quantity of the system (25) and relations between the quantities  $G_k$  and the focal quantities  $L_k$  of the system (24)
8. Determining the quantities  $G_1, G_2, G_3$  from the systems (36)–(41) and the corresponding focal quantities
9. General type of comitants which have as coefficients expressions with generalized focal pseudo-quantities of the system (34)
10. Comitants which have as coefficients expressions with generalized focal pseudo-quantities of the system  $s(1, 2, 3)$
11. Graded algebra of comitants whose spaces contain comitants associated to generalized focal pseudo-quantities of the systems (34) and (62)
12. Main results

## 1 Introduction

The nonlinear differential system

$$\frac{dx}{dt} = \sum_{i=0}^{\ell} P_{m_i}(x, y), \quad \frac{dy}{dt} = \sum_{i=0}^{\ell} Q_{m_i}(x, y) \quad (1)$$

is considered, where  $P_{m_i}$  and  $Q_{m_i}$  are homogeneous polynomials of degree  $m_i \geq 1$  in  $x$  and  $y$ ,  $m_0 = 1$ . The set  $\{1, m_1, m_2, \dots, m_\ell\}$  consists of a finite number ( $l < \infty$ ) of distinct natural numbers. The coefficients and variables in polynomials  $P_{m_i}$  and  $Q_{m_i}$  take values from the field of the real numbers  $\mathbb{R}$ .

It is known that if the roots of characteristic equation of the singular point  $O(0, 0)$  of the system (1) are imaginary, then the singular point  $O$  is a center (surrounded by closed trajectories) or a focus (surrounded by spirals) [1,5]. In this case the origin of coordinates is a singular point of the second type.

Hereafter we denote the system (1) by  $s(1, m_1, m_2, \dots, m_\ell)$ .

The center-focus problem can be formulated as follows: *Let for the system  $s(1, m_1, m_2, \dots, m_\ell)$  the origin of coordinates be a singular point of the second type (center or focus). Find the conditions which distinguish center from focus.* This problem was posed by A. Poincaré [1,2]. The basic results were obtained by A. M. Lyapunov [5]. It was shown that the conditions for center are the vanishing of an infinite sequence of polynomials (focal quantities)

$$L_1, L_2, \dots, L_k, \dots \quad (2)$$

in the coefficients of right side of the system (1). If at least one of the quantities (2) is not zero, then the origin of coordinates for the system (1) is a focus. These conditions are necessary and sufficient.

In the case of the system (1) from Hilbert's theorem on the finiteness of basis of polynomial ideals it follows that in the mentioned sequence (2) only a finite number of conditions for center *are essential*, the rest are consequences of them. Then the center-focus problem for the system (1) takes the following formulation: *How many polynomials (essential conditions for center)*

$$L_{n_1}, L_{n_2}, \dots, L_{n_\omega}, \dots \quad (n_i \in \{1, 2, \dots, k, \dots\}; i = \overline{1, \omega}; \omega < \infty) \quad (3)$$

*from (2) must be equal to zero in order that all other polynomials (2) would vanish?*

The problem of determining the number  $\omega$  of essential conditions for center (3) is complicated. It is completely solved for the systems  $s(1, 2)$  and  $s(1, 3)$  [8,11], for which we have respectively  $\omega = 3$  and 5. Until now  $\omega$  has not been known for the system  $s(1, 2, 3)$ . There exists only a Zolâdek hypothesis, which is based mostly on intuition, that for the system  $s(1, 2, 3)$  the number  $\omega \leq 13$ . To the present day this hypothesis has not been disproved. But in [12] it is proved that for the system  $s(1, 2, 3)$  12 focal quantities are not enough for solving the center-focus problem in the complex plane.

It is natural to ask why there is still no answer about the value of  $\omega$  from (3) for any system  $s(1, m_1, m_2, \dots, m_\ell)$ ?

We can explain this failure as follows: searching for a finite  $\omega$  from (3), till now the researchers have used basically a known approach, i. e. with the help of certain calculations they constructed the explicit form of the first focal quantities from (2), without knowing a priori the number  $\omega$ . Sometimes the existence of some geometric properties for the system (1) was assumed, for example, the existence of integral straight lines, conics and other curves. Then with their help the attempts were made to show that the vanishing of the available quantities implies the vanishing of other members of the sequence (2), often there was only a vague idea about their expressions.

This approach gave quite unsatisfactory results. One of the reasons is due to the enormous computing for focal quantities, which can not be overcome using super-computers even for the system  $s(1, 2, 3)$ , not to mention more complicated systems  $s(1, m_1, m_2, \dots, m_\ell)$ . Therefore, it is clear that the results obtained in this direction refer more to the systems (1) of special forms.

From what has been said above the following conclusion can be drawn: solving the center-focus problem is equivalent to finding the essential conditions for center (3), that requires knowledge of the number  $\omega$ , the finiteness of which follows from Hilbert's theorem on the finiteness of basis of polynomial ideals.

Therefore, the problem of finding the number  $\omega < \infty$  or obtaining for it an argued numerical upper bound (even as a hypothesis), which is still absent, is a very important condition of the complete solving of the center-focus problem for the system (1).

The last affirmation can be considered as a generalized center-focus problem for the systems  $s(1, m_1, m_2, \dots, m_\ell)$ , and obtaining an answer to it will be qualified as perhaps one of the sufficient conditions in solving the mentioned problem.

## 2 Graded algebras of comitants of the system (1)

In [5,6,7] the type of center-affine polynomial comitant with respect to the center-affine group  $GL(2, \mathbb{R})$  for any differential system  $s(m_0, m_1, m_2, \dots, m_\ell)$  was determined, and it is denoted as follows:

$$(d) = (\delta, d_0, d_1, \dots, d_\ell), \tag{4}$$

where  $\delta$  is the degree of homogeneity of this comitant in phase variables  $x, y$ , and  $d_i$  ( $i = \overline{1, \ell}$ ) is the degree of homogeneity of the same comitant in the coefficients of the polynomials  $P_{m_i}(x, y)$ ,  $Q_{m_i}(x, y)$  from the right side of the system (1).

In [7] the following affirmations were proved:

**Proposition 1.** *The set of center-affine comitants of the system (1) of the same type (4) forms a finite linear space  $V_{m_0, m_1, m_2, \dots, m_\ell}^{(d)}$ , i. e. it has a finite maximal system of linearly independent comitants of the given type (linear basis), all the rest are linearly expressed through them.*

**Proposition 2.** *In order that any homogeneous polynomial of the type (4) in phase variables and coefficients of the system (1) would be a center-affine comitant of this system, it is necessary and sufficient that it be an unimodular comitant (invariant polynomial with respect to the unimodular group  $SL(2, \mathbb{R})$ ) of the same type (4) for the given system.*

**Proposition 3.** [6] *For any center-affine comitant of differential system of the type (4) the following equality holds:*

$$2g = \sum_{i=0}^{\ell} d_i(m_i - 1) - \delta, \quad (5)$$

where  $g$  is usually called the weight of given comitant, and it is an integer number.

Following Propositions 1–2 and according to [7] we denote the space of unimodular comitants of the type (4) for the system  $s(1, m_1, m_2, \dots, m_\ell)$  by

$$S_{1, m_1, m_2, \dots, m_\ell}^{(d)} \cong V_{1, m_1, m_2, \dots, m_\ell}^{(d)}.$$

Let us consider the linear space

$$S_{1, m_1, m_2, \dots, m_\ell} = \sum_{(d)} S_{1, m_1, m_2, \dots, m_\ell}^{(d)}, \quad (6)$$

which is a graded algebra of comitants of the system  $s(1, m_1, m_2, \dots, m_\ell)$ , where its components satisfy the inclusion

$$S_{1, m_1, m_2, \dots, m_\ell}^{(d)} S_{1, m_1, m_2, \dots, m_\ell}^{(e)} \subseteq S_{1, m_1, m_2, \dots, m_\ell}^{(d+e)}, \quad S_{1, m_1, m_2, \dots, m_\ell}^{(0)} = \mathbb{R}.$$

We denote by  $SI_{1, m_1, m_2, \dots, m_\ell}$  a graded algebra of unimodular invariants (comitants that do not depend on the phase variables  $x, y$ ) of the system  $s(1, m_1, m_2, \dots, m_\ell)$ , which satisfies the inclusion

$$SI_{1, m_1, m_2, \dots, m_\ell} \subset S_{1, m_1, m_2, \dots, m_\ell}. \quad (7)$$

As for the first time the comitants and invariants for systems of the form (1) were introduced by K. S. Sibirsky [14], hereafter we will refer to these and similar algebras as *Sibirsky algebras*.

### 3 Krull dimension for Sibirsky graded algebras

From the theory of invariants and tensors [5,13] it results that the Sibirsky graded algebras  $S_{1, m_1, m_2, \dots, m_\ell}$  and  $SI_{1, m_1, m_2, \dots, m_\ell}$  are commutative and finitely determined algebras. If for these algebras we introduce a unified notation  $A$ , then the last affirmation can be written as

$$A = \langle a_1, a_2, \dots, a_m \mid f_1 = 0, f_2 = 0, \dots, f_n = 0 \rangle \quad (m, n < \infty), \quad (8)$$

where  $a_i$  are generators for this algebra, and  $f_j$  are defining relations (syzygies).

It is known from [7] that for the simplest differential system  $s(0, 1)$  of the form

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + ex + fy \quad (9)$$

the finitely defined graded algebras of comitants  $S_{0,1}$  and invariants  $SI_{0,1}$  can be written respectively

$$\begin{aligned} S_{0,1} &= \langle i_1, i_2, i_3, k_1, k_2, k_3 \mid (i_1k_1 - k_3)^2 + k_3^2 - i_2k_1^2 - 2i_3k_2 = 0 \rangle, \\ SI_{0,1} &= \langle i_1, i_2, i_3 \rangle, \end{aligned} \quad (10)$$

where

$$\begin{aligned} i_1 &= c + f, \quad i_2 = c^2 + 2de + f^2, \quad i_3 = -ea^2 + (c - f)ab + db^2, \\ k_1 &= -bx + ay, \quad k_2 = -ex^2 + (c - f)xy + dy^2, \\ k_3 &= -(ea + fb)x + (ca + db)y. \end{aligned} \quad (11)$$

We note that using the system (9) the whole theory of center-affine (unimodular) comitants and invariants for two-dimensional polynomial differential systems can be illustrated.

