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Constantin Sergeevich Sibirsky
(1928 – 1990)

This issue is a tribute in honor of his 85th birthday

New developments based on the mathematical legacy of C. S. Sibirschi

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Abstract. In this article we survey new developments which occurred during the past ten years on planar polynomial differential equations, developments based on the theory of algebraic invariants founded by C. S. Sibirschi for such systems.

In 2003 on the occasion of the 75th birthday of C. S. Sibirschi, my article entitled "The mathematical legacy of C. S. Sibirsky, basis for future work" appeared in the Bulletin of the Academy of Sciences of Moldova [29]. Ten years have since passed and it is now time to cast a glance over the work based on Sibisrchi's mathematical legacy which has been done in these years. On the occasion of the 85th anniversary of his birthday this year, there cannot be a better way of honoring his memory than by showing that the field founded by him, the invariant theory of polynomial differential equations, is an active area of research today and that many new results were obtained during these past ten years in this area.

Planar polynomial differential equations are systems of the form:

$$(S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where p, $q \in \mathbb{R}[x, y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$. We call *degree* of a system (1) the integer $n = \max(\deg p, \deg q)$. In particular we call *quadratic*, respectively *cubic*, a differential system (1) with $n = 2$, respectively $n = 3$, and we denote by **QS** the class of all quadratic systems.

Problems on polynomial differential systems are usually easy to state but extremely difficult to solve. Thus of the three famous classical problems on these differential systems which have been open for over a hundred years, Hilbert's 16th problem (1900,[18]), Poincaré's problem of the center (1885,[23]) and Poincaré's problem of algebraic integrability (1891,[24,25]), only the problem of the center was solved and this only for linear and quadratic differential systems or some very particular cases of higher degrees. These problems in their general context are daunting at this stage and for this reason let us recall the following words from Hilbert's address at the Paris International Congress of Mathematicians in 1900:

"In dealing with mathematical problems, specialization plays, as I believe, a still more important part than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been not at all or incompletely solved."

Considering the three classical problems mentioned above, stated in the context of general polynomial differential equations, the simplest case is clearly the quadratic one for which the problem of the center was already solved.

At this stage however, the global study of the quadratic class is still a very hard problem. There are several reasons which support this statement. One of them is the elusive nature of limit cycles. Indeed, unlike singularities, limit cycles are usually very hard if not impossible to pin down and the history of their study for the quadratic class includes some notorious errors. Another reason is the large number of parameters involved in the study of this class. Indeed, planar quadratic differential systems depend on 12 parameters, the coefficients of the systems. On **QS** acts the group of affine transformations and time homotheties and due to this action, the study of **QS** ultimately depends on five parameters. To obtain the bifurcation diagram for this class thus means that we have to work in this 5-dimensional topological space which is not $\mathbb{R}^5$ but a much more complicated space.

The third reason for the difficulties in this study lies in the necessity to perform complicated calculations. Indeed, consider for example the study of the bifurcation hypersurfaces of singularities of quadratic systems. These bifurcation hypersurfaces are algebraic but they sit in a 12-dimensional space or in a 5-dimensional space if we use the group action. Some of these hypersurfaces are of a high degree. Finding the singularities of these hypersurfaces means solving systems of polynomial equations of high degrees. Also we need to know the intersection points of these hypersurfaces and even more, namely how they intersect, in other words their intersection numbers. Of course, studying the singularities is just the beginning. What comes afterwards is not less complicated, namely the study of the analytic (non-algebraic) bifurcation hypersurfaces. This is mainly done by numerical analysis.

In our work on the quadratic class (see for example [2, 20]), the computations were done by using Mathematica, Maple or the program P4 (see [14]). There are also other computer programs such as Macauley 2, CoCoa and Singular. These high level programming languages are used for Commutative Algebra and Algebraic Geometry but some begin to be used also for Dynamical Systems. One of the ingredients occurring in these specialized programs is the theory of Gröbner bases and the Buchberger's algorithm for computing them. In the future it would be wise to appropriate these programs for problems on polynomial differential systems. However, faced with the challenges mentioned above it seems that the computer programs we have, come still short of expectations.

This however does not mean that we should give up. Indeed, fortunately we can point out some achievements in the direction of computations. An example is the successful computer program P4 (see [14]) allowing us to construct phase portraits and determine the nature of singularities for individual polynomial differential systems.

The first subclass studied globally was the family of all of quadratic systems with a center. The phase portraits for this class were obtained by N. Vulpe (see [39]) followed by the bifurcation diagram of this class (see [22, 28, 41]).

The next subclass studied globally, using global geometric concepts was the class **QW3** of quadratic systems with a third order weak focus (see [20]). This family depends on two parameters. Systems in **QW3** are important for Hilbert's 16th problem since weak foci of third order produce up to a maximum of three limit cycles, close to the foci, in quadratic perturbations. The work on **QW3** in [20] was based on the theorem saying that no limit cycle could surround a weak focus of third order of a quadratic system (see [19]) and on work (see [1]) done with usual techniques which do not involve global geometric concepts. In our study [20], global topological invariants were used for the classification.

A family which is again important for Hilbert's 16th problem is the class **QW2** of quadratic differential systems with a weak focus of second order. Indeed, a quadratic system with a second order weak focus could produce up to a maximum of two limit cycles close to the focus in quadratic perturbations of the system. The study of this family was more challenging since this is a three parameter subclass of **QS**, modulo the group action. In this study both topological and polynomial invariants were used in the classification.

So far no global studies of families of quadratic systems which depend on four or five parameters were done, using global concepts and in particular topological and polynomial invariants.