**Definition 1.** [15] *Elements  $a_1, a_2, \dots, a_r$  of the algebra  $A$  are called algebraically independent if for any non-trivial polynomial  $F$  in these  $r$  elements the following inequality holds:*

$$F(a_1, a_2, \dots, a_r) \neq 0.$$

**Definition 2.** *The maximal number of algebraically independent elements of an graded algebra  $A$  is called the Krull dimension of this algebra and is denoted by  $\varrho(A)$ .*

It is known [7] that for an algebra  $A$  of the form (8) the equality  $n = m - \varrho(A)$  holds. However, this equality is not very effective because it is impossible to determine the numbers  $m$  and  $n$  for most algebras of invariants and comitants for systems of the form (1).

In the classical theory of invariants [16] a set of elements  $a_1, a_2, \dots, a_{\varrho(A)}$  from  $A$  which define the Krull dimension of the algebra  $A$  is called *an algebraic basis*. This means that for any  $a \in A$  ( $a \neq a_j$ ) there exists a natural number  $p$  such that the following identity holds:

$$P_0 a^p + P_1 a^{p-1} + \dots + P_p = 0, \quad (12)$$

where  $P_k$  ( $k = \overline{0, p}$ ) are polynomials in  $a_j$  ( $j = \overline{1, \varrho(A)}$ ). We note that in general  $P_0 \neq 1$ .

If for any  $a \in A$  in (12) we have  $P_0 \equiv 1$ , then this basis is called *integer algebraic basis*. The existence a basis was shown by D. Hilbert (see [16]). We denote the number of its elements by  $\varrho'(A)$ .

We note that in general the numbers of elements in the mentioned bases does not always coincide. For example, from [7] we have that for the system  $s(4)$  the Krull dimension  $\varrho(SI_4) = 7$ , but from [17] for the same system we obtain that the number of elements in the integer algebraic basis of the same algebra is  $\varrho'(SI_4) = 9$ ,

i. e.  $\varrho(SI_4) < \varrho'(SI_4)$ . From [7] we have that for the system  $s(0,1)$  the equality  $\varrho(S_{0,1}) = \varrho'(S_{0,1}) = 5$  holds, and  $\varrho(SI_{0,1}) = \varrho'(SI_{0,1}) = 3$ . Also from [5,6,7] and [18] it follows that for the systems  $s(2)$  and  $s(3)$  we have  $\varrho(SI_2) = \varrho'(SI_2) = 3$ ,  $\varrho(SI_3) = \varrho'(SI_3) = 5$ . From [7] and [19] we find that for the system  $s(1,2)$  the equalities  $\varrho(SI_{1,2}) = \varrho'(SI_{1,2}) = 7$  are valid. However, for the system  $s(1,2,3)$  according to [7,20] we have that  $\varrho(SI_{1,2,3}) = 15$ , but  $\varrho'(SI_{1,2,3}) = 21$ .

The mentioned examples lead us to the relation

$$\varrho(A) \leq \varrho'(A).$$

This inequality accentuates that the integer algebraic basis contains an algebraic basis of an algebra  $A$ . The proof of this fact can be easily obtained by an indirect proof.

*Remark 1.* The main property of an integer algebraic basis of an algebra  $A$  of invariants is that it is the minimum number of elements of the algebra  $A$  such that if they are equal to zero, all elements of the algebra  $A$  vanish.

Hereafter we need some evident affirmations:

**Proposition 4.** *If  $B$  is a graded subalgebra of an algebra  $A$ , then between the Krull dimensions of these algebras the following inequality holds:*

$$\varrho(B) \leq \varrho(A).$$

It is evident

**Proposition 5.** *If the Krull dimension of an algebra  $A$  is  $\varrho(A)$ , then on any variety  $V = \{a = 0, b < 0\}$  with fixed  $a, b \in A$  ( $b$  has no effect on the mentioned variety) in the algebra  $A$  there are not more than  $\varrho(A)$  algebraically independent elements (possibly no more than  $\varrho(A)$  elements which form an integer algebraic basis) of this algebra.*

#### 4 Hilbert series for Sibirsky graded algebras $S_{1,m_1,m_2,\dots,m_\ell}$ and $SI_{1,m_1,m_2,\dots,m_\ell}$

According to Proposition 1 for the spaces of the algebra  $S_{1,m_1,m_2,\dots,m_\ell}$  from (5) we have  $\dim_{\mathbb{R}} S_{1,m_1,m_2,\dots,m_\ell}^{(d)} < \infty$ . Then, following [7], by the *generalized Hilbert series* of the algebra  $S_{1,m_1,m_2,\dots,m_\ell}$  we mean a formal series

$$H(S_{1,m_1,m_2,\dots,m_\ell}; u, z_0, z_1, \dots, z_\ell) = \sum_{(d)} \dim_{\mathbb{R}} S_{1,m_1,m_2,\dots,m_\ell}^{(d)} u^\delta z_0^{d_0} z_1^{d_1} \dots z_\ell^{d_\ell}, \quad (13)$$

which is said to reflect a  $u, z$ -graduation of the considered algebra.

From the definition of the algebra of invariants  $SI_{1,m_1,m_2,\dots,m_\ell}$  and (13) it follows that

$$H(SI_{1,m_1,m_2,\dots,m_\ell}, z_0, z_1, \dots, z_\ell) = H(S_{1,m_1,m_2,\dots,m_\ell}, 0, z_0, z_1, \dots, z_\ell), \quad (14)$$

and the *common Hilbert series* will be written respectively

$$\begin{aligned} H_{S_{1,m_1,m_2,\dots,m_\ell}}(u) &= H(S_{1,m_1,m_2,\dots,m_\ell}, u, u, u, \dots, u), \\ H_{SI_{1,m_1,m_2,\dots,m_\ell}}(z) &= H(SI_{1,m_1,m_2,\dots,m_\ell}, z, z, \dots, z). \end{aligned} \quad (15)$$

The last series contain meaningful information about asymptotic character of the behavior of the considered algebras.

The method of construction of the generalized Hilbert series (13)–(15) for the algebras  $S_{1,m_1,m_2,\dots,m_\ell}$  and  $SI_{1,m_1,m_2,\dots,m_\ell}$  was developed in [7].

For example, the generalized Hilbert series for the algebras  $S_{0,1}$  and  $SI_{0,1}$  of unimodular comitants and invariants of the system  $s(0,1)$  have, respectively, the forms

$$H(S_{0,1}, u, z_0, z_1) = \frac{1 + uz_0z_1}{(1 - uz_0)(1 - z_1)(1 - z_1^2)(1 - z_0^2z_1)(1 - u^2z_1)},$$

$$H(SI_{0,1}, z_0, z_1) = \frac{1}{(1 - z_1)(1 - z_1^2)(1 - z_0^2z_1)},$$

and the corresponding common Hilbert series will be written as

$$H_{S_{0,1}}(u) = \frac{1 - u + u^2}{(1 - u)^2(1 - u^2)(1 - u^3)^2}, \quad H_{SI_{0,1}}(z) = \frac{1}{(1 - z)(1 - z^2)(1 - z^3)}.$$

*Remark 2.* We note, following [21], that the Krull dimension  $\varrho(S_{1,m_1,m_2,\dots,m_\ell})$  (respectively  $\varrho(SI_{1,m_1,m_2,\dots,m_\ell})$ ) of the graded algebra  $S_{1,m_1,m_2,\dots,m_\ell}$  (respectively  $SI_{1,m_1,m_2,\dots,m_\ell}$ ) is equal to the multiplicity of the pole of the common Hilbert series  $H_{S_{1,m_1,m_2,\dots,m_\ell}}(u)$  (respectively  $H_{SI_{1,m_1,m_2,\dots,m_\ell}}(z)$ ) at the unit.

For example, considering the above mentioned common Hilbert series  $H_{S_{0,1}}(u)$  and  $H_{SI_{0,1}}(z)$  for the Krull dimension of the algebras  $S_{0,1}$  and  $SI_{0,1}$  we obtain  $\varrho(S_{0,1}) = 5$  and  $\varrho(SI_{0,1}) = 3$ , respectively.

In other cases, when there is no explicit form of the common Hilbert series, but the power series expansion is known, then we can use the following

*Remark 3.* Accept that the comparison of series with non-negative coefficients is performed coefficient-wise ( $\sum a_n t^n \leq \sum b_n t^n \Leftrightarrow a_n \leq b_n; \forall n$ ). Taking this into account, if for commutative graded algebras  $A$  and  $B$  we have

$$H_A(t) \leq H_B(t), \tag{16}$$

then for their Krull dimensions we also have  $\varrho(A) \leq \varrho(B)$ .

It is also evident that if for the common Hilbert series of a commutative graded algebra  $A$  we have

$$H_A(t) \leq \frac{C}{(1 - t)^m}, \tag{17}$$

where  $C$  is a fixed constant, then we obtain  $\varrho(A) \leq m$ .

The extended theory and bibliography about Hilbert series for graded algebras can be found in [22].

## 5 Lie algebras of operators admitted by polynomial differential systems

It is shown in [7] that any differential system  $s(m_0, m_1, m_2, \dots, m_\ell)$  from (1) admits a four-dimensional reductive Lie algebra  $L_4$ , which consists of operators

$$X_1 = x \frac{\partial}{\partial x} + D_1, \quad X_2 = y \frac{\partial}{\partial x} + D_2, \quad X_3 = x \frac{\partial}{\partial y} + D_3, \quad X_4 = y \frac{\partial}{\partial y} + D_4, \quad (18)$$

where the differential operators  $D_1, D_2, D_3, D_4$  are operators of the representation of the center-affine group  $GL(2, \mathbb{R})$  in the space of the coefficients of the polynomials  $P_{m_i}$  and  $Q_{m_i}$  ( $i = \overline{1, \ell}$ ) of the system (1).

In [7] it is proved

**Theorem 1.** *For a polynomial  $k$  in the coefficients of the system  $s(m_0, m_1, m_2, \dots, m_\ell)$  from (1) and phase variables  $x, y$  to be a center-affine comitant of this system with the weight  $g$ , it is necessary and sufficient that it satisfies the equations*

$$X_1(k) = X_4(k) = -gk, \quad X_2(k) = X_3(k) = 0.$$

With the help of this theorem and properties of rational absolute center-affine comitants of the system (1) from [7], following the classical theory of these invariants [16], it can be shown that for the number of elements in an algebraic basis of center-affine comitants of the system  $s(m_0, m_1, m_2, \dots, m_\ell)$  the following formula holds:

$$\varrho = 2 \left( \sum_{i=0}^{\ell} m_i + \ell \right) + 1. \quad (19)$$

In the theory of center-affine comitants of polynomial differential systems [6] it is shown that if  $S$  is a *semi-invariant* in the center-affine comitant  $k$ , then

$$k = Sx^\delta - D_3(S)x^{\delta-1}y + \frac{1}{2!}D_3^2(S)x^{\delta-2}y^2 + \dots + \frac{(-1)^\delta}{\delta!}D_3^\delta(S)y^\delta, \quad (20)$$

where  $D_3$  is defined in [7].

*Remark 4.* [6] With the help of this equality it can be shown that the center-affine comitants  $k_1, k_2, \dots, k_{\varrho(S_{m_0, m_1, m_2, \dots, m_\ell})} \in S_{m_0, m_1, m_2, \dots, m_\ell}$  which belong to the system  $s(m_0, m_1, m_2, \dots, m_\ell)$  are algebraically independent if and only if their semi-invariants are algebraically independent.

## 6 An invariant variety in the center-focus problem of the system $s(1, m_1, m_2, \dots, m_\ell)$

The center-focus problem for systems of the form (1) has the following classical formulation: for an infinite system of polynomials

$$\{(x^2 + y^2)^k\}_{k=1}^{\infty} \quad (21)$$



there exists a function

$$U(x, y) = x^2 + y^2 + \sum_{k=3}^{\infty} f_k(x, y), \quad (22)$$

where  $f_k(x, y)$  are homogeneous polynomials of degree  $k$  in  $x, y$ , and such constants

$$L_1, L_2, \dots, L_k, \dots \quad (2)$$

that the identity

$$\frac{dU}{dt} = \sum_{k=1}^{\infty} L_k (x^2 + y^2)^{k+1} \quad (23)$$

(with respect to  $x$  and  $y$ ) holds along the trajectories of the system

$$\dot{x} = y + \sum_{i=1}^{\ell} P_{m_i}(x, y), \quad \dot{y} = -x + \sum_{i=1}^{\ell} Q_{m_i}(x, y). \quad (24)$$

The constants (2) are polynomials in coefficients of the system (24), and are called *focal quantities*.

We note that the algebra  $S_{1, m_1, m_2, \dots, m_\ell}$  for any differential system  $s(1, m_1, m_2, \dots, m_\ell)$ , written in the form

$$\dot{x} = cx + dy + \sum_{i=1}^{\ell} P_{m_i}(x, y), \quad \dot{y} = ex + fy + \sum_{i=1}^{\ell} Q_{m_i}(x, y) \quad (25)$$

contains among its generators the polynomials

$$i_1 = c + f, \quad i_2 = c^2 + 2de + f^2, \quad k_2 = -ex^2 + (c - f)xy + dy^2, \quad (26)$$

which are given already in (11).

*Remark 5.* We note that the set

$$\mathcal{V} = \{i_1 = c + f = 0, \text{ Discr}(k_2) = 2i_2 - i_1^2 < 0\} \quad (27)$$

is a Sibirsky invariant variety for center and focus for the system (25), because the comitant  $k_2$  from (26) through a real center-affine transformation of the plane  $xOy$  can be brought to the form

$$x^2 + y^2, \quad (28)$$

and the system (25) can be brought to the form (24) [5], for which the roots of the characteristic equation are imaginary, i. e. the origin of coordinates for this system is a singular point of the second type (center or focus).

Considering Remark 5 we have

*Remark 6.* Taking into account the comitant  $k_2$  from (26) and the fact that its expression through a real center-affine transformation on the invariant variety  $\mathcal{V}$  can be brought to the form (28), then formally this variety for the system (25) can be written as

$$\mathcal{V} = \{f = -c\} \cup \{c = 0, d = -e = 1\}. \quad (29)$$

## 7 Null focal pseudo-quantity of the system (25) and relations between the quantities $G_k$ and the focal quantities $L_k$ of the system (24)

Let us consider for the system (25) the identity

$$\left[ cx + dy + \sum_{i=1}^{\ell} P_{m_i}(x, y) \right] \frac{\partial U}{\partial x} + \left[ ex + fy + \sum_{i=1}^{\ell} Q_{m_i}(x, y) \right] \frac{\partial U}{\partial y} = \sum_{k=1}^{\infty} G_k k_2^{k+1}, \quad (30)$$

where

$$U(x, y) = k_2 + \sum_{r=3}^{\infty} F_r(x, y), \quad (31)$$

( $k_2 \neq 0$  from (26)), which splits by powers of  $x$  and  $y$  into an infinite number of algebraic equations, where the variables are the coefficients of the homogeneous polynomials  $F_r(x, y)$  of degree  $r$  in  $x, y$ , and also the quantities  $G_1, G_2, \dots, G_k, \dots$

For any system (25) from the identity (30) with  $k_2$  from (26) we find that the first three equations have the following form:

$$x^2 : e(c + f) = 0, \quad xy : (c - f)(c + f) = 0, \quad y^2 : d(c + f) = 0.$$

These equalities are equivalent to one of two sets of the conditions: 1)  $c + f = 0$ ; 2)  $e = c - f = d = 0$ . Since  $k_2 \neq 0$ , then, according to (26), these conditions are equivalent to the condition  $c + f = 0$ , which is contained in the variety  $\mathcal{V}$  from (27).

In this way from Definition 3 and formulation of the center-focus problem for the system (24) we conclude: for  $L_k$  from (2) and  $G_k$  from (30) the following equalities take place:

$$L_k = G_k|_{\mathcal{V}} \quad (k = 1, 2, \dots), \quad (32)$$

where  $\mathcal{V}$  is from (27).

Hereafter some concretizations for these equalities will be done.

From the above mentioned follows

*Remark 7.* The identity (30) with function (31) on the variety  $\mathcal{V}$  from (27) guarantees that the system (25) has at the origin of coordinates a singular point of the second type (center or focus).

We denote the expression  $c + f$ , which is contained in the variety  $\mathcal{V}$  from (27), by

$$G_0 \equiv i_1 = c + f, \quad (33)$$

and will call it *the null focal pseudo-quantity*. We note that  $G_0$  from (33) is a center-affine (unimodular) invariant of the system  $s(1, m_1, m_2, \dots, m_{\ell})$  of the type

$$(0, 1, \underbrace{0, \dots, 0}_{\ell}).$$

To get a more clear idea about the quantities  $G_1, G_2, \dots, G_k, \dots$  from the identity (30) with the function (31), we write the remaining equations, in which this identity is

splitted by powers  $x^3, x^2y, xy^2, y^3, \dots$  without taking into consideration the equality  $i_1 = c + f = 0$  on the variety  $\mathcal{V}$ .

To explain the further way of implementation of this scenario, we consider the identity (30) with unknown constants  $G_1, G_2, \dots$  for the example of the simplest differential system  $s(1, 2)$  with the quadratic nonlinearities

$$\begin{aligned} \dot{x} &= cx + dy + gx^2 + 2hxy + ky^2, \\ \dot{y} &= ex + fy + lx^2 + 2mxy + ny^2, \end{aligned} \quad (34)$$

with the finitely defined graded algebra of unimodular comitants  $S_{1,2}$  [7]. For this algebra we write the function (31) as

$$\begin{aligned} U(x, y) &= k_2 + a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3 + b_0x^4 + 4b_1x^3y + \\ &\quad + 6b_2x^2y^2 + 4b_3xy^3 + b_4y^4 + c_0x^5 + 5c_1x^4y + 10c_2x^3y^2 + \\ &\quad + 10c_3x^2y^3 + 5c_4xy^4 + c_5y^5 + d_0x^6 + 6d_1x^5y + 15d_2x^4y^2 + \\ &\quad + 20d_3x^3y^3 + 15d_4x^2y^4 + 6d_5xy^5 + d_6y^6 + e_0x^7 + 7e_1x^6y + \\ &\quad + 21e_2x^5y^2 + 35e_3x^4y^3 + 21e_5x^2y^5 + 7e_6xy^6 + e_7y^7 + f_0x^8 + \\ &\quad + 8f_1x^7y + 28f_2x^6y^2 + 56f_3x^5y^3 + 70f_4y^4 + 56f_5x^3y^5 + \\ &\quad + 28f_6x^2y^6 + 8f_7xy^7 + f_8y^8 + \dots, \end{aligned} \quad (35)$$

where  $k_2$  is from (26) and  $a_0, a_1, \dots, f_7, f_8, \dots$  are unknown constants. Then without taking into consideration the variety  $\mathcal{V}$ , the identity (30) along the trajectories of the system (34) with the function (35) splits into the following systems of equations