However a large number of articles on classification problems for quadratic families of systems were done, but not from a global geometric viewpoint. These studies are tied down to fixed normal forms and cannot readily be applied to other presentations of the systems. They employ usual techniques which are not global, and in particular they do not use topological and polynomial invariants. This is a major drawback for several reasons. Firstly because the results cannot be applied in other contexts, for different presentations (normal forms) of the systems. Secondly, in a study several normal forms could occur but no mention is made of how the results obtained in one specific normal form relate to those obtained in another normal form. There is no global viewpoint tying up the results in a single whole so as to lead us to a global understanding of the phenomena occurring in the specific family.

In recent years progress has however been made and the mathematical tools developed by the school of Sibirschi, the theory of polynomial invariants, played a major role. They are important because they allow us to study a family in its full parameter space independent of the several particular normal forms in which the systems are presented and which are necessary for their study. In particular they allow us to pass easily from one normal form to another and thus glue results obtained with respect to several such normal forms in a single global picture. Examples or works where this approach was taken are: [30, 32–37]. The family of Lotka-Volterra differential systems is important since these systems occur in many areas of applied mathematics. The two studies [36] and [37] of this family not only produced the only complete and correct list of phase portraits of this family known in the literature but also characterized each one of the phase portraits in terms of invariant conditions with respect to the affine group and time homotheties. This was possible since the topological study done in [37] was based on the study of all possible configurations of invariants straight lines of this family which was done in [36]. The configuration of invariant lines is a concept introduced by the authors and this notion turned out to be a powerful geometric classification tool for this family. This last study was achieved because of the series of articles [30, 32–35] where the classification of all quadratic systems possessing invariant lines of total multiplicity at least four was achieved.

We mentioned above subclasses of **QS** for which we have obtained the topological classification and in some cases also the characterization of phase portraits in terms of invariant conditions.

We now turn to work done on classifying the *whole class* **QS** according to a specific feature such as for example according to their singularities. Recently the study of the *whole class* **QS** according to the *global geometric configurations of singularities at infinity* was completed (see [3]). This work was based on [31]. In [7], Artés, Llibre and Vulpe classified **QS** according to their finite singularities. They did not distinguish among the strong or weak foci, or among weak foci of various orders, or among the strong or weak saddles. Hence this work needs to be augmented so as to include all these distinctions which are important in the production of limit cycles. This is going to be done within the larger frame of classifying **QS** with respect to the *global geometric configurations* of both finite and infinite singularities. Work in this direction has already begun. Thus in the two articles (see [4, 5]) the *global geometric configurations* of both finite and infinite singularities were given for quadratic systems having the total multiplicity m_f of finite singularities less than three. Work is now in progress for the remaining two cases, i.e. $m_f = 3$ and $m_f = 4$.

We pass now to the second classical problem, namely the problem of the center. Invariant conditions with respect to the general linear group $GL(2, \mathbb{R})$ were given first by Sibirschi [38] for having quadratic systems with a center, when the center is placed at the origin. Then the invariant conditions with respect to the group $\mathrm{Aff}\,(2, \mathbb{R})$ of affine transformations were determined by Boularas, Sibirschi and Vulpe [12] for having quadratic systems with a center (or two centers) arbitrarily located on the phase plane.

Romanovski and Shafer wrote a book (see [27]) on the problem of the center which takes a computational approach. The work of Sibirschi is cited in their book which contains a chapter on invariants of the rotational group.

Three years ago the complete characterization of all weak singularities (foci, centers and saddles) via invariant theory, for the family of quadratic systems was done by Vulpe [40]. In this paper necessary and sufficient conditions for a real quadratic system to possess a fixed number of weak singularities of a specific order are given. The conditions are stated in terms of affine invariant polynomials in the 12-dimensional space of the coefficients. These results play an important role in the determination of *global geometric configurations* of singularities mentioned above.

The third classical problem mentioned at the beginning is Poincaré's problem of algebraic integrability. This problem, stated by Poincaré in [25], asks for giving necessary and sufficient conditions for a planar polynomial system (1) to have a rational first integral. Such a system generates a foliation with singularities on the plane such that all its leaves are algebraic. This is a special case of the theory of invariant algebraic curves of polynomial differential equations developed by Darboux. Poincaré was very enthusiastic about this theory as it can be seen from the following lines of Poincaré which appeared in [24]:

"La question de l'intégrabilité algégrique des équations différentielles du premier ordre et du premier degré n'a pas attiré l'attention des géomètres autant qu'elle méritait. la voie a été ouverte, il y a vingt ans, par un admirable travail de M. Darboux;..."

In recent years the theory of Darboux has flourished and numerous new results were obtained on algebraic curves of differential equations. The theory of polynomial invariants has begun to intervene in some of the publications on this theme. We only mention here some of them.

The problem of characterizing in terms of invariant polynomials the class of quadratic systems which possess a polynomial first integral was completely solved in [8].

The problem of determining necessary and sufficient conditions for quadratic systems to possess a rational first integral of degree two was completely solved in terms of polynomial invariants in [6] where the first integral is a quotient of invariant polynomials.

In [11] the algebraic theory of invariants of differential equations is applied to construct the first integrals for the family of real polynomial differential systems of the form $x' = cx + dy + xC_r(x, y)$, $y' = ex + fy + yC_r(x, y)$, where $C_r(x, y)$ is a real homogeneous polynomial of degree $r \geq 1$.

Within the mathematical school created by Sibirschi we observe a new direction of studies on applications of Lie algebras to the study of differential systems. For example we have [26] where such applications are developed. Using this theory a set of new results for various families of systems of differential equations where obtained. We mention here the articles [13, 16, 17, 21].

In conclusion we can safely say that during the ten years which have passed since the publication of [29], a wealth of new material appeared in print in which the invariant theory of planar polynomial differential systems founded by C. S. Sibirschi has played a major role. This field of studies is alive and other works are now in progress.

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