$$\begin{aligned} x^3 &: 3ca_0 + 3ea_1 = 2eg - (c - f)l, \\ x^2y &: 3da_0 + 3(2c + f)a_1 + 6ea_2 = (f - c)(g + 2m) - 2dl + 4eh, \\ xy^2 &: 6da_1 + 3(2f + c)a_2 + 3ea_3 = (f - c)(2h + n) + 2ek - 4dm, \\ y^3 &: 3da_2 + 3fa_3 = (f - c)k - 2dn; \end{aligned} \quad (36)$$

$$\begin{aligned} x^4 &: 4cb_0 + 4eb_1 - e^2G_1 = -3ga_0 - 3la_1, \\ x^3y &: 4db_0 + 4(f + 3c)b_1 + 12eb_2 + 2e(c - f)G_1 = -6ha_0 - \\ &\quad - 6(g + m)a_1 - 6la_2, \\ x^2y^2 &: 12db_1 + 12(c + f)b_2 + 12eb_3 + [2de - (c - f)^2]G_1 = \\ &\quad = -3ka_0 - 3(4h + n)a_1 - 3(g + 4m)a_2 - 3la_3, \\ xy^3 &: 12bd_2 + 4(3f + c)b_3 + 4eb_4 + 2d(f - g)G_1 = -6ka_1 - \\ &\quad - 6(h + n)a_2 - 6ma_3, \\ y^4 &: 4db_3 + 4fb_4 - d^2G_1 = -3ka_2 - 3na_3; \\ x^5 &: 5cc_0 + 5ec_1 = -4gb_0 - 4lb_1, \\ x^4y &: 5dc_0 + 5(4c + f)c_1 + 20ec_2 = -8hb_0 - 4(3g + 2m)b_1 - \\ &\quad - 12lb_2, \end{aligned} \quad (37)$$

$$\begin{aligned}
x^3y^2 &: 20dc_1 + 10(3c + 2f)c_2 + 30ec_3 = -4kb_0 - 4(6h + n)b_1 - \\
&\quad - 12(g + 2m)b_2 - 12lb_3, \\
x^2y^3 &: 30dc_2 + 10(2c + 3f)c_3 + 20ec_4 = -12kb_1 - 12(2h + \\
&\quad + n)b_2 - 4(g + 6m)b_3 - 4lb_4, \\
xy^4 &: 20dc_3 + 5(c + 4f)c_4 + 5ec_5 = -12kb_2 - 4(2h + 3n)b_3 - \\
&\quad - 8mb_4, \\
y^5 &: 5dc_4 + 5fc_5 = -4kb_3 - 4nb_4;
\end{aligned} \tag{38}$$

$$\begin{aligned}
x^6 &: 6cd_0 + 6ed_1 + e^3G_2 = -5gc_0 - 5lc_1, \\
x^5y &: 6dd_0 + 6(5c + f)d_1 + 30ed_2 + 3e^2(f - c)G_2 = -10hc_0 - \\
&\quad - 10(2g + m)c_1 - 20lc_2, \\
x^4y^2 &: 30dd_1 + 30(2c + f)d_2 + 60ed_3 + 3e[(c - f)^2 - de]G_2 = \\
&\quad = -5kc_0 - 5(8h + n)c_1 - 10(3g + 4m)c_2 - 30lc_3, \\
x^3y^3 &: 60dd_2 + 60(c + f)d_3 + 60ed_4 + (f - c)[(c - f)^2 - \\
&\quad - 6de]G_2 = -20kc_1 - 20(3h + n)c_2 - 20(g + 3m)c_3 - 20lc_4, \\
x^2y^4 &: 60dd_3 + 30(c + 2f)d_4 + 30ed_5 + 3d[de - (c - f)^2]G_2 = \\
&\quad = -30kc_2 - 10(4h + 3n)c_3 - 5(g + 8m)c_4 - 5lc_5, \\
xy^5 &: 30dd_4 + 6(c + 5f)d_5 + 6ed_6 + 3d^2(f - c)G_2 = -20kc_3 - \\
&\quad - 10(h + 2n)c_4 - 10mc_5, \\
y^6 &: 6dd_5 + 6fd_6 - d^3G_2 = -5kc_4 - 5nc_5;
\end{aligned} \tag{39}$$

$$\begin{aligned}
x^7 &: 7ce_0 + 7ee_1 = -6gd_0 - 6ld_1, \\
x^6y &: 7de_0 + 7(6c + f)e_1 + 42ee_2 = -12hd_0 - 6(5g + 2m)d_1 - \\
&\quad - 30ld_2, \\
x^5y^2 &: 42de_1 + 7(15c + 6f)e_2 + 105ee_3 = -6kd_0 - 6(10h + \\
&\quad + n)d_1 - 60(g + m)d_2 - 60ld_3, \\
x^4y^3 &: 105de_2 + 5(28c + 21f)e_3 + 140ee_4 = -30kd_1 - 30(4h + \\
&\quad + n)d_2 - 60(g + 2m)d_3 - 60ld_4, \\
x^3y^4 &: 140de_3 + 35(3c + 4f)e_4 + 105ee_5 = -60kd_2 - 60(2h + \\
&\quad + n)d_3 - 30(g + 4m)d_4 - 30ld_5, \\
x^2y^5 &: 105de_4 + 7(6c + 15f)e_5 + 42ee_6 = -60kd_3 - 60(h + \\
&\quad + n)d_4 - 6(g + 10m)d_5 - 6ld_6, \\
xy^6 &: 42de_5 + 7(c + 6f)e_6 + 7ee_7 = -30kd_4 - 6(2h + 5n)d_5 - \\
&\quad - 12md_6, \\
y^7 &: 7de_6 + 7fe_7 = -6kd_5 - 6nd_6;
\end{aligned} \tag{40}$$

$$x^8 : 8cf_0 + 8ef_1 - e^4G_3 = -7ge_0 - 7le_1,$$

$$\begin{aligned}
 x^7y &: 8df_0 + 8(7c + f)f_1 + 56ef_2 + 4e^3(c - f)G_3 = \\
 &= -14he_0 - 14(3g + m)e_1 - 42le_2, \\
 x^6y^2 &: 56df_1 + 56(3c + f)f_2 + 168ef_3 + 2e^2[2de - 3(c - f)^2]G_3 = \\
 &= -7ke_0 - 7(12h + n)e_1 - 21(5g + 4m)e_2 - 105le_3, \\
 x^5y^3 &: 168df_2 + 56(5c + 3f)f_3 + 280ef_4 + 4e(f - c)[3de - (c - \\
 &f)^2]G_3 = -42ke_1 - 42(5h + n)e_2 - 70(2g + 3m)e_3 - 140le_4, \\
 x^4y^4 &: 280df_3 + 280(c + f)f_4 + 280ef_5 + [12de(c - f)^2 - 6d^2e^2 - \\
 &(c - f)^4]G_3 = -105ke_2 - 35(8h + 3n)e_3 - 35(3g + \\
 &+ 8m)e_4 - 105le_5, \\
 x^3y^5 &: 280df_4 + 56(3c + 5f)f_5 + 168ef_6 + 4d(f - c)[(c - f)^2 - \\
 &- 3de]G_3 = -140ke_3 - 70(3h + 2n)e_4 - 42(g + 5m)e_5 - 42le_6, \\
 x^2y^6 &: 168df_5 + 56(c + 3f)f_6 + 56ef_7 + 2d^2[2de - 3(c - f)^2]G_3 = - \\
 &- 105ke_4 - 21(4h + 5n)e_5 - 7(g + 12m)e_6 - 7le_7, \\
 xy^7 &: 56df_6 + 8(c + 7f)f_7 + 8ef_8 + 4d^3(f - c)G_3 = -42ke_5 - \\
 &- 14(h + 3n)e_6 - 14me_7, \\
 y^8 &: 78df_7 + 8ff_8 - d^4G_3 = -7ke_6 - 7ne_7.
 \end{aligned} \tag{41}$$

It is evident that the linear systems of equations (36)–(41) in variables  $a_0, a_1, a_2, a_3, b_0, b_1, \dots, b_4, c_0, c_1, \dots, c_5, d_0, d_1, \dots, d_6, e_0, e_1, \dots, e_7, f_0, f_1, \dots, f_8, \dots, G_1, G_2, G_3, \dots$  can be extended by adding, after the last equation from (41), an infinite number of equations, obtained from the equality of coefficients of  $x^\alpha y^\beta$  for  $\alpha + \beta > 8$  in the identity (30) along the trajectories of the system (34).

## 8 Determining the quantities $G_1, G_2, G_3$ from the systems (36)–(41) and the corresponding focal quantities

To obtain the quantity  $G_1$  we write the equations (36)–(37) in the matrix form

$$A_1 B_1 = C_1, \tag{42}$$

where

$$A_1 = \begin{pmatrix} 3c & 3e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3d & 3(2c+f) & 6e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6d & 3(2c+f) & 3e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3d & 3f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3g & 3l & 0 & 0 & 4c & 4e & 0 & 0 & 0 & 0 & -e^2 \\ 6h & 6(g+m) & 6l & 0 & 4d & 4(f+3c) & 12e & 0 & 0 & 2e(c-f) & \\ 3k & 3(4h+n) & 3(g+4m) & 3l & 0 & 12d & 12(c+f) & 12e & 0 & 2de-(d^2) & \\ 0 & 6k & 6(h+n) & 6m & 0 & 0 & 12d & 4(3f+c) & 4l & 2d(f-c) & \\ 0 & 0 & 3k & 3n & 0 & 0 & 0 & 4d & 4f & -d^2 & \end{pmatrix},$$

$$B_1 = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ G_1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 2eg + (f-c)l \\ (f-c)(g+2m) - 2dl + 4eh \\ (f-c)(2h+n) + 3ek - 4dm \\ (f-c)k - 2dn \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (43)$$

Since the dimension of the matrix  $A_1$  is  $9 \times 10$ , clearly we have at least one free parameter. Therefore choosing as a free parameter  $b_i$  ( $i \in \{0, 1, \dots, 4\}$ ) with the help of Cramer's rule from the system (42) for each fixed  $i$  we obtain

$$G_1 = \frac{G_{1,i} + B_{1,i}b_i}{\sigma_{1,i}} \quad (44)$$

where  $G_{1,i}, B_{1,i}, \sigma_{1,i}$  are polynomials in the coefficients of the system (34), and  $b_i$  are undetermined coefficients of the function  $U(x, y)$  from (35).

By studying the matrices (43) of the system (42) we conclude that  $G_{1,i}$  from (44) are homogeneous polynomials of degree 8 with respect to the linear part, and of degree 2 with respect to the quadratic part of the system (34).

Because  $G_{1,i}$  from (44) are homogeneous polynomials in the coefficients of the system (34), then, according to [6,23], for  $i = 0, 1, 2, 3, 4$  we can determine respectively and isobarity

$$(3, -1), (2, 0), (1, 1), (0, 2), (-1, 3).$$

According to the formula (5) (for the system (34) and the theory of invariants of differential systems [5,6]) it suggests that the numerators of the fractions (44) can be coefficients in comitants of the weight  $-1$  of the type  $(4, 8, 2)$ . This means that according to (20) with the help of the Lie differential operator  $D_3$  for the system (34) from [7] and the numerator of the fraction (44) we obtain a redefined system of four linear non-homogeneous differential equations

$$\begin{aligned} D_3(G_{1,0} + B_{1,0}b_0) &= G_{1,1} + B_{1,1}b_1, & D_3(G_{1,1} + B_{1,1}b_1) &= -G_{1,2} - B_{1,2}b_2, \\ -D_3(G_{1,2} + B_{1,2}b_2) &= G_{1,3} + B_{1,3}b_3, & D_3(G_{1,3} + B_{1,3}b_3) &= -G_{1,4} - B_{1,4}b_4 \end{aligned} \quad (45)$$

with five unknowns  $b_0, b_1, b_2, b_3, b_4$ . We can note that a particular solution to this system is  $b_0 = b_1 = b_2 = b_3 = b_4 = 0$ , for which the polynomial

$$f'_4(x, y) = G_{1,0}x^4 + 4G_{1,1}x^3y + 2G_{1,2}x^2y^2 + 4G_{1,3}xy^3 + G_{1,4}y^4 \quad (46)$$

is a center-affine comitant of the system (34). This fact is also confirmed by Theorem 1 with the operators  $X_1 - X_4$  from [7] for the system (34), for which

$$X_1(f'_4) = X_4(f'_4) = f'_4, \quad X_2(f'_4) = X_3(f'_4) = 0.$$

Similarly, one can see that another particular solution for the system (45) is given by the following expressions:

$$\begin{aligned} b_0 &= \frac{-e(g^2 + 2hl + m^2)}{3c^2 - 4de + 10cf + 3f^2}, \\ b_1 &= \frac{(c-f)(g^2 + 2hl + m^2) - 2e(gh + kl + hm + mn)}{4(3c^2 - 4de + 10cf + 3f^2)}, \\ b_2 &= \frac{2(c-f)(gh + kl + hm + mn) - e(h^2 + 2km + n^2) + d(g^2 + 2hl + m^2)}{6(3c^2 - 4de + 10cf + 3f^2)}, \\ b_3 &= \frac{(c-f)(h^2 + 2km + n^2) + 2d(gh + kl + hm + mn)}{4(3c^2 - 4de + 10cf + 3f^2)}, \\ b_4 &= \frac{d(h^2 + 2km + n^2)}{3c^2 - 4de + 10cf + 3f^2}, \end{aligned}$$

whose denominators are different from zero on the variety  $\mathcal{V}$  from (27). They define the center-affine comitant

$$\begin{aligned} f_4''(x, y) &= (G_{1,0} + B_{1,0}b_0)x^4 + 4(G_{1,1} + B_{1,1}b_1)x^3y + 2(G_{1,2} + \\ &+ B_{1,2}b_2)x^2y^2 + 4(G_{1,3} + B_{1,3}b_3)xy^3 + (G_{1,4} + B_{1,4}b_4)y^4. \end{aligned} \quad (47)$$

It is evident that the differential system (45) has an infinite number of solutions  $b_0, b_1, b_2, b_3, b_4$ , which define center-affine comitants of the type (47).

In view of the above, the comitants (46)–(47) belong to the space

$$S_{1,2}^{(4,8,2)}.$$

Remark that the comitants (46)–(47) on the variety  $\mathcal{V}$  from (27) for the system (34) have the following form:

$$f_4'(x, y)|_{\mathcal{V}} = f_4''(x, y)|_{\mathcal{V}} = -8L_1(x^2 + y^2), \quad (48)$$

where

$$L_1 = \frac{1}{2} [g(l-h) - k(h+n) + m(l+n)]$$

is the first focal quantity of the system (34) on the invariant variety  $\mathcal{V}$  (see [4, p. 110]).

Similarly to the previous case, for determining the quantity  $G_2$  we write the equations (36)–(39) in the matrix form

$$A_2 B_2 = C_2, \quad (49)$$

from which we find

$$G_2 = \frac{G_{2,i,j} + B_{2,i,j}b_i + D_{2,i,j}d_j}{\sigma_{2,i,j}}, \quad (i = \overline{0,4}, j = \overline{0,6}). \quad (50)$$

By studying the matrix equality (49) we obtain that  $\deg G_{2,i,j} = 24$ , and using the system (36)–(39) we obtain that  $G_{2,i,j}$  from (50) has the type  $(0, 20, 4)$ , i. e.  $G_{2,i,j}$

are homogeneous polynomials of degree 20 in coefficients of the linear part and of degree 4 in coefficients of the quadratic part of the system  $s(1, 2)$  from (34).

Computing the expressions  $G_{2,i,j}$  for each  $i = \overline{0, 4}$  and  $j = \overline{0, 6}$ , according to [6,23], we obtain for their isobarity the following table:

Table 1

| $G_{2,i,j}$ | $d_0$  | $d_1$  | $d_2$  | $d_3$ | $d_4$  | $d_5$  | $d_6$  |
|-------------|--------|--------|--------|-------|--------|--------|--------|
| $b_0$       | (7,-3) | (6,-2) | (5,-1) | (4,0) | (3,1)  | (2,2)  | (1,3)  |
| $b_1$       | (6,-2) | (5,-1) | (4,0)  | (3,1) | (2,2)  | (1,3)  | (0,4)  |
| $b_2$       | (5,-1) | (4,0)  | (3,1)  | (2,2) | (1,3)  | (0,4)  | (-1,5) |
| $b_3$       | (4,0)  | (3,1)  | (2,2)  | (1,3) | (0,4)  | (-1,5) | (-2,6) |
| $b_4$       | (3,1)  | (2,2)  | (1,3)  | (0,4) | (-1,5) | (-2,6) | (-3,7) |

By studying the isobarity of  $G_{2,i,j}$  top-down for each line of this table, according to the theory of invariants of differential systems [5,6], we find that the numerators of the fraction (50) can be coefficients in center-affine comitants with the corresponding weights  $-3, -2, -1, 0, 1$ . Using these weights and the formula (5) for the system (34), as well as the fact that  $G_{2,i,j}$  have the type  $(0, 20, 4)$ , we obtain that the mentioned comitants correspond to the types

$$(10, 20, 4), (8, 20, 4), (6, 20, 4), (4, 20, 4), (2, 20, 4). \quad (51)$$

As the quantity  $G_2$  in (30) is the coefficient in front of the homogeneity of degree 6 in the phase variables  $x$  and  $y$ , then it is logical to choose from (51) the type

$$(6, 20, 4), \quad (52)$$

which corresponds to the expression  $G_{2,2,j}$  ( $j = \overline{0, 6}$ ) in Table 1.

This means that according to (20) using the Lie differential operator  $D_3$  for the system (34) from [7] and the numerator of the fraction (50) for fixed  $i = 2$ , we obtain one redefined system of six linear non-homogeneous differential equations

$$\begin{aligned}
D_3(G_{2,2,0} + B_{2,2,0}b_0 + D_{2,2,0}d_0) &= -(G_{2,2,1} + B_{2,2,1}b_1 + D_{2,2,1}d_1), \\
-D_3(G_{2,2,1} + B_{2,2,1}b_1 + D_{2,2,1}d_1) &= G_{2,2,2} + B_{2,2,2}b_2 + D_{2,2,2}d_2, \\
D_3(G_{2,2,2} + B_{2,2,2}b_2 + D_{2,2,2}d_2) &= -(G_{2,2,3} + B_{2,2,3}b_3 + D_{2,2,3}d_3), \\
-D_3(G_{2,2,3} + B_{2,2,3}b_3 + D_{2,2,3}d_3) &= G_{2,2,4} + B_{2,2,4}b_4 + D_{2,2,4}d_4, \\
D_3(G_{2,2,4} + B_{2,2,4}b_4 + D_{2,2,4}d_4) &= -(G_{2,2,5} + B_{2,2,5}b_5 + D_{2,2,5}d_5), \\
-D_3(G_{2,2,5} + B_{2,2,5}b_5 + D_{2,2,5}d_5) &= G_{2,2,6} + B_{2,2,6}b_6 + D_{2,2,6}d_6,
\end{aligned} \quad (53)$$

with eight unknowns  $b_2, d_0, d_1, \dots, d_6$ . From these six equations it results that the expressions contained in them can be coefficients in comitants of the type  $(6, 20, 4)$ . Observe that obtaining an explicit form for solutions of the system (53) is a difficult task. We will show the importance of homogeneities of  $G_{2,2,j}$  from (50) in obtaining the focal quantities for the system (34) on the invariant variety  $\mathcal{V}$  for center and



focus from (27). According to (52) the system (53) defines center-affine comitants belonging to the space

$$S_{1,2}^{(6,20,4)}. \quad (54)$$

According to (20) and (53) such a comitant, belonging to this space, can be written as

$$\begin{aligned} f'_6(x, y) = & (G_{2,2,0} + B_{2,2,0}b_2 + D_{2,2,0}d_0)x^6 - (G_{2,2,1} + B_{2,2,1}b_2 + D_{2,2,1}d_1)x^5y + \\ & + \frac{1}{2!}(G_{2,2,2} + B_{2,2,2}b_2 + D_{2,2,2}d_2)x^4y^2 - \frac{1}{3!}(G_{2,2,3} + B_{2,2,3}b_2 + D_{2,2,3}d_3)x^3y^3 + \\ & + \frac{1}{4!}(G_{2,2,4} + B_{2,2,4}b_2 + D_{2,2,4}d_4)x^2y^4 - \frac{1}{5!}(G_{2,2,5} + B_{2,2,5}b_2 + D_{2,2,5}d_5)xy^5 + \\ & + \frac{1}{6!}(G_{2,2,6} + B_{2,2,6}b_2 + D_{2,2,6}d_6)y^6. \end{aligned}$$

Observe that on the variety  $\mathcal{V}$  from (27) for the system (34) the expressions  $G_{2,2,j}$  ( $j = \overline{0,6}$ ) have the following expressions:

$$\begin{aligned} G_{2,2,0}|_{\mathcal{V}} = G_{2,2,2}|_{\mathcal{V}} = G_{2,2,4}|_{\mathcal{V}} = G_{2,2,6}|_{\mathcal{V}} = -2304L_2, \\ G_{2,2,1}|_{\mathcal{V}} = G_{2,2,3}|_{\mathcal{V}} = G_{2,2,5}|_{\mathcal{V}} = 0 \end{aligned} \quad (55)$$

where

$$\begin{aligned} 24L_2 = & 62g^3h - 2gh^3 + 95g^2hk - 2h^3k + 38ghk^2 + 5hk^3 - 62g^3l + \\ & + 27gh^2l - 39g^2kl + 29h^2kl - 15gk^2l - 8ghl^2 + 15hkl^2 - 5gl^3 + \\ & + 53g^2hm + 66ghkm + 13hk^2m - 127g^2lm - 6h^2lm - 68gklm - \\ & - 15k^2lm - 13hl^2m - 5l^3m + 6ghm^2 + 6hkm^2 - 63glm^2 - 29klm^2 + \\ & + 2lm^3 + 6g^3n + 61gh^2n + 72g^2kn + 63h^2kn + 33gk^2n + 5k^3n - \\ & - 10ghln + 68hklm - 33gl^2n + 15kl^2n - 72g^2mn - 6h^2mn + \\ & + 10gkmm + 8k^2mn - 66hlmn - 38l^2mn - 61gm^2n - 27km^2n + \\ & + 2m^3n + 72ghn^2 + 127hkn^2 - 72gln^2 + 39kln^2 - 53hmn^2 - \\ & - 95lmn^2 - 6gn^3 + 62kn^3 - 62mn^3 \end{aligned}$$

is the second focal quantity of the system (34) on the invariant variety  $\mathcal{V}$  for center and focus (see [4, p. 110]).

Now we concentrate our attention to the construction of the quantity  $G_3$  which is in front of the homogeneity of degree 8 in  $x$  and  $y$  in (50). Writing the system (36)–(41) in the matrix form

$$A_3B_3 = C_3,$$

we obtain

$$G_3 = \frac{G_{3,i,j,k} + B_{3,i,j,k}b_i + D_{3,i,j,k}d_j + F_{3,i,j,k}f_k}{\sigma_{3,i,j,k}}, \quad (i = \overline{0,4}; j = \overline{0,6}; k = \overline{0,8}). \quad (56)$$

Similarly to the previous case, we choose a comitant of the weight  $-1$  of the system  $s(1, 2)$  from (34) which contains as semi-invariant the expression  $G_{3,2,j,k} + B_{3,2,j,k}b_2 + D_{3,2,j,k}d_j + F_{3,2,j,k}f_k$  ( $k = \overline{0, 8}$ ), and we find that it belongs to the space

$$S_{1,2}^{(8,37,6)}.$$

## 9 General type of comitants which have as coefficients expressions with generalized focal pseudo-quantities of the system (34)

Let's consider the extension of the system (36)–(41) obtained from the identity (30) for the system (34) and the function (35) which contains the quantity  $G_k$ , which we write in a matrix form as follows  $A_k B_k = C_k$ . We denote by  $m_{G_k}$  the number of equations and by  $n_{G_k}$  the number of unknowns of this system. Observe that these numbers can be written as

$$\begin{aligned} m_{G_k} &= \underbrace{4+5}_{G_1} + \underbrace{6+7}_{G_2} + \underbrace{8+9}_{G_3} + \cdots + \underbrace{(2k+2) + (2k+3)}_{G_k}, \quad (k = 1, 2, 3, \dots), \\ n_{G_k} &= \underbrace{4+6}_{G_1} + \underbrace{6+8}_{G_2} + \underbrace{8+10}_{G_3} + \cdots + \underbrace{(2k+2) + (2k+4)}_{G_k}. \end{aligned}$$

Hence we obtain

$$m_{G_k} = k(2k+7), n_{G_k} = m_{G_k} + k > m_{G_k}. \quad (57)$$

Similarly to the previous cases, from this system we have

$$G_k = \frac{G_{k,i_1,i_2,\dots,i_k} + B_{k,i_1,i_2,\dots,i_k}b_{i_1} + \cdots + Z_{k,i_1,i_2,\dots,i_k}z_{i_k}}{\sigma_{k,i_1,i_2,\dots,i_k}}, \quad (58)$$

Now it is important to determine the degree of the polynomial  $G_{k,i_1,i_2,\dots,i_k}$  in coefficients of the differential system (34).

Observe that the degree of non-zero polynomial coefficient of  $G_i$  ( $i = \overline{1, k}$ ) in coefficients of the system (34) in the matrix of Cramer's determinant of the order  $m_{G_k}$ , when the column corresponding to the last quantity  $G_k$  is replaced with the column corresponding to free members, forms the following diagram (the last quantity  $G_k$  has the degree 2 according to the substitution):

$$\begin{array}{cccccc} G_1, & G_2, & G_3, & \dots, & G_{k-1}, & G_k. \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ 2 & 3 & 4 & & k & 2 \end{array}$$

Then the degree of the polynomial  $G_{k,i_1,i_2,\dots,i_k}$  in coefficients of the system (34), denoted by  $N_{G_k}$ , can be written as

$$N_{G_k} = m_{G_k} - k + \frac{k(k+1)}{2} + 1,$$

hence according to (57) we have

$$N_{G_k} = \frac{1}{2}(5k^2 + 13k + 2). \quad (59)$$

It is the degree of homogeneity of  $G_{k,i_1,i_2,\dots,i_k}$  in coefficients of the linear and the quadratic parts of the differential system (34) which is contained in a polynomial of the type  $(d) = (\delta, d_1, d_2)$ . Since  $\delta = 2(k+1)$  and  $d_2 = 2k$ , then  $d_1 = N_{G_k} - 2k$ . So we obtain that a comitant of the weight  $-1$  of the system  $s(1, 2)$  from (34), containing the semi-invariant  $G_{k,i_1,i_2,\dots,i_k} + B_{k,i_1,i_2,\dots,i_k} b_{i_1} + \dots + Z_{k,i_1,i_2,\dots,i_k} z_{i_k}$ , which corresponds to the quantity  $G_k$  for  $k = 1, 2, 3, \dots$ , belongs to the type

$$\left( 2(k+1), \frac{1}{2}(5k^2 + 9k + 2), 2k \right), \quad (60)$$

where  $2(k+1)$  is the degree of homogeneity of the comitant in phase variables  $x, y$ ;  $\frac{1}{2}(5k^2 + 9k + 2)$  is the degree of homogeneity of the comitant in coefficients of the linear part  $c, d, e, f$  and  $2k$  is the degree of homogeneity of the comitant in coefficients of the quadratic part of the system (34).

Hereafter the expressions  $G_{k,i_1,i_2,\dots,i_k}$ , which determine the types of comitants (60) corresponding to the quantity  $G_k$  ( $k = 1, 2, 3, \dots$ ) will be called *the defining focal quantities*. The comitants of the type (60) for  $k = 1, 2, 3, \dots$  which contains as the coefficients expressions with the generalized focal pseudo-quantities

$$G_{k,i_1,i_2,\dots,i_k} + B_{k,i_1,i_2,\dots,i_k} b_{i_1} + \dots + Z_{k,i_1,i_2,\dots,i_k} z_{i_k}.$$

will be called *the comitants associated to generalized focal pseudo-quantities*.

For  $G_0$  from (32), which for the system  $s(1, 2)$  from (34) has the type  $(0, 1, 0)$ , we retain the name *a null focal pseudo-quantity*.

The space of comitants of the system  $s(1, 2)$  from (34), corresponding to the type (60), will be denoted by

$$S_{1,2}^{(2(k+1), \frac{1}{2}(5k^2+9k+2), 2k)}. \quad (61)$$

## 10 Comitants which have as coefficients expressions with generalized focal pseudo-quantities of the system $s(1,2,3)$

Let us consider the system  $s(1, 2, 3)$  of the form

$$\begin{aligned} \dot{x} &= cx + dy + gx^2 + 2hxy + kx^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \dot{y} &= ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3cxy^2 + wy^3 \end{aligned} \quad (62)$$

with finitely determined Sibirsky graded algebra of unimodular comitants  $S_{1,2,3}$  [7], for which the function (31) will be write in the form (35), where  $K_2$  is from (26) and  $a_0, a_1, \dots, f_7, f_8, \dots, G_1, G_2, \dots$  are unknowns. Similarly as in the Sections 6 and 7 for

determining the quantity  $G_1$  we write the equations in which splits the identity (30) in the case of the system (62) in the matrix form

$$\tilde{A}_1 \tilde{B}_1 = \tilde{C}_1, \quad (63)$$

where

$$\tilde{A}_1 = \begin{pmatrix} 3c & 3e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3d & 6c+3f & 6e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6d & 3c+6f & 3e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3d & 3f & 0 & 0 & 0 & 0 & 0 & 0 \\ 3g & 3l & 0 & 0 & 4c & 4e & 0 & 0 & 0 & -e^2 \\ 6h & 6g+6m & 6l & 0 & 4d & 12c+4f & 12e & 0 & 0 & 2ce-2ef \\ 3k & 12h+3n & 3g+12m & 3l & 0 & 12d & 12c+12f & 12e & 0 & -c^2+2de+2cf-f^2 \\ 0 & 6k & 6h+6n & 6m & 0 & 0 & 12d & 4c+12f & 4e & -2cd+2df \\ 0 & 0 & 3k & 3n & 0 & 0 & 0 & 4d & 4f & -d^2 \end{pmatrix},$$

$$\tilde{B}_1 = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ G_1 \end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix} 2eg - cl + fl \\ -cg + fg + 4eh - 2dl - 2cm + 2fm \\ -2ch + 2fh + 2ek - 4dm - cn + fn \\ -ck + fk - 2dn \\ 2ep - ct + ft \\ -cp + fp + 6eq - 2dt - 3cu + 3fu \\ -3cq + 3fq + 6er - 6du - 3cv + 3fv \\ -3cr + 3fr + 2es - 6dv - cw + fw \\ -cs + fs - 2dw \end{pmatrix} \quad (64)$$

For each fixed  $i \in \{0, 1, \dots, 4\}$  using the Cramer's rule from the system (63) we find

$$\tilde{G}_1 = \frac{\tilde{G}_{1,i} + \tilde{B}_{1,i} b_i}{\tilde{\sigma}_{1,i}}, \quad (65)$$

where  $\tilde{G}_{1,i}, \tilde{B}_{1,i}, \tilde{\sigma}_{1,i}$  are polynomials in the coefficients of the system (62) and  $b_i$  are undetermined coefficients of the function  $U(x, y)$  from (35).

By studying the matrices (63)–(64) of the system (62) we conclude that the focal pseudo-quantity  $\tilde{G}_{1,i}$  for fixed  $i$  from (65) can be write as

$$\tilde{G}_{1,i} = \tilde{G}'_{1,i} + \tilde{G}''_{1,i}, \quad (i = 0, 1, 2, 3, 4), \quad (66)$$

where  $\tilde{G}'_{1,i}$  (respectively  $\tilde{G}''_{1,i}$ ) are homogeneous polynomials of degree 8 (respectively 9) in coefficients of the linear part and of degree 2 in the coefficients of the quadratic part (respectively of degree 1 in the coefficients of the cubic part) of the differential system (62).

Using here the operators (18) of Lie algebra  $L_4$  from [7] for the system (62) we construct the corresponding operators which we denote respectively by  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$ . Applying these operators under the expressions from (66) we find

$$\begin{aligned} \mathcal{X}_1(\tilde{f}'_4) &= \mathcal{X}_4(\tilde{f}'_4) = \tilde{f}'_4, & \mathcal{X}_2(\tilde{f}'_4) &= \mathcal{X}_3(\tilde{f}'_4) = 0, \\ \mathcal{X}_1(\tilde{f}''_4) &= \mathcal{X}_4(\tilde{f}''_4) = \tilde{f}''_4, & \mathcal{X}_2(\tilde{f}''_4) &= \mathcal{X}_3(\tilde{f}''_4) = 0, \end{aligned}$$

where

$$\begin{aligned}\tilde{f}'_4(x, y) &= \tilde{G}'_{1,0}x^4 - 4\tilde{G}'_{1,1}x^3y + 2\tilde{G}'_{1,2}x^2y^2 + 4\tilde{G}'_{1,3}xy^3 - \tilde{G}'_{1,4}y^4, \\ \tilde{f}''_4(x, y) &= \tilde{G}''_{1,0}x^4 - 4\tilde{G}''_{1,1}x^3y + 2\tilde{G}''_{1,2}x^2y^2 + 4\tilde{G}''_{1,3}xy^3 - \tilde{G}''_{1,4}y^4,\end{aligned}\quad (67)$$

are comitants of the weight  $-1$  of the system (62) and  $\tilde{G}'_{1,i}$ ,  $\tilde{G}''_{1,i}$  are from (65).

According to the above mentioned and (4) the given comitants (67) belongs respectively to the linear spaces

$$S_{1,2,3}^{(4,8,2,0)}, S_{1,2,3}^{(4,9,0,1)}, \quad (68)$$

which are components of Sibirsky graded algebra of comitants  $S_{1,2,3}$  for the system (62).

Taking into account (65) for  $b_i = 0$  ( $i = \overline{0,4}$ ) on the variety  $\mathcal{V}$  from (27) and also (66), (67) we find out that the first focal quantity  $L_1$  of the system (62) is related to the comitants (67) as follows

$$[\tilde{f}'_4(x, y) + \tilde{f}''_4(x, y)]|_{\mathcal{V}} = 8L_1(x^2 + y^2)^2,$$

where

$$L_1 = \frac{1}{4} \{[g(l-h) - k(h+n) + m(l+n)] - 3[p+r+u+v]\}.$$

Similarly to the previous case, for determining the quantity  $G_2$  for the system (62), from the identity (30) we obtain the following equation in the matrix form

$$\tilde{A}_2\tilde{B}_2 = \tilde{C}_2. \quad (69)$$

For each fixed  $i \in \{0, 1, \dots, 4\}$ ,  $j \in \{0, 1, 2, \dots, 6\}$  we find the expression

$$\tilde{G}_2 = \frac{\tilde{G}_{2,i,j} + \tilde{B}_{2,i,j}b_i + \tilde{D}_{2,i,j}d_j}{\tilde{\sigma}_{2,i,j}}. \quad (70)$$

By studying the matrix equality (69) we find that the focal pseudo-quantity from (70) can be written in the form of homogeneity of degree 24 that can be represented in the form

$$\tilde{G}_{2,i,j} = \tilde{G}'_{2,i,j} + \tilde{G}''_{2,i,j} + \tilde{G}'''_{2,i,j}, \quad (71)$$

where  $\tilde{G}'_{2,i,j}$ ,  $\tilde{G}''_{2,i,j}$  and  $\tilde{G}'''_{2,i,j}$  are homogeneity of the type (4) respectively of the form  $(0, 20, 4, 0)$ ,  $(0, 21, 2, 1)$  and  $(0, 22, 0, 2)$ . We note that on the variety  $\mathcal{V}$  from (27) for the system (62) the quantities  $\tilde{G}_{2,2,j}$  ( $j = \overline{0,6}$ ) have the expressions

$$\tilde{G}_{2,2,j}|_{\mathcal{V}} = 2304L_2, \quad (j = 0, 2, 4, 6), \quad \tilde{G}_{2,2,j}|_{\mathcal{V}} = 0, \quad (j = 1, 3, 5).$$

On the other hand, the second focal quantity  $L_2$  of the system (62) can be written with the terms from (71) as follows

$$24L_2 = \tilde{G}'_{2,2,j}|_{\mathcal{V}} + \tilde{G}''_{2,2,j}|_{\mathcal{V}} + \tilde{G}'''_{2,2,j}|_{\mathcal{V}}, \quad (j = 0, 2, 4, 6),$$

where

$$\begin{aligned} \tilde{G}'_{2,2,j}|_{\mathcal{V}} = & 4(62g^3h - 2gh^3 + 95g^2hk - 2h^3k + 38ghk^2 + 5hk^3 - 62g^3l + 27gh^2l - \\ & - 39g^2kl + 29h^2kl - 15gk^2l - 8ghl^2 + 15hkl^2 - 5gl^3 + 53g^2hm + 66ghkm + \\ & + 13hk^2m - 127g^2lm - 6h^2lm - 68gklm - 15k^2lm - 13hl^2m - 5l^3m + \\ & + 6ghm^2 + 6hkm^2 - 63glm^2 - 29klm^2 + 2lm^3 + 6g^3n + 61gh^2n + 72g^2kn + \\ & + 63h^2kn + 33gk^2n + 5k^3n - 10ghln + 68hklm - 33gl^2n + 15kl^2n - 72g^2mn - \\ & - 6h^2mn + 10gkmn + 8k^2mn - 66hlmn - 38l^2mn - 61gm^2n - 27km^2n + \\ & + 2m^3n + 72ghn^2 + 127hkn^2 - 72gln^2 + 39kln^2 - 53hmn^2 - 95lmn^2 - \\ & - 6gn^3 + 62kn^3 - 62mn^3), \end{aligned}$$

$$\begin{aligned} \tilde{G}''_{2,2,j}|_{\mathcal{V}} = & -2(186g^2p + 10h^2p + 117gkp + 45k^2p + 59hlp + 15l^2p + 159gmp + \\ & + 75kmp + 18m^2p + 143hnp + 89lnp + 196n^2p - 69ghq - 57hkq + 69glq + \\ & + 12klq + 9lmq + 60gnq + 3knq + 21mnq + 168g^2r - 6h^2r + 69gkr + 15k^2r + \\ & + 87hlr + 45l^2r + 123gmr + 39kmr + 18m^2r + 171hnr + 129lnr + 222n^2r - \\ & - 13ghs - 17hks - 15gls - 16hms - 15lms - 16gns - 17kns - 19mns - \\ & - 19ght - 15hkt - 17glt - 16hmt - 17lmt - 16gnt - 15knt - 13mnt + \\ & + 222g^2u + 18h^2u + 129gku + 45k^2u + 39hlu + 15l^2u + 171gmu + 87kmu - \\ & - 6m^2u + 123hnu + 69lnu + 168n^2u + 21ghv + 9hkv + 3glv + 12klv - 57lmv + \\ & + 60gnv + 69knv - 69mnv + 196g^2w + 18h^2w + 89gkw + 15k^2w + 75hlw + \\ & + 45l^2w + 143gmw + 59kmw + 10m^2w + 159hnw + 117lnw + 186n^2w), \end{aligned}$$

$$\begin{aligned} \tilde{G}'''_{2,2,j}|_{\mathcal{V}} = & -9(11pq + 15qr - 5ps - rs + pt + 5rt + 3qu - 5su + tu - 7pv - 3rv - \\ & - 15uv + 7qw - sw + 5tw - 11vw). \end{aligned}$$

Similarly to the technique described in the Sections 6 and 7 we choose a comitant of the weight  $-1$  of the system  $s(1, 2, 3)$  from (62) which contains as a semi-invariant the expression  $\tilde{G}_{2,i,j} + \tilde{B}_{2,i,j}b_i + \tilde{D}_{2,i,j}d_j$ . According to the decomposition (71) and the types shown below we find that this comitant is a sum of comitants belonging to the spaces

$$S_{1,2,3}^{(6,20,4,0)}, S_{1,2,3}^{(6,21,2,1)}, S_{1,2,3}^{(6,22,0,2)}. \quad (72)$$

Following this process with the help of the matrix equation

$$\tilde{A}_3\tilde{B}_3 = \tilde{C}_3$$

for each fixed  $i \in \{0, 1, \dots, 4\}$ ,  $j \in \{0, 1, \dots, 6\}$ ,  $k \in \{0, 1, \dots, 8\}$  we obtain

$$\tilde{G}_3 = \frac{\tilde{G}_{3,i,j,k} + \tilde{B}_{3,i,j,k}b_i + \tilde{D}_{3,i,j,k}d_j + \tilde{F}_{3,i,j,k}f_j}{\tilde{\sigma}_{3,i,j,k}}. \quad (73)$$

Similarly to the previous case we find that the focal pseudo-quantity  $\tilde{G}_{3,i,j,k}$  splits into a sum of four terms of the same degree 43 in the coefficients of the system (62),

which according to (4) belongs to the types  $(0, 37, 6, 0)$ ,  $(0, 38, 4, 1)$ ,  $(0, 39, 2, 2)$  and  $(0, 40, 0, 3)$ . Then it results that the comitant of the weight  $-1$  having as a semi-invariant one of the expressions (73) consists of the sum of comitants of the system (62) which belongs to the spaces

$$S_{1,2,3}^{(8,37,6,0)}, S_{1,2,3}^{(8,38,4,1)}, S_{1,2,3}^{(8,39,2,2)}, S_{1,2,3}^{(8,40,0,3)}. \quad (74)$$

Following this process we obtain the sequence of linear spaces (68), (72), (74) etc. of comitants of the system (62). It remain to underline that the generalized focal pseudo-quantities corresponding to  $G_k$  of the given system is exactly a sum of coefficients of these comitants.

It is not difficult to deduce that the generic formula of the types of the comitants in which the generalized focal pseudo-quantities corresponding to  $G_k$  splits as a sum, has the form:

$$\left( 2(k+1), \frac{1}{2}(5k^2 + 9k + 2) + i, 2(k-i), i \right), \quad (i = \overline{0, k}).$$

## 11 Graded algebra of comitants whose spaces contain comitants associated to generalized focal pseudo-quantities of the system (34) and (62)

Thus we obtain for the system (34) the set of spaces of center-affine (unimodular) comitants

$$\mathbb{R} = S_{1,2}^{(0,0,0)}, S_{1,2}^{(0,1,0)}, S_{1,2}^{(4,8,2)}, S_{1,2}^{(6,20,4)}, \dots, S_{1,2}^{(2(k+1), \frac{1}{2}(5k^2+9k+2), 2k)}, \dots \subset S_{1,2}, \quad (75)$$

where  $S_{1,2}$  is Sibirsky graded algebra of the system (34).

Let's consider the graded algebra  $S'_{1,2}$ , generated by the space  $S_{1,2}^{(\delta', d'_1, d'_2)}$  from (75), which can be written as

$$S'_{1,2} = \bigoplus_{(d')} S_{1,2}^{(d')}. \quad (76)$$

Here  $S_{1,2}^{(d')}$  denote linear spaces, contained in  $S_{1,2}^{(\delta', d'_1, d'_2)}$  for all  $(d')$ , as well as the spaces from  $S_{1,2}$  which contains all possible products of spaces (75).

Since the algebra  $S'_{1,2}$  is a graded subalgebra of the algebra  $S_{1,2}$  for the system (34), according to Proposition 4, we obtain that for the Krull dimensions of these algebras the following inequality takes place:

$$\varrho(S'_{1,2}) \leq \varrho(S_{1,2}). \quad (77)$$

Taking into account this inequality, and the fact that from [7] we have  $\varrho(S_{1,2}) = 9$ , according to Definition 2, we have

**Lemma 1.** *The maximal number of algebraically independent generalized focal pseudo-quantities in the center-focus problem for the system (34) does not exceed 9.*

According to the equalities (32), (48), (55) etc. and the conclusion, resulting from Proposition 5, that the number of algebraically independent focal quantities  $L_k$  ( $k = \overline{0, \infty}$ ) can not exceed the maximal number of algebraically independent generalized focal pseudo-quantities, using Lemma 1, we have

**Theorem 2.** *The maximal number of algebraically independent focal quantities of the system (34) on the variety  $\mathcal{V}$  from (27) or, equivalently, from (29), that take part in solving the center-focus problem, does not exceed 9.*

With the help of Hilbert series of the algebras  $S_{1,2}$ ,  $S'_{1,2}$ ,  $SI_{1,2}$  [23] and Remark 3 it can be shown that the predicted upper bound of algebraically independent focal quantities of the system (34) on the variety  $\mathcal{V}$  from (27) ((29)) can be much less than 9, and can be equal to 7 or, may be, even 5.

We note that the similar studies that for the system  $s(1, 2)$  from (34) were realized in the works [27,28,31] for the systems  $s(1, 3)$ ,  $s(1, 4)$ ,  $s(1, 5)$  respectively. This scenario is confirmed for the system  $s(1, 2, 3)$  by studies in the case 9 which allow to form the algebra  $S'_{1,2,3}$  with the same properties as the algebra  $S'_{1,2}$ .

## 12 Main results

Similarly to the considered cases, for any system  $s(1, m_1, m_2, \dots, m_\ell)$  from (1) we have that the algebras similar to the obtained in the above mentioned examples satisfy the inclusion condition

$$S'_{1,m_1,m_2,\dots,m_\ell} \subset S_{1,m_1,m_2,\dots,m_\ell},$$

hence according to Proposition 5, for their Krull dimensions we have

$$\varrho(S'_{1,m_1,m_2,\dots,m_\ell}) \leq \varrho(S_{1,m_1,m_2,\dots,m_\ell}). \quad (78)$$

By the formula (19) we obtain

$$\varrho(S_{1,m_1,m_2,\dots,m_\ell}) = 2 \left( \sum_{i=1}^{\ell} m_i + \ell \right) + 3. \quad (79)$$

Analogously to Lemma 1 and other considered examples, with the help of (78) and (79) it can be shown that the following lemma is true:

**Lemma 2.** *The maximal number of algebraically independent generalized focal pseudo-quantities in the center-focus problem for the system (1) does not exceed the number from (79).*

*Remark 8.* According to the Remarks 5 and 6 and formulation of center-focus problem given in Section 5, as well as the identities (32) we can say that the generalized focal pseudo-quantities, being semi-invariants in above mentioned comitants, have as projections on the variety  $\mathcal{V}$  from (27) ((29)) the focal quantities  $L_k$  ( $k = 1, 2, \dots$ ).

From identity (30) and Lyapunov's function (35) it results that for any system  $s(1, m_1, m_2, \dots, m_\ell)$  we can write the identities of the type (58) for quantities  $G_k$  ( $k = 1, 2, \dots$ ), which have as numerators the generalized focal pseudo-quantities.



Using these quantities and the operator  $D_3$  from (18) of system  $s(1, m_1, m_2, \dots, m_\ell)$  we can determine comitants of the given system, having as coefficients the above mentioned focal pseudo-quantities.

According to the Remark 8 we conclude that the following statement take place:  
**Theorem 3.** *The maximal number of algebraically independent focal quantities of the system (1) on the variety  $\mathcal{V}$  from (27) or, equivalently, from (29), that take part in solving the center-focus problem does not exceed the number from (79).*

We recall that in the introduction it was told that for the systems  $s(1, 2)$  and  $s(1, 3)$  the number of essential conditions for center  $\omega = 3$  and 5, respectively, but for the system  $s(1, 2, 3)$  there is an assumption that  $\omega \leq 13$ .

From Theorem 3 we obtain that the maximal number of algebraically independent focal quantities for the system  $s(1, 2)$  does not exceed 9, for  $s(1, 3)$  does not exceed 11, and for  $s(1, 2, 3)$  does not exceed 17.

These arguments and Proposition 5 with  $\mathcal{V}$  from (27) or, equivalently, from (29), and the defined above algebra  $S'_{1, m_1, m_2, \dots, m_\ell}$  suggest that is true

**The main hypothesis.** *The number  $\omega$  of essential conditions for center from (3) which solve the center-focus problem for the system (1), having at the origin of coordinates a singular point of the second type, does not exceed the number from (79).*

*Remark 9.* The equality (79) shows that the quantity  $\varrho$  is equal to the number of coefficients of the right parts of the system (1) minus one.

Besides [23], the authors have published their vision of the center-focus problem in the theses [24-33].

